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Aggregation-Theoretic Monetary Aggregation over the Euro Area, when Countries are Heterogeneous

by

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Contents

Abstract

We derive fundamental new theory for measuring monetary service flows aggregated over countries within the European Monetary Union (EMU). We develop three increasingly restrictive approaches: (1) the heterogeneous agents approach, (2) the multilateral representative agent approach, and (3) the unilateral representative agent approach. Our heterogeneous agents approach contains our multilateral representative agent approach as a special case.

In our most general approach, we assume the existence of a representative consumer within each country to aggregate within each country. We use a stochastic approach to aggregation across countries over the heterogeneous representative agents, and we derive the resulting formulas for stochastic aggregation over countries. Our theory permits monitoring the effects of policy at the aggregate level over the euro area, while also monitoring the distribution effects of policy among the countries of the euro area. Our approach requires the simultaneous use of two inflation indexes over the euro area.

JEL Classifications: C43, C82, E41, E51, F31. Keywords: Monetary Aggregation, Aggregation over Countries, Heterogeneous Agents, Multilateral Aggregation; Euro Area.

Non-Technical Summary

The aggregation theory for aggregating over monetary assets, their dual user cost prices, and interest rates has been available for a closed economy, since the theory was first derived by Barnett (1980,1987) and organized within the book, *The Theory of Monetary Aggregation*, edited by Barnett and Serletis (2000). More recently there has been growing interest in the extension of that theory to the multicountry case, especially for purposes of aggregation of monetary service flows and prices within the euro area. The assumptions needed to treat a group of countries as a single country are not easily accepted, since the existence of multiple countries within an area tends to contradict the demographic and taste distribution assumptions accepted in the closed economy theory. The purpose of this paper is to produce the direct, rigorous extension of the single country aggregation theory to the open economy, multicountry case under reasonable assumptions.

As a result of the particular relevancy for the euro area, the theory is derived in a form applicable both before and after the appearance of the euro, with the historical data containing exchange rates among the area's legacy currencies. The results are derived under three sets of increasingly strong assumptions. In the first case, the aggregation theory permits very general forms of heterogeneity among countries and uses stochastic heterogeneous agents theory. This approach is needed to permit aggregation of the historical data prior to the existence of a common currency and prior to progress towards convergence within the area. Under this most general theory, no representative agent is assumed to exist for the euro area, although a representative agent is assumed to exist within each country. This theory not only permits aggregation under reasonable assumptions, but also permits stochastic monitoring of progress towards convergence, using second moment dispersion measures.

Under somewhat stronger assumptions, we find that our stochastic heterogeneous agents approach converges to a new multilateral representative agent approach, permitting recursive aggregation first within countries and then over countries in a manner fully consistent with deterministic economic theory. Significant heterogeneity of tastes remains possible across countries under our multilateral representative agent approach. Since the multilateral representative agent approach is strictly nested within the heterogeneous agents approach, the heterogeneous agents approach, in practice, would converge to the multilateral representative agent approach on its own, when the necessary assumptions become satisfied.[2]

The third approach, although potentially convenient in practice, requires very strong assumptions. We call this most restrictive case the unilateral representative agent approach, within which the country of residence of a consumer becomes irrelevant to the person's consumption decisions. At the present time, this most restrictive case seems

[2] Our use of the terms unilateral and multilateral representative agents should not be confused with the unrelated concept of unilateral and bilateral index numbers, as in Anderson, Jones, and Nesmith (1997a, p. 75).

primarily of interest in theory, since the two less-restrictive approaches can be implemented in practice without unreasonable difficulty.

Our theory produces a number of surprising results. For example, we find that there is a need for two different consumer price indexes: one for deflation of nominal to real money balances and another for deflation of nominal to real consumer expenditure. The two consumer price indexes become the same only under our strongest assumption structure, such that the residents of the euro area behave as if they were residents of the same country. The existence of the two consumer price indexes did not appear in earlier theory derived for a single closed economy. We also find that the Divisia second moments, which play only a minor role in the closed economy aggregation theory, can be useful in our heterogeneous agents theory for monitoring progress in many dimensions, including (1) convergence progress towards the more restrictive approaches to aggregation, (2) monitoring distribution effects of policy across countries within the euro area, and (3) exploration of information loss from aggregation, when some of the underlying assumptions are violated.

In addition to deriving the implied formulas for aggregation, we also derive the dual user cost price aggregation formulas and the interest rate aggregation formulas. We find that the current approach to aggregation over interest rates is not consistent with the relevant aggregation theory.

1. Introduction

The fields of monetary aggregation and index number theory, and the broader fields of general financial aggregation and index number theory, were first rigorously connected with the long literature on microeconomic aggregation and index number theory by Barnett (1980,1987). A collection of his most important contributions to that field is available in Barnett and Serletis (2000). That book contains extensions in many directions, including introduction of risk, demand by firms as well as consumers, and production of monetary services by financial firms. But Barnett's work in those publications has been based upon the assumption that the data was produced by a single closed economy.

The purpose of this paper is to extend that theory to the multicountry case in a form that would be applicable to the euro area both prior to and after the introduction of the euro. Progress towards convergence among the euro area economies has occurred, and further progress is expected into the future. As a result, our results are produced under a sequence of increasingly strong assumptions, beginning with (1) a heterogeneous agents approach applicable to the past under reasonable assumptions, and then to (2) a new multilateral representative agent approach applicable to the area under reasonable convergence assumptions, and finally to (3) a unilateral representative agent approach requiring very strong assumptions, perhaps relevant to the very distant future, if at all.

Prior to the introduction of the euro, our heterogeneous agents approach provides a substantial generalization of our multilateral representative agent approach. At some date following the introduction of the euro, our heterogeneous agents approach could become mathematically equivalent to the multilateral representative agent approach, since the assumptions necessary for equivalency of the two approaches are reasonably related to objectives of the EMU. But the far more restrictive unilateral representative agent approach requires very strong assumptions. In particular the unilateral representative agent approach would require convergence of inflation rates and interest rates across countries and would imply demographic convergence to a homogeneous population, such that the country of residence of a consumer would become irrelevant to the unilateral representative agent's decisions.[3]

We prove that identical tastes across countries are not sufficient for the existence of a unilateral representative agent, since tastes specific to a country do not exist for a unilateral representative agent, who does not recognize the country of residence of a consumer. Under the assumptions required for the existence of a unilateral representative agent, the allocation of goods, assets, and services over countries is indeterminate. In contrast to the very restrictive unilateral economic agent approach, our heterogeneous agents approach can be used both before and after the introduction of the euro, with recognition of the potential equivalence to our multilateral representative agent approach at some time after the introduction of the euro.

[3] There would have to be convergence of all kinds of rates of return on financial assets, including bond yields and bank interest rates across countries. This convergence could not occur without fiscal harmonization and full completion of a single market for each financial and banking service.

Since the proposal for a common European currency first arose, a number of researchers have sought to determine how to measure monetary service flows aggregated over the proposed euro area in a manner that would be consistent with aggregation theory.[4] Two approaches have been proposed and applied by other researchers.[5] One has been called the direct approach and the other the indirect approach. We show that the direct approach implies the existence of our unilateral representative agent, which requires assumptions that we consider to be very restrictive.[6] Under this approach, assets of each type are first aggregated over countries by simple sum aggregation. Divisia aggregation then is used to aggregate over each internationally-aggregated asset type. The alternative indirect approach uses Divisia aggregation within countries and then *ad hoc* weighting of those within-country indexes to aggregate over countries.[7] The indirect approach produces a result that is disconnected from theory and does not produce nesting of the multilateral or unilateral representative agent approaches. But the indirect approach's intent and objectives are similar to those of our rigorously derived heterogeneous agents approach.

This paper's direct extensions of Barnett's earlier work produce a number of unexpected innovations, including the need for simultaneous use of two different consumer price indexes for internal consistency of the theory. The current paper is intended to solve the central theoretical problems associated with monetary aggregation over countries. This paper is likely to be the first in a series of papers. Later papers are planned to incorporate risk aversion along with other extensions. The extension to risk aversion should be jointly applicable both to monetary and nonmonetary assets. The resulting extended theory will not only be relevant to aggregation over risky monetary assets but also to modeling substitution among both monetary and nonmonetary assets, such as common stock. The solutions of the fundamental problems addressed in the current paper are logically prior to our planned future work on this subject.

2. Definition of Variables.

[4] See, e.g., M. M. G. Fase and C. C. A. Winder (1994), Spencer (1997), Wesche (1997), Fase (2000), Beyer, Doornik, and Hendry (2001), Stracca (2001), and Reimers (2002). For general information on empirical applications of this approach in the single-country as well as international contexts, see Barnett and Serletis (2000), Belongia and Binner (2000), and Serletis (2001). For a convenient overview of the relevant theory in the single country case, see Anderson, Jones, and Nesmith (1997b).

[5] Some of those studies were applied to the ERM ("exchange rate mechanism") countries. The European Monetary System (EMS) preceded the EMU, where the EMS comprised the ERM together with the ecu (European currency unit) market basket of currencies. The ERM countries included the UK and were a superset of the EMU countries.

[6] Those studies often have used *ad hoc* weighted averages of interest rates or of inflation rates over countries to produce one interest rate for each asset type and one inflation rate for the euro area. This computational approach does not solve the theoretical problems associated with implicitly assuming identical interest rates and inflation rates across countries in an area with heterogeneous tastes. In addition, the use of those *ad hoc* weighted averages of inflation rates or interest rates is not consistent with index number theory and hence produces theoretical internal inconsistencies.

[7] GDP weights have often been used.

All results are in continuous time, so that all variables should be viewed as functions of time.[8] In addition, the current analysis assumes certainty equivalence within the decisions of each consumer. Under risk neutrality, contemporaneously random rates of return need only be replaced by their expectations to attain certainty equivalence.

Let K be the number of countries in the European Monetary Union (the "EMU"), i.e. in the "euro area." We let $p_k^* = p_k^*(\mathbf{p}_k)$ be the true cost of living index in country $k \in \{1,\ldots,K\}$, where $\mathbf{p}_k = \mathbf{p}_k(t)$ is the vector of prices of consumer goods at time t and $\mathbf{x}_k = \mathbf{x}_k(t)$ is the vector of per-capita real rates of consumption of those goods in country k at time t.[9] Let $H_k = H_k(t)$ be the population of country k at time t, and let m_{kji} be the nominal per capita holdings of asset type i located or purchased in country j but owned by economic agents in country k.[10] The holdings are per capita relative to country k's own population, H_k. We present all results in per capita form, since the per capita variables are the ones that are needed in demand functions at the aggregate level. In addition the correlation with inflation tends to be in terms of per capita flows, since increases in monetary services that produce no change in per capita monetary services just accommodate population growth.

Assume that asset holders within the euro area also sometimes hold assets in Z countries that are outside the euro area. Let N_j be the number of asset types available within country j, and let N be the total number of asset types available within all of the relevant countries $j \in \{1,\ldots,K+Z\}$, where clearly $N \geq N_j$ for all $j \in \{1,\ldots,K+Z\}$. Then the subscripts of m_{kji} have range: $k \in \{1,\ldots,K\}$, $j \in \{1,\ldots,K+Z\}$, $i \in \{1,\ldots,N\}$. We are not limiting i to be within $\{1,\ldots,N_j\}$, since we wish to associate a unique numerical value of i to each asset type, regardless of country j within which the asset is located.[11] As a result, for each (k,j) there will necessarily be zero values of m_{kji} for $N - N_j$ values of i. If countries j and k do not share the same currency, then nominal holdings are converted to units of country k's currency using the exchange rate between country k's and country j's currencies.[12] Then $m_{kji}^* = m_{kji}/p_k^*$ is the real per capita holdings of asset i located or purchased in country j but owned by economic agents in country k.[13]

Let $r_{kji} = r_{kji}(t)$ be the holding-period after-tax yield on asset i located or purchased in country j and owned by an economic agent in country k at instant of time t, where all

[8] In a later section, we provide the procedure for conversion of the continuous time formulas to discrete time formulas, as is required to operationalize the formulas for use with data acquired in discrete time.
[9] If the aggregation conditions for the existence of a representative consumer do not apply, relevant theory for computing a consumer price index for a country can be found in Diewert (2001).
[10] In the case of retail deposits in banks in country j, the asset would be located in country j, regardless of the country of residence, k, of the depositor. But if the asset is a negotiable security, such as commercial paper, an asset purchased in country j could be held in country k.
[11] This ability becomes necessary when we define and derive the unilateral representative agent approach.
[12] Similarly we assume that prices of consumer goods are converted to units of country k's currency. Since aggregation over consumer goods is not the primary subject of this paper, our notation for consumer goods quantities, expenditures, and prices is less formal than for monetary assets.
[13] Note that deflation of nominal balances is relative to prices in the country of the asset's owner, regardless of the country within which the asset is located.

asset rates of return are yield-curve adjusted to the same holding period (e.g., 30 days).[14] It is important to recognize that the subscript k identifies the country of residence of the asset holder, and not necessarily the country of location of the asset. Rates of return on foreign denominated assets owned by residents of country k are understood to be effective rates of return, net of the instantaneous expected percentage rate of change in the exchange rate between the domestic and foreign currency.[15] At some time following the introduction of the euro, the dependency of rates of return upon k is expected to end, and the dependency upon j will be relevant only to holdings within the euro area of assets located in the Z countries outside the euro area.[16] Hence at some time after the introduction of the euro, it follows that r_{kji} will be independent of (j,k) for all j, $k \in \{1,\ldots,K\}$.

Let $R_k = R_k(t)$ be the benchmark rate of return in country k at instant of time t, where the benchmark rate of return is the rate of return received on a pure investment providing no services other than its yield.[17] Then

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[14] In most cases below, the adjustment for taxation will have no effect, unless the marginal tax rate is not the same on assets appearing in the numerator and denominator of the shares. See Barnett and Serletis (2000, p. 20). The yield curve adjustment of rates of return of different maturities is acquired by subtracting from the asset's yield the country's Treasury security yield of the same maturity and then adding that yield differential onto the Treasury security yield of the chosen holding period. The same holding period should be used for all assets. Unlike risk premia, maturity premia exist even if the economic agents are risk neutral. There are many other relevant details to the use of this theory with actual data, such as the procedure for introducing new goods through imputation of a reservation price and switching temporarily to the Fisher ideal index. Excellent sources of information on handling those matters are Anderson, Jones, and Nesmith (1997a,b) and Barnett and Serletis (2000). For example, regarding the yield curve adjustment procedure, see Anderson, Jones, and Nesmith (1997a, p. 70-71). Own rate adjustment formulas can be found in Table 7 of that same article, and regression based proxies for own-rates on pp. 65-68 of that article.

[15] The forward premium or discount for the percentage expected rate of change in exchange rates can be computed using spot and forward exchange rates. In applications in discrete time, the adjustment added onto the foreign interest rate is (F-E)/E, where E is the spot exchange rate (domestic currency per unit of foreign currency) and F is the forward exchange rate. If the spot and forward rate data are not available, then uncovered interest rate parity could be assumed to impute the domestic rate of return on an asset to foreign holdings of the same asset net of expected variation in the spot exchange rate. But the services of like assets might not be identical in different countries and governmental regulation of interest rates and risk aversion could damage the uncovered interest rate parity theory. In addition, if there are holdings in country k or more than one asset type in the currency of country j, then the imputed expected spot exchange rate variation between the two currencies could be inconsistent across the two asset types. Such inconsistencies can result (1) because of differences in transactions costs to arbitrage the violations of interest rate parity, or (2) because of differences of risk, or (3) because of interest rate regulation of some assets.

[16] Dependency upon k will continue so long as retail accounts in some countries in the euro area remain available only to citizens of those countries.

[17] See the Appendix regarding construction of a proxy for the benchmark rate. It is often stated that the benchmark asset's rate of return must be "capital certain," i.e. risk free. This conclusion, although producing the correct result, should be interpreted carefully. Under risk neutrality, e.g., the benchmark rate stochastic process need only be replaced by its mean function. Barnett (1995, section 5) has proven that certainty equivalence applies in the risk neutral case, so long as preferences are intertemporally separable and all variables are replaced by their expectations. Although the benchmark rate in the risk neutral case is not risk free, its mean is nonstochastic and contains no risk premium, and it is that risk free mean that is used in our formulas under risk neutrality. In the risk averse case, the benchmark rate must be replaced by

$$\pi^*_{kji}(t) = R_k(t) - r_{kji}(t)$$

is the real user cost price of asset i located or purchased in country j and owned by residents of country k at time t, and $\pi_{kji} = p^*_k \pi^*_{kji}$ is the corresponding nominal user cost.[18] It does not matter whether real or nominal interest rates are used, since the inflation rate conversion between nominal and real applies to both terms in the user cost formula and hence cancels out between the two terms.

Technically speaking, whenever m_{kji} is zero, as often will happen when a particular asset type i is not available within country j, the user cost price should be the asset's reservation price in country j. But in practice, terms containing assets having zero quantity will drop out of all of our formulas, except when the asset's quantity becomes nonzero in the next period. In such cases, the reservation price must be imputed during the period preceding the innovation and the new goods introduction procedure must be used.[19] Since such innovations are infrequent, it usually will not be necessary to impute a reservation price or interest rate to asset holdings for which $m_{kji} = 0$.[20]

We now define

$$\mathbf{m}^*_{kj} = (m^*_{kj1}, \ldots, m^*_{kji}, \ldots, m^*_{kjN})',$$

$$\mathbf{m}_{kj} = (m_{kj1}, \ldots, m_{kji}, \ldots, m_{kjN})',$$

$$\mathbf{r}_{kj} = (r_{kj1}, \ldots, r_{kji}, \ldots, r_{kjN})',$$

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its mean minus a deterministic adjustment for risk aversion. In short, the rate of return on the benchmark asset need not itself be nonstochastic, but in our user cost formulas, the stochastic benchmark rate must be replaced by a nonstochastic risk-adjusted property of the stochastic process. For example, in the risk neutral case, $\pi^*_{kji}(t) = E[R_k(t)] - E[r_{kji}(t)]$, where E is the expectation operator. While $R_k(t)$ need not be risk free, E[R,(t)} is risk free, and it is that risk free expectation that is entered into the user cost formula. See, e.g., Barnett and Serletis (2000, chapter 12).

[18] For these formulas and results, see Barnett (1978, section 3; 1980, section 3.2; or 1987, section 2.1). In discrete time, it is necessary to discount to the beginning of the period all interest paid at the end of the period. This requires dividing nominal and real user costs by $1+R_k$. The dependency upon that denominator cancels out in most applications, since that denominator does not depend upon i, while the user costs appear in both the numerators and denominators of all share weights.

[19] For the new goods introduction procedure, see Barnett and Serletis (2000, p. 77, footnote 25) and Anderson, Jones, and Nesmith (1997a, pp. 77-78), who in turn cite Diewert (1980, pp. 498-501).

[20] In practice, when $m_{kji} = 0$ for some (k,j,i) and remains at 0 into the next time period, r_{kji}, π_{kji}, and π^*_{kji} can be left in symbolic notation in any vectors in which they appear, since there will be no need to impute numerical values to them.

$$\boldsymbol{\pi}^{*}_{kj} = (\pi^{*}_{kj1},\ldots,\pi^{*}_{kji},\ldots,\pi^{*}_{kjN})',$$

$$\boldsymbol{\pi}_{kj} = (\pi_{kj1},\ldots,\pi_{kji},\ldots,\pi_{kjN})',$$

and let

$$\mathbf{m}^{*}_{k} = (\mathbf{m}^{*}_{k1},\ldots,\mathbf{m}^{*}_{kj},\ldots,\mathbf{m}^{*}_{k,K+Z})',$$

$$\mathbf{m}_{k} = (\mathbf{m}_{k1},\ldots,\mathbf{m}_{kj},\ldots,\mathbf{m}_{k,K+Z})',$$

$$\mathbf{r}_{k} = (\mathbf{r}_{k1},\ldots,\mathbf{r}_{kj},\ldots,\mathbf{r}_{k,K+Z})',$$

$$\boldsymbol{\pi}^{*}_{k} = (\boldsymbol{\pi}^{*}_{k1},\ldots,\boldsymbol{\pi}^{*}_{kj},\ldots,\boldsymbol{\pi}^{*}_{k,K+Z})',$$

$$\boldsymbol{\pi}_{k} = (\boldsymbol{\pi}_{k1},\ldots,\boldsymbol{\pi}_{kj},\ldots,\boldsymbol{\pi}_{k,K+Z})'.$$

3. Aggregation within Countries[21]

Aggregation within countries uses the existing theory developed by Barnett (1980, 1987).[22] That theory uses the economic approach to index number theory and assumes the existence of a representative agent within each country.[23] To avoid the unnecessary imputation of reservation prices to assets not being held by residents of country k, we shall restrict most of our computations to the index set

$$S_k = \{(j,i): m_{kji}>0, j\in\{1,\ldots,K+Z\}, i\in\{1,\ldots,N\}\}$$

for all $k\in\{1,\ldots,K\}$.

[21] We present our results for monetary asset holdings by consumers. But Barnett (2000, p. 63, equations 40 and 41) proved that it makes no difference for the aggregation theory whether the asset demand is by consumers or by firms or by a combination of both. The issues for aggregation over economic agents is no more or less difficult, if some of the economic agents are consumers and some are firms, all are consumers, or all are firms. A possible exception regards the measurement of the inflation rate for consumers versus firms. If aggregate markets are not cleared and incentive compatibility fails for firms, the inflation rate for firms can differ from that for consumers. But as we shall see, the price index used to deflate nominal to real money balances will not be the usual consumer price index and will not be affected by problems regarding market clearing and incentive compatibility. We consider these matters further in the section on possible future extensions.

[22] We shall introduce that relevant economic decision problem, when needed below, in Decision 4 of Section 5.2.

[23] The same results could be produced within countries by using the stochastic approach to aggregation that we use over countries in the next section. The stochastic approach does not require the existence of a representative agent and is best understood as a heterogeneous agents approach.

Definition 1: Within each country $k \in \{1,\ldots,K\}$, define the monetary real user-cost price aggregate Π_k^*, the monetary nominal user-cost price aggregate $\prod_k$, the real per-capita monetary services aggregate M_k^*, and the nominal per-capita monetary services aggregate M_k by the following Divisia indices:

$$d \log \Pi_k^* = \sum_{(j,i)\in S_k} w_{kji}\, d \log \pi_{kji}^*,$$

$$d \log \prod_k = \sum_{(j,i)\in S_k} w_{kji}\, d \log \pi_{kji},$$

$$d \log M_k^* = \sum_{(j,i)\in S_k} w_{kji}\, d \log m_{kji}^*,$$

$$d \log M_k = \sum_{(j,i)\in S_k} w_{kji}\, d \log m_{kji},$$

where

$$w_{kji} = \frac{\pi_{kji} m_{kji}^*}{\boldsymbol{\pi}_k' \mathbf{m}_k^*} = \frac{\pi_{kji}^* m_{kji}^*}{\boldsymbol{\pi}_k^{*\prime} \mathbf{m}_k^*}$$

$$= \frac{(R_k - r_{kji}) m_{kji}^*}{\sum_{(j,i)\in S_k} (R_k - r_{kji}) m_{kji}^*} = \frac{(R_k - r_{kji}) m_{kji}}{\sum_{(j,i)\in S_k} (R_k - r_{kji}) m_{kji}}.$$

Observe that $0 \le w_{kji} \le 1$ for all $k \in \{1,\ldots,K\}$, $j \in \{1,\ldots,K+Z\}$, and $i \in \{1,\ldots,N\}$. Also observe that $\sum_{(j,i)\in S_k} w_{kji} = 1$ for all $k \in \{1,\ldots,K\}$. Hence the shares, w_{kji}, have the properties of a probability distribution for each $k \in \{1,\ldots,K\}$, and we could interpret our Divisia indexes above as Divisia growth rate means. But since it is convenient to assume the existence of a representative agent within each country, the statistical interpretation as a mean is not necessary. We instead can appeal to the Divisia index's known ability to track the aggregator function of the country's representative consumer.

The following result relating nominal to real values follows immediately.

Lemma 1: $M_k = M_k^* p_k^*$ and $\prod_k = \Pi_k^* p_k^*$.

Proof: Follows from the known linear homogeneity of the Divisia index. *Q.E.D.*

4. Aggregation Over Countries

Our heterogeneous agents approach to aggregation over countries is based upon the stochastic convergence approach to aggregation, championed by Theil (1967) and developed further by Barnett (1979a; 1979b; 1980, chap. 2). This approach not only can be used to aggregate over heterogeneous consumers, but also jointly over consumers and firms. Hence the approach is not only a heterogeneous consumers approach, but more generally is a true heterogeneous agents approach. See, e.g., Barnett and Serletis (2000, pp. 88-90 and chapter 9). By assuming the existence of a representative agent within each country, and treating those representative agents as heterogeneous agents, we produce a heterogeneous countries approach to aggregation over countries.

In aggregating within the euro area, this approach implies that the countries' characteristics, including cultures, tastes, languages, etc., were sampled from underlying theoretical populations consistent with the climates, histories, resources, geographies, neighboring population characteristics, etc. All time varying variables then become stochastic processes. Each Divisia index aggregating over component stochastic processes becomes the sample mean of realizations of those stochastic processes, and thereby an estimate of the mean function of the underlying unknown population stochastic process. The distributions of those stochastic processes are derived distributions induced by the random sampling from country characteristics. The derived empirical distributions of the countries' solution stochastic-process growth rates impute probabilities to countries equal to their relevant expenditure shares in euro area expenditure.

Let e_k be the exchange rate of country k's currency relative to a market basket of currencies, such as the ecu (European currency unit), where e_k is defined in units of the market basket currency per unit of country k's currency.[24] When extending the data backwards to before the introduction of the euro, the exchange rates can play an important role in our results.

The stochastic convergence approach to aggregation over heterogeneous agents has traditionally been based more on statistical theory than on economic theory. But a

[24] We use the ecu as the benchmark exchange rate prior to the introduction of the euro only for expository convenience. Our derived formulas remain valid relative to any definition of the benchmark exchange rate. While the ecu can be viewed as a forerunner of the euro, the choice of the exchange rate to use for the conversion of historical data in legacy currencies into euros is not unambiguous. In particular, the use of the ecu, as opposed to a basket of currencies restricted to euro area countries, can produce paradoxical implications. For example, currency revaluation by one of the countries participating in the ecu but external to the euro area (e.g., the UK), would lead to a variation in euro area inflation, even in the absence of changes in domestic inflation for any country k.

rigorous connection with economic theory has been provided by Barnett (1979). We shall use that interpretation in our heterogeneous agents approach, as we now explain.

Consider a possible country with representative consumer c, having utility function $U_c = U_c[u_c(\mathbf{m}_c^*), g_c(\mathbf{x}_c)]$. Assume that the differences in tastes across possible countries can be explained in terms of a vector of taste-determining variables, $\boldsymbol{\phi}_c$. The dimension of the vector of taste-determining variables must be finite, but otherwise is irrelevant to the theory.[25] Then there must exist functions U, u, and g, such that

$$U_c = U_c[u_c(\mathbf{m}_c^*), g_c(\mathbf{x}_c)] = U[u(\mathbf{m}_c^*, \boldsymbol{\phi}_c), g(\mathbf{x}_c, \boldsymbol{\phi}_c), \boldsymbol{\phi}_c]$$

for all possible countries' tastes, $\boldsymbol{\phi}_c$. Although U, u, and g are fixed functions, the random vector $\boldsymbol{\phi}_c$ of taste determining variables causes U_c, u_c, and g_c to become random functions reflecting the possible variations of tastes and their probabilities, conditionally upon their given environmental, demographic, historical, resource, and other factors in the euro area.

Assume that each possible country c's representative consumer solves the following decision problem for $(\mathbf{m}_c^*, \mathbf{x}_c)$ at each instant of time t:[26]

$$\text{maximize } U[u(\mathbf{m}_c^*, \boldsymbol{\phi}_c), g(\mathbf{x}_c, \boldsymbol{\phi}_c), \boldsymbol{\phi}_c]$$

$$\text{subject to } \mathbf{m}_c^{*\prime}\boldsymbol{\pi}_c + \mathbf{x}_c'\mathbf{p}_c = I_c.$$

Assume that the euro area countries and their representative agents are about to be drawn from the theoretically possible populations, but have not yet been drawn. Assume that there is an infinite number of possible countries in the euro area, so that there exists a continuous joint distribution of the random variables $(I_c, \mathbf{p}_c, e_c, \boldsymbol{\pi}_c, \boldsymbol{\phi}_c)$ at any time t. We assume that $\boldsymbol{\phi}_c$ is sampled at birth and does not change during lifetimes, so that $\boldsymbol{\phi}_c$ is not time dependent. But $\{I_c(t), \mathbf{p}_c(t), e_c(t), \boldsymbol{\pi}_c(t)\}$ are stochastic processes. Hence at any time t we can write the theoretical population distribution function of $\{I_c(t), \mathbf{p}_c(t), e_c(t), \boldsymbol{\pi}_c(t), \boldsymbol{\phi}_c\}$ at t as F_t. It follows that at any t, the following are random variables with distributions derived from F_t:

$$d \log (p_c^* e_c),\ d \log (M_c e_c),\ d \log (M_c^*),\ d \log (\Pi_c e_c), \text{ and } d \log (\Pi_c^*).$$

[25] The assumption of finite dimensionality of $\boldsymbol{\phi}_c$ is only for notational convenience. Without that assumption, $\boldsymbol{\phi}_c$ could not be written as a vector. A sequence or continuum of taste-determining variables would not alter any of our conclusions, but would complicate the notation.

[26] Although not known to us, all variables in the decision are assumed to be known to the representative consumer at time instant t, and hence the decision is under perfect certainty for the representative consumer.

Using the derived distribution of those random variables, we can define their theoretical population means by:

$$\theta_1 = E[d \log (p_c^* e_c)],$$
$$\theta_2 = E[d \log (M_c e_c)],$$
$$\theta_3 = E[d \log (M_c^*)],$$
$$\theta_4 = E[d \log (\Pi_c e_c)],$$
$$\theta_5 = E[d \log (\Pi_c^*)],$$

where $(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) = (\theta_1(t), \theta_2(t), \theta_3(t), \theta_4(t), \theta_5(t))$ is a nonstochastic function of time. Now consider sampling from the theoretical population K times to draw the $k \in \{1,\ldots,K\}$ actual countries. The countries are assumed to have representative consumers having characteristics that are produced from the continuous theoretical population distribution F_t at t.

Definition 2: Let $s_k = H_k / \sum_{\kappa=1}^{K} H_\kappa$ be country k's fraction of total euro area population.

Define the k^{th} country's expenditure share W_k of the EMU's monetary service flow by:

$$W_k = \frac{M_k^* \Pi_k^* p_k^* e_k s_k}{\sum_{\kappa=1}^{K} M_\kappa^* \Pi_\kappa^* p_\kappa^* e_\kappa s_\kappa} = \frac{M_k \Pi_k^* e_k s_k}{\sum_{\kappa=1}^{K} M_\kappa \Pi_\kappa^* e_\kappa s_\kappa} = \frac{M_k^* \Pi_k e_k s_k}{\sum_{\kappa=1}^{K} M_\kappa^* \Pi_\kappa e_\kappa s_\kappa}.$$

The fact that this definition is in terms of total national expenditure shares, rather than per capita shares, is evident from the fact that:

$$\frac{M_k^* \Pi_k^* p_k^* e_k s_k}{\sum_{\kappa=1}^{K} M_\kappa^* \Pi_\kappa^* p_\kappa^* e_\kappa s_\kappa} = \frac{M_k^* \Pi_k^* p_k^* e_k H_k}{\sum_{\kappa=1}^{K} M_\kappa^* \Pi_\kappa^* p_\kappa^* e_\kappa H_\kappa}.$$

Observe that $0 \leq W_k \leq 1$ for all k, and $\sum_{k=1}^{K} W_k = 1$. We thereby can treat the $\{W_1,\ldots,W_K\}$ as a probability distribution in computing the following Divisia means by our stochastic heterogeneous-countries approach to aggregation over countries.[27]

Definition 3: Aggregating over countries, define the monetary-sector-weighted Divisia consumer price index, $p^* = p^*(t)$, by:

$$d \log p^* = \sum_{k=1}^{K} W_k \, d \log \, p_k^* e_k \qquad (1)$$

Definition 4: Define the euro area's nominal, M, and real, M*, per-capita monetary service flows by:

$$d \log M = \sum_{k=1}^{K} W_k \, d \log (s_k M_k e_k)$$

and

$$d \log M^* = \sum_{k=1}^{K} W_k \, d \log (s_k M_k^*).$$

Definition 5: Define the euro area's nominal, Π, and real, Π^*, monetary user-cost prices by

$$d \log \Pi = \sum_{k=1}^{K} W_k \, d \log (\Pi_k e_k)$$

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[27] In our formulas, we treat the probability of drawing $d \log M_k$ to be the share of monetary expenditure in country k. It is not inconceivable that for some currently overlooked purposes, it might be preferable to assume that probability to be proportional to the per-capita share of expenditure in country k. In that case, one need only drop s_k from the formulas. But this possibility is not consistent with past uses of this approach (e.g., Theil (1967) and Barnett and Serletis (2000, pp 88-90 and chapter 9)), and it is not presently clear under what circumstances, if any, this latter sampling assumption would be relevant. We do not advocate this alternative sampling assumption for aggregation within the euro area.

and

$$d \log \prod^* = \sum_{k=1}^{K} W_k \, d \log (\prod_k^*).$$

When we draw from the derived population distributions, the frequency with which we draw $d \log p_k^* e_k$, $d \log (s_k M_k e_k)$, $d \log (s_k M_k^*)$, $d \log (\prod_k e_k)$, and $d \log (\prod_k^*)$ is W_k. From Khinchine's theorem, assuming independent sampling, we find that $d \log p^*$, $d \log M$, $d \log M^*$, $d \log \prod$, and $d \log \prod^*$ are sample means of distributions having population means equal to $\theta_1(t)$, $\theta_2(t)$, $\theta_3(t)$, $\theta_4(t)$, and $\theta_5(t)$, respectively. In addition, $d \log p^*$, $d \log M$, $d \log M^*$, $d \log \prod$, and $d \log \prod^*$ converge in probability as $K \to \infty$ to $\theta_1(t)$, $\theta_2(t)$, $\theta_3(t)$, $\theta_4(t)$, and $\theta_5(t)$, respectively. It is this convergence to theoretical population properties that accounts for this aggregation approach's name, "the stochastic convergence approach," in Barnett (1979).

Observe that there is no assumption that a representative agent exists over countries. We assume in this heterogeneous agents approach only that representative agents exist within countries. Aggregation over countries is defined to be estimation of the moments of the stochastic processes produced by sampling from the underlying theoretical population that produces the countries' representative agents. When in later sections we consider the existence of multilateral and unilateral representative agents over countries, we add strong assumptions about the *realized* tastes after sampling from the theoretical population.

In summary, the perspective from which our heterogeneous agents approach is produced is prior to the drawing from the theoretical distribution, so that random variables have not yet been realized and all dynamic solution paths are stochastic processes induced by the randomness of $\{I_c(t), \mathbf{p}_c(t), e_c(t), \boldsymbol{\pi}_c(t), \boldsymbol{\phi}_c\}$. No assumptions are made about the precise form in which realized tastes relate to each other across countries. The heterogeneous-agents approach tracks aggregator functions within countries. But this approach does not require assumptions sufficient for the existence of microeconomic aggregator functions over countries. After aggregating over countries, this approach tracks moments of aggregate stochastic processes and is interpreted relative to the underlying population distributions.

In contrast, our multilateral and unilateral representative agent approaches add assumptions regarding the functional relationship among realized tastes of countries already in existence, and seek to track the realized aggregator function over countries. Under those additional assumptions producing the existence of an aggregator function over the euro area, the heterogeneous agents approach reduces to the multilateral representative agent approach as a special case. Although the two approaches have different interpretations, because of the difference in perspective regarding prior versus post sampling, the multilateral economic agent approach is nevertheless mathematically a nested special case of the heterogeneous agents approach.

It is important to recognize the following proof's dependence upon the definition of p* in equation 1, with the share weights determined by Definition 2. If any other weights, such as consumption-expenditure share or GDP weights, had been used in defining p*, then Theorem 1 would not hold.

Theorem 1: $M = M^*p^*$ and $\prod = \prod^*p^*$.

Proof: The method of proof is proof by contradiction.

First consider M, and suppose that $M \neq M^*p^*$. Then

$$d \log M \neq d \log (M^*p^*) = d \log M^* + d \log p^*.$$

So by Lemma 1, $\sum_{k=1}^{K} W_k \, d \log (s_k M_k e_k) \neq \sum_{k=1}^{K} W_k \, d \log (s_k M_k / p_k^*) + d \log p^*.$

$$= \sum_{k=1}^{K} W_k \, d \log (s_k M_k) - \sum_{k=1}^{K} W_k \, d \log p_k^* + d \log p^*.$$

Hence $\sum_{k=1}^{K} W_k \, d \log (s_k M_k) \neq \sum_{k=1}^{K} W_k \, d \log (s_k M_k) - \sum_{k=1}^{K} W_k \, d \log p_k^*$

$$+ d \log p^* - \sum_{k=1}^{K} W_k \, d \log e_k$$

$$= \sum_{k=1}^{K} W_k \, d \log (s_k M_k) - \sum_{k=1}^{K} W_k \, d \log (p_k^* e_k) + d \log p^*$$

$$= \sum_{k=1}^{K} W_k \, d \log (s_k M_k),$$

which is a contradiction. The last equality follows from equation (1) in Definition 3.

Now consider $\prod$, and suppose that $\prod \neq \prod^*p^*$. Then

$$d \log \prod \neq d \log (\prod^*p^*) = d \log \prod^* + d \log p^*.$$

By Definitions 3 and 5, it follows that

$$\sum_{k=1}^{K} W_k \, d \log (\Pi_k e_k) \neq \sum_{k=1}^{K} W_k \, d \log (\Pi_k^*) + \sum_{k=1}^{K} W_k \, d \log (p_k^* e_k).$$

Hence by Lemma 1, we have that

$$\sum_{k=1}^{K} W_k \, d \log (\Pi_k^* p_k^* e_k) \neq \sum_{k=1}^{K} W_k \, d \log (\Pi_k^*) + \sum_{k=1}^{K} W_k \, d \log (p_k^* e_k),$$

or

$$\sum_{k=1}^{K} W_k \, d \log (\Pi_k^*) + \sum_{k=1}^{K} W_k \, d \log (p_k^* e_k) \neq \sum_{k=1}^{K} W_k \, d \log (\Pi_k^*) + \sum_{k=1}^{K} W_k \, d \log (p_k^* e_k),$$

which is a contradiction. *Q. E. D.*

The following theorem proves Fisher's factor reversal property for the monetary quantity and user cost aggregates over countries. In particular, we prove that total expenditure on monetary services aggregated over countries is the same, whether computed from the product of the euro-area quantity and user cost aggregates or from the sum of the products within countries. The multiplications by s_k convert to per capita values relative to total euro-area population, while the within-country aggregates, M_k^*, remain per capita relative to each country's own population.

Theorem 2: $M^* \Pi = \sum_{k=1}^{K} \left(M_k^* s_k \right) \Pi_k e_k$.

Proof: The method of proof is proof by contradiction. So assume that

$$d \log (M^*) + d \log (\Pi) \neq d \log \left(\sum_{k=1}^{K} M_k^* s_k \Pi_k e_k \right)$$

$$= \frac{d \left(\sum_{k=1}^{K} M_k^* s_k \Pi_k e_k \right)}{\sum_{k=1}^{K} M_k^* s_k \Pi_k e_k}.$$

Hence by Definitions 4 and 5, it follows that

$$\sum_{k=1}^{K} W_k \, d \log (s_k M_k^*) + \sum_{k=1}^{K} W_k \, d \log (\Pi_k e_k) \neq \frac{d\left(\sum_{k=1}^{K} M_k^* s_k \Pi_k e_k \right)}{\sum_{k=1}^{K} M_k^* s_k \Pi_k e_k}.$$

Multiplying through by $\sum_{k=1}^{K} M_k^* s_k \Pi_k e_k$ and using Definition 2, we get

$$\sum_{k=1}^{K} \left(M_k^* s_k \Pi_k e_k \right) d \log (s_k M_k^*) + \sum_{k=1}^{K} \left(M_k^* s_k \Pi_k e_k \right) d \log (\Pi_k e_k) \neq d\left(\sum_{k=1}^{K} M_k^* s_k \Pi_k e_k \right).$$

So

$$\sum_{k=1}^{K} \left(M_k^* s_k \Pi_k e_k \right) \frac{d\left(s_k M_k^* \right)}{s_k M_k^*} + \sum_{k=1}^{K} \left(M_k^* s_k \Pi_k e_k \right) \frac{d\left(\Pi_k e_k \right)}{\Pi_k e_k} \neq \sum_{k=1}^{K} d\left(M_k^* s_k \Pi_k e_k \right).$$

Hence

$$\sum_{k=1}^{K} \left(\Pi_k e_k \right) d\left(s_k M_k^* \right) + \sum_{k=1}^{K} \left(M_k^* s_k \right) d\left(\Pi_k e_k \right) \neq \sum_{k=1}^{K} d\left(M_k^* s_k \Pi_k e_k \right). \qquad (2)$$

But taking the total differential of $M_k^* s_k \Pi_k e_k$, we have

$$d(M_k^* s_k \Pi_k e_k) = (\Pi_k e_k) d(M_k^* s_k) + (M_k^* s_k) d(\Pi_k e_k).$$

Substituting that total differential into the right hand side of equation (2), we get

$$\sum_{k=1}^{K} \left(\Pi_k e_k \right) d\left(s_k M_k^* \right) + \sum_{k=1}^{K} \left(M_k^* s_k \right) d\left(\Pi_k e_k \right) \neq \sum_{k=1}^{K} \left(\Pi_k e_k \right) d(M_k^* s_k) + \sum_{k=1}^{K} \left(M_k^* s_k \right) d(\Pi_k e_k),$$

which is a contradiction. *Q.E.D.*

5. Special Cases

We now consider some special cases of our results. First we consider the case of purchasing power parity. While the purchasing power parity assumption is not applicable to the euro area data, this special case is useful in understanding the forms of the more general formulas we have derived without purchasing power parity.

5.1 Purchasing Power Parity

Definition 6: We define $E = \{e_k: k = 1,\ldots, K\}$ to satisfy purchasing power parity, if $p_j^*/p_i^* = e_i/e_j$ for all countries $i,j \in \{1,\ldots, K\}$. Under this definition, it equivalently follows that there exists a price p^0 such that $p^0 = p_i^* e_i = p_j^* e_j$ for all $i,j \in \{1,\ldots, K\}$.

Observation 1: If E and the European currency unit (ecu) had been chosen to satisfy purchasing power parity, then p^0 would have been determined by the ecu prior to the introduction of the euro and could be designated as p_{ecu}.

Although the following two theorems are not relevant to the way in which the ecu evolved into the euro, the theorem nevertheless provides an interesting special case of Definition 2 and can help to clarify and illustrate the form of Definition 2.

Theorem 3: If E satisfies purchasing power parity, then

$$W_k = \frac{M_k^* \Pi_k^* s_k}{\sum_{\kappa=1}^{K} M_\kappa^* \Pi_\kappa^* s_\kappa}.$$

Proof: From definition 2, we have in general that

$$W_k = \frac{M_k^* \Pi_k^* p_k^* e_k s_k}{\sum_{\kappa=1}^{K} M_\kappa^* \Pi_\kappa^* p_\kappa^* e_\kappa s_\kappa}.$$

$$= \frac{M_k^* \Pi_k^* s_k}{\sum_{\kappa=1}^{K} M_\kappa^* \Pi_\kappa^* \left(\frac{p_\kappa^* e_\kappa}{p_k^* e_k}\right) s_\kappa}. \qquad (3)$$

But by Definition 6, it follows under purchasing power parity that

$$\frac{p_\kappa^* e_\kappa}{p_k^* e_k} = 1 \qquad (4)$$

for all countries κ, $k \in \{1,\ldots, K\}$. Hence the theorem follows by substitution of equation (4) into equation (3). *Q.E.D.*

The following theorem is immediate from the linear homogeneity of p*. But because of the unusual weights in p*, we nevertheless provide a formal proof of the following simple theorem.

Theorem 4: If E satisfies purchasing power parity, then the growth rate of p* would equal the growth rate of $p^*_k e_k$ for all countries $k \in \{1,\ldots, K\}$.

Proof: By the definition of p* in equation (1), we know that

$$d \log p^* = \sum_{\kappa=1}^{K} W_\kappa \; d \log \; p^*_\kappa e_\kappa .$$

But under purchasing power parity, we have that

$$p^*_\kappa e_\kappa = p^*_k e_k \text{ for all } \kappa, k \in \{1,\ldots, K\}.$$

Hence, it follows immediately by substitution that

$$\begin{aligned} d \log p^* &= \sum_{\kappa=1}^{K} W_\kappa \; d \log \; p^*_k e_k \\ &= d \log \; p^*_k e_k \sum_{\kappa=1}^{K} W_\kappa \\ &= d \log \, (p^*_k e_k) \end{aligned}$$

for all $k \in \{1,\ldots, K\}$. *Q.E.D.*

The following corollary demonstrates that the inflation rate based upon p* cannot be expected to equal that of p_{ecu}, unless there is purchasing power parity.

Corollary 1 to Theorem 4: If E satisfies purchasing power parity and if $p_{ecu} = p^0$, as defined in Definition 6, then the inflation rate of p* would equal that of p_{ecu}, as defined in Observation 1.

Proof: The proof of this corollary follows immediately from Theorem 4 and Definition 6. *Q.E.D.*

5.2. **Existence of a Multilateral Representative Agent over the Euro Area**

In this section, we define the concept of a multilateral representative agent. In the next section, we define a unilateral representative agent over countries to be a representative agent who considers the same goods in different countries to be perfect substitutes, regardless of the country of residence of the purchaser or the country within which the good or asset is acquired. The existence of a unilateral representative agent has been implicit in the existing studies using the "direct method" of aggregation over monetary assets in the euro area. As we shall show, the existence of a unilateral representative agent requires extremely strong assumptions. Without a homogeneous culture within the euro area and the vast population migrations that could produce that uniformity, this assumption will not apply. The existence of a multilateral representative agent requires far more reasonable assumptions.

If tastes across countries do converge into the distant future, the convergence is more likely to be towards a homogeneous multilateral representative agent, which we shall define, rather than towards a unilateral representative agent. A homogeneous multilateral representative agent recognizes the existence of country specific tastes, but equates those tastes across countries. A unilateral representative agent does not recognize the relevancy of countries at all and thereby does not recognize the existence of country specific tastes. Country specific utility functions cannot be factored out of euro area tastes (i.e., weak separability of country tastes fails); and the country subscripts, j and k, disappear from the decision of the unilateral representative agent. The allocation of goods across countries is indeterminate in that case.

5.2.1 A Multilateral Representative Agent with Heterogeneous Tastes

We begin by defining relevant assumptions and produce the theory of a multilateral representative agent. We show that the existence of a multilateral representative agent is a special case of our heterogeneous countries theory. We further show that a homogeneous multilateral representative agent exists under stronger assumptions.

As described in the previous section, our representative agent approach for aggregating over countries treats countries as already realized, so that variables and functions no longer are random. Hence we can consider realized functional structure aggregated over realized countries. The following assumption is needed and begins to become weak only after the introduction of the euro.

Assumption 1: Suppose there is convergence over the euro area in the following sense. Let there exist R = R(t) such that $R_k = R(t)$ for all $k \in \{1,\ldots, K\}$ and all t.[28]

The existence of a representative agent is necessary and sufficient for the nonexistence of distribution effects.[29] Distribution effects introduce second moments and possibly higher order moments into demand functions aggregated over consumers. The existence of such second and higher order moments in the macroeconomy can cause policy to influence distributions of income and wealth across consumers. Assumption 1 rules out certain possible distribution effects. Additional assumptions ruling out other sources of distribution effects will be needed as we consider further special cases.

By its definition, the benchmark asset, unlike "monetary" assets, provides no services other than its investment rate of return, and hence cannot enter the utility function of an infinitely lived representative agent.[30] Therefore, differences in tastes across countries play no role in decisions regarding benchmark asset holdings by a euro area representative agent. For that reason, the existence of a common benchmark rate for all countries is necessary for a representative agent over countries. A euro area representative agent would hold only the highest yielding of multiple possible benchmark assets. This conclusion is not necessary in our thereby-more-general heterogeneous countries approach.

With Assumption 1, we also can consider the following stronger assumption. We assume that all K countries have already been drawn from their theoretical population of potential countries. Then the tastes of the representative consumers in each country are realized and are no longer random. The following assumption produces the existence of aggregator functions, (U, V, G), over the individual realized countries' tastes, (u_k, g_k), for $k \in \{1,\ldots,K\}$.

Assumption 2a: Assume that there exists a representative consumer over the euro area.[31] Within that representative agent's intertemporal utility function, assume that

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[28] As explained in the appendix, the benchmark rate R in theory is the rate of return on an illiquid pure investment. If for some i, asset i is denominated in a foreign currency, then the rate of return r_{kji}, as defined in Section 1, is the effective rate of return net of expected appreciation or depreciation in the foreign currency relative to the domestic currency. Hence both the benchmark rate and all own rates on monetary assets held within country k are effective rates relative to the domestic currency. Therefore, there is no need also to adjust for expected variation of exchange rates relative to the market basket currency, since that adjustment would be from the domestic currency to the ecu for all assets, including the benchmark asset. Hence that adjustment would cancel out of the two terms in $\pi_{kji}(t) = R(t) - r_{kji}(t)$, and hence in all weights in our indexes.

[29] See Gorman (1953).

[30] See, e.g., Barnett and Serletis (2000, p. 53). In the finite planning horizon case, the benchmark asset enters utility only in the terminal period to produce a savings motive to endow the next planning horizon.

[31] In accordance with Gorman's (1953) theorem on the representative consumer, a representative consumer exists within an area only if the Engel curves of all consumers within the area are linear and parallel across consumers for each good. Equivalently all consumers within the area must have linear Engel curves, and

$(\mathbf{m}_1^*(t),\ldots,\mathbf{m}_K^*(t),\mathbf{x}_1(t),\ldots,\mathbf{x}_K(t))$ is intertemporally weakly separable from $(\mathbf{m}_1^*(\tau),\ldots,\mathbf{m}_K^*(\tau),\mathbf{x}_1(\tau),\ldots,\mathbf{x}_K(\tau))$ for all $t \neq \tau$, and also assume that monetary assets are weakly separable from consumer goods. As defined in Section 1, $\mathbf{x}_k$ is the vector of instantaneous per-capita goods consumption rates in country k relative to the population of country k. Then $s_k\mathbf{x}_k$ is the per-capita real consumption vector relative to total euro-area population, $H = \sum_{k=1}^{K} H_k$. Since contemporaneous consumption of goods and services is weakly separable from future consumption, a contemporaneous category utility function exists of the form

$$U = U[\breve{V}(s_1\mathbf{m}_1^*,\ldots,s_K\mathbf{m}_K^*),\breve{G}(s_1\mathbf{x}_1,\ldots,s_K\mathbf{x}_K)], \tag{5a}$$

where $\breve{V}$ and $\breve{G}$ are linearly homogeneous.

Assumption 2b: Assume further that consumption of monetary assets and goods are weakly separable among countries, so that the contemporaneous utility function has the blockwise weakly separable form

$$U = U\{V[s_1u_1(\mathbf{m}_1^*),\ldots,s_Ku_K(\mathbf{m}_K^*)],G[s_1g_1(\mathbf{x}_1),\ldots,s_Kg_K(\mathbf{x}_K)]\}. \tag{5b}$$

Assume that the functions V, G, u_k, and g_k do not change over time and are linearly homogeneous for all $k \in \{1,\ldots,K\}$.[32] The dependency of u_k and g_k on k permits heterogeneity of tastes across countries. In the next subsection, we shall explore the special case of homogeneity of tastes across countries.

As in our heterogeneous agents approach, the subscript k identifies the country of residence of the owner of the asset and not necessarily the country within which the asset is purchased or located. Hence, equation (5b) requires that the tastes that determine the utility functions, u_k and g_k, are those of the residents of country k, regardless of the country within which the residents have deposited their assets. Note that equation (5a) does not require that tastes of consumer's residing in country k exist independently of the tastes of consumers residing in other countries. The existence of stable country-specific tastes, u_k and g_k, exist only under the stronger assumption (5b).[33]

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tastes can differ across consumers within the area only from a vertical translations of utility functions. For any good, marginal propensities to consume must be identical for all consumers. The uniformity of tastes within the area must be very high.

[32] The relevance of weak separability has been recognized by Drake, Mullineux, and Agung (1997) and Swofford (2000). The assumption that the functions do not move over time does not preclude subjective discounting of future utility within the integrand of the intertemporal utility integral.

[33] Under Assumption (5a) the marginal rate of substitution between goods or assets within country k can depend upon consumption of goods or assets by consumers residing in other countries.

Equation (5b) could equivalently be written as

$$U = U\{V[u_1(s_1 \mathbf{m}_1^*),\ldots, u_K(s_K \mathbf{m}_K^*)],G[g_1(s_1\mathbf{x}_1),\ldots, g_K(s_K\mathbf{x}_K)]\},$$

because of the linear homogeneity of the utility functions, u_k and g_k. But we prefer the form of equation (5b), since it makes clear our ability to aggregate first within countries to acquire the within-country monetary aggregates, $M_k^* = u_k(\mathbf{m}_k^*)$, and the within-country consumer goods aggregates, $X_k = g_k(\mathbf{x}_k)$. Note that M_k^* and X_k are in per capita terms relative to country k's population. We then can aggregate over countries to acquire the euro area monetary aggregate over countries, $M^* = V[s_1u_1(\mathbf{m}_1^*),\ldots,s_Ku_K(\mathbf{m}_K^*)] = V[s_1 M_1^*,\ldots, s_K M_K^*]$, and euro area consumer goods aggregate over countries, $X = G[s_1g_1(\mathbf{x}_1),\ldots, s_Kg_K(\mathbf{x}_K)] = G[s_1X_1,\ldots, s_KX_K]$. Note that M* and X are in per capita terms relative to total euro area population. Our proofs below demonstrate the capability to aggregate recursively in that manner.

Under Assumptions 1, 2a, and 2b, let I = I(t) be the instantaneous rate of expenditure. It is budgeted to t by the representative consumer in a prior stage intertemporal allocation. Then we can define the following contemporaneous, conditional decision at instant of time t.

Decision 1: Choose $(\mathbf{m}_1^*,\ldots, \mathbf{m}_K^*, \mathbf{x}_1,\ldots, \mathbf{x}_K)$ to

$$\text{maximize } U\{V[s_1u_1(\mathbf{m}_1^*),\ldots, s_Ku_K(\mathbf{m}_K^*)],G[s_1g_1(\mathbf{x}_1), \ldots , s_Kg_K(\mathbf{x}_K)]\}$$

$$\text{subject to } \sum_{k=1}^{K} s_k \mathbf{m}_k^{*\prime} \boldsymbol{\pi}_k e_k + \sum_{k=1}^{K} s_k \mathbf{x}_k' \mathbf{p}_k e_k = I.$$

Definition 7: We define a multilateral representative consumer to be an economic agent who solves Decision 1 under Assumptions 1, 2a, and 2b.

Note that our definitions of real and nominal money balances have not changed from those in Section 1. Nominal balances owned by residents of country k are deflated by p_k^* to acquire real balances, where p_k^* is the unit cost function dual to the consumer goods quantity aggregator function, $g_k(\mathbf{x}_k)$, within country k. We are not yet accepting assumptions that would be sufficient for existence of a single consumer-price index that could be used to deflate nominal balances within all euro area countries to real balances

in those countries.[34] Hence p_k^* is not independent of k. Our euro-area consumer goods price aggregate, p*, is relevant to deflation of monetary balances only after monetary balances have been aggregated over countries.

Observe that Assumption 1 does not require convergence of rates of return on all monetary assets across countries. To produce the multilateral representative consumer, Assumption 1 requires only that consumers in all countries of the euro area have access to the same benchmark rate of return on pure investment. We now consider the implications of a multilateral representative agent. In the next section, we then focus on the case of a unilateral representative agent, requiring the adoption of very strong assumptions.

The following lemmas now are immediate.

Lemma 2: Under Assumptions 1, 2a, and 2b, the representative consumer's allocation of I(t) over goods and monetary services will solve Decision 1.

Proof: Follows immediately from known results on two stage budgeting, where the first stage decision is intertemporal. One need only redefine the variables in the continuous time analog to Barnett (1978, section 3; 1980, section 3.1; or 1987, sections 2.1-2.2).

Q.E.D.

Lemma 3: Under Assumptions 1, 2a, and 2b, let $X_k = g_k(\mathbf{x}_k)$ be the exact consumer goods per-capita quantity aggregate over $\mathbf{x}_k$ for country k, relative to the population of country k, and let $X = G(s_1X_1, \ldots, s_KX_K)$ be the exact consumer goods per capita quantity aggregate over countries, relative to total euro-area population. Then p_k^* is the exact price dual to X_k, and P* is the exact price dual to X, where P* is defined such that

[34] We thereby assume that the preference preordering over monetary real balances owned in each country is over the space of real balances deflated by that country's own consumer price index. Our assumption permits us to derive a coherent second stage decision that is within each country. The duality of price and quantity aggregator functions applies only at the same level of aggregation over economic agents, or in this case over countries. The duality of the unit cost function, $p_k^*(\mathbf{p}_k)$, to the consumer goods quantity aggregator function, $g_k(\mathbf{x}_k)$, requires p_k^* to serve as deflator of monetary balances within country k. Attempts to impute the same consumer price index to each country for deflation of its domestic monetary assets, as in Wesche (1997), are difficult to justify without accepting Assumption 3 that we introduce in Section 5.3 and thereby the existence of a unilateral representative agent. In the next section, we shall determine the implications of that Assumption 3 for the representative agent.

$$d \log P^* = \sum_{k=1}^{K} \left(\frac{X_k p_k^* e_k}{\sum_{\kappa=1}^{K} X_\kappa p_\kappa^* e_\kappa} \right) d \log p_k^* e_k . \qquad (6)$$

Proof: The result regarding p_k^* follows, since it was defined in Section 1 to be the true cost of living index of X_k. The result on P* follows by a proof analogous to that of Theorem 2, since duality of P* and X implies, from factor reversal, that

$$XP^* = \sum_{k=1}^{K} X_k p_k^* e_k . \qquad (7)$$

This equation accounts for the form of the share weights in equation (6). *Q.E.D.*

Note that P*, defined by equation 6, and p*, defined by equation 1, are not the same. Both consumer price indexes are needed for different purposes, as we shall discuss further below. Now consider the following decision, within which aggregation over consumer goods has already occurred.

Decision 2: Choose $(\mathbf{m}_1^*, \ldots, \mathbf{m}_K^*, X)$ to

$$\text{maximize } U\{V[s_1 u_1(\mathbf{m}_1^*), \ldots, s_K u_K(\mathbf{m}_K^*)], X\}$$

$$\text{subject to } \sum_{k=1}^{K} s_k \mathbf{m}_k^{*\prime} \boldsymbol{\pi}_k e_k + XP^* = I.$$

The following theorem establishes the connection between Decisions 1 and 2.

Theorem 5: Under Assumptions 1, 2a, and 2b, let $(\mathbf{m}_1^*,\ldots,\mathbf{m}_K^*, \mathbf{x}_1,\ldots,\mathbf{x}_K)$ solve Decision 1, and let X and P* be defined as in Lemma 3. Then $(\mathbf{m}_1^*,\ldots,\mathbf{m}_K^*, X)$ will solve Decision 2.

Proof: Follows from Lemma 3 and well known results on two stage budgeting. *Q.E.D.*

Theorem 5 permits us to concentrate on aggregation over monetary assets within countries and then over countries, while using a quantity and price aggregate for consumer goods. Theorem 5 also demonstrates our need for the P* price index in the prior aggregation over consumer goods.

In Decision 3, we now define a "second stage" decision, in which funds preallocated to monetary-services expenditure within the euro area are allocated over countries. In Decision 4, we then define a "third stage" decision, in which funds preallocated to monetary-services expenditure within the each country are allocated over assets in the country.

Let Π_k for each $k \in \{1,\ldots,K\}$ be as in Definition 1. We then can define the following decision.

Decision 3: For given value of X, choose $(M_1^*,\ldots,M_K^*)$ to

$$\text{maximize } V(s_1 M_1^*,\ldots,s_K M_K^*)$$

$$\text{subject to } \sum_{k=1}^{K} s_k M_k^* \Pi_k e_k = I - XP^*. \tag{8}$$

Decision 4: For each $k \in \{1,\ldots,K\}$ choose $\mathbf{m}_k^*$ to

$$\text{maximize } u_k(\mathbf{m}_k^*)$$

$$\text{subject to } \mathbf{m}_k^{*\prime}\boldsymbol{\pi}_k = M_k^* \Pi_k.$$

The following two corollaries to Theorem 5 relate to Decisions 3 and 4.

Corollary 1 to Theorem 5: Under Assumptions 1, 2a, and 2b, let $(\mathbf{m}_1^*, \ldots, \mathbf{m}_K^*, X)$ solve Decision 2. Define P* as in equation 6 and the vector of user costs $\prod = (\prod_1, \ldots, \prod_K)$ as in Definition 1. Then $(M_1^*, \ldots, M_K^*)$ will solve Decision 3, where $M_k^* = u_k(\mathbf{m}_k^*)$ for all $k \in \{1, \ldots, K\}$.

Proof: Follows from well known results on two stage budgeting. *Q.E.D.*

Corollary 2 to Theorem 5: Under Assumptions 1, 2a, and 2b, let $(\mathbf{m}_1^*, \ldots, \mathbf{m}_K^*, X)$ solve Decision 2, and let $M_k^* = u_k(\mathbf{m}_k^*)$ for all $k \in \{1, \ldots, K\}$. Define Π_k as in Definition 1. Then $\mathbf{m}_k^*$ also will solve Decision 4 for all $k \in \{1, \ldots, K\}$.

Proof: Follows from well known results on two stage budgeting and a simple proof by contradiction. Suppose $M_k^* = u_k(\mathbf{m}_k^*)$, but $\mathbf{m}_k^*$ does not solve Decision 4 for all $k \in \{1, \ldots, K\}$. Then $(\mathbf{m}_1^*, \ldots, \mathbf{m}_K^*, X)$ cannot solve Decision 2. *Q.E.D.*

Decision 4 defines the representative consumers assumed to exist within countries in Section 2. Under the assumptions in Definition 7 for the existence of a multilateral representative consumer, Corollary 2 to Theorem 5 proves that the decisions of the representative consumers in Section 2 are nested as conditional decisions within the decision of the multilateral representative consumer. Hence our results in Section 2 can be used to aggregate within countries, regardless of whether aggregation over countries is by our heterogeneous countries approach or by our multilateral representative consumer approach.

After the aggregation within countries is complete, Corollary 1 to Theorem 5 demonstrates that Decision 3 can be used to aggregate over countries, if we accept the assumptions necessary for the existence of a multilateral representative agent. The monetary quantity aggregator function for aggregation over countries then is V, and a Divisia index can be used to track V in the usual manner.

Observe that Decision 4 would be unaffected, if the vector of within-country user costs $\boldsymbol{\pi}_k$ and the aggregate within-country user cost Π_k were changed to real user costs, since all that would be involved is the division of each constraint by p_k^*. Hence that constraint would continue to hold, if all values in the constraint were in real terms.

But observe that in Decisions 2 and 3, the consumer price index P* on the right hand side of equation 8 is not the same as the consumer price index p_k^* needed to deflate the user costs on the left hand side to real value. In addition, the consumer price index p_k^* used to deflate each term on the left hand side is different for each $k \in \{1, \ldots, K\}$. Hence the constraint would be broken, if all variables on both sides of the constraint were replaced by real values. This would amount to dividing each term by a different price index. Also recall that conversion of $\mathbf{m}_k^*$ to nominal balances requires multiplication by p_k^*, which is different for each country k.

The following illustration can further clarify the need for two price indexes in modeling. Consider the following decision using the exact aggregates both over monetary assets and goods within the euro area.

Decision 5: Choose (M*,X) to

$$\text{maximize } U(M^*,X)$$

$$\text{subject to } M^*\Pi + XP^* = I.$$

The solution will be of the form

$$\begin{pmatrix} M^* \\ X \end{pmatrix} = D(I,\Pi,P^*) = D(I,\Pi^* p^*,P^*)\,. \qquad (9a)$$

But by Lemma 1 and the homogeneity of degree zero of demand, we equivalently can write:

$$\begin{pmatrix} M^* \\ X \end{pmatrix} = D(\frac{I}{p^*},\Pi^*,\frac{P^*}{p^*}), \qquad (9b)$$

or

$$\begin{pmatrix} M^* \\ X \end{pmatrix} = \widehat{D}(\frac{I}{P^*},\frac{\Pi^* p^*}{P^*}), \qquad (9c)$$

where

$$\widehat{D}(\frac{I}{P^*},\frac{\Pi^* p^*}{P^*}) = D(\frac{I}{P^*},\frac{\Pi^* p^*}{P^*},1).$$

As can be seen from equations (9b) and (9c), there is no way to remove the simultaneous dependence of the solution demand function systems upon the two price indexes, P* and p*. The form of the demand system in (9a) is in terms of nominal total expenditure ("income"), I. The form of the demand system in (9b) is in terms of real income relative to p* aggregate prices. The form in (9c) is in terms of real income relative to P* aggregate prices. None of the three possible forms results in either p* or P* canceling out. In addition, Lemma 1 requires that conversion of M* to nominal balances must be relative to p* prices.

The following theorem establishes the relationship between our heterogeneous countries approach and our multilateral representative agent approach.

Theorem 6: Under Assumptions 1, 2a, and 2b, let $(M_1^*,\ldots,M_K^*)$ solve Decision 3, and let M* be as defined in Definition 4. Then

$$d \log M^* = d \log V(s_1 M_1^*,\ldots,s_K M_K^*).$$

Proof: Follows from the exact tracking of the Divisia index in continuous time. *Q.E.D.*

Our multilateral representative agent theory produces conditions under which an economic (rather than statistical) monetary aggregate exists over countries. When an economic monetary aggregator function, V, exists over countries, Theorem 6 shows that our index number M*, introduced in Definition 4, will exactly track the theoretical aggregate. In particular, we have demonstrated that our heterogeneous agents approach for aggregating over countries reduces to the multilateral approach under assumptions 1, 2a, and 2b, since both approaches then produce the same monetary aggregate, M*, over countries. In addition, Π_k and Π_k^* defined in Definition 5 will remain dual to M*, since the proofs of Theorems 1 and 2 remain valid under Assumptions 1, 2a, and 2b.

We have demonstrated at all stages of aggregation that our multilateral representative agent approach is nested within our heterogeneous countries approach as a special case under Assumptions 1, 2a, and 2b. Theorem 6 is the result at the level of aggregation over countries, while Corollary 2 to Theorem 5 is the result for aggregation within countries.

Also observe that since the proofs of Theorems 1 and 2 remain valid under our additional assumptions in this section, it follows that we must continue to deflate nominal M aggregated over countries to real M* using p*, not P*. The correct dual to aggregate real consumption X is P*, which should be used to deflate nominal to real consumption expenditure. Regarding the computation of P* and its possible use as an inflation target, see Diewert (2002).[35] It is important to recognize that p* and P* both play important roles in this theory, and neither is an acceptable substitute for the other.[36] These conclusions hold in both our heterogeneous countries approach and in the multilateral representative agents special case acquired when the benchmark rate is the same for all countries in the euro area.

5.2.2 A Multilateral Representative Agent with Homogeneous Tastes

We now proceed to the far more restrictive case of a homogeneous multilateral representative agent who imputes identical tastes to the residents of all countries in the euro area. An initial necessary assumption is Assumption 1. As shown by Theorem 7 below, the seeming paradox of the existence of two consumer price indices---p* to deflate nominal money balances to real balances and P* to deflate nominal consumption expenditure to real aggregate consumption---disappears under the following additional important assumption.

Assumption 3: Suppose there is convergence over the euro area in the following strong sense. Let there exist $\hat{P} = \hat{P}(t)$ such that

$$\frac{d}{dt}[\log (p_k^*(t)\, e_k(t))] = \frac{d}{dt}[\log \hat{P}(t)] \qquad (10)$$

for all $k \in \{1,\ldots, K\}$ and all t.

The following theorem is immediate.

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[35] Although our paper is about measurement and is not intended to imply advocacy of any particular policy, it is nevertheless worth observing that p* would not be a suitable price index for use as an inflation target. The role of p* is specific to deflation of aggregate monetary service flows.

[36] Although perhaps somewhat surprising, the need for two different consumer price indexes is not entirely without precedent. The theory that produces the relative price version of Theil's (1971, p. 578, eq. 6.19) Rotterdam consumer demand system model also requires two consumer price indexes: the Divisia price index with average share weights to deflate nominal income to real income and the Frisch consumer price index with marginal budget share weights to deflate nominal to real relative prices. But that Rotterdam model phenomenon has a different source, since it applies to modeling the demand of one consumer. Our need for two consumer price indexes results from aggregation over consumers, when consumers in different groups have different true cost of living indexes.

Theorem 7: Under Assumption 3, the following equation holds for all nonnegative $(e_1,\ldots, e_K)$ and all nonnegative $(p_1^*,\ldots, p_K^*)$:

$$d \log p^*(p_1^* e_1,\ldots, p_K^* e_K) = d \log P^*(p_1^* e_1,\ldots, p_K^* e_K).$$

Proof: By equation (10), $d \log (p_k^* e_k) = d \log \hat{P}$ for all $k \in \{1,\ldots, K\}$ and all t. Hence $d \log p^* = d \log \hat{P}$ by equation (1), and $d \log P^* = d \log \hat{P}$ by equation (10). So $d \log p^* = d \log P^*$. *Q.E.D.*

We now consider further the case of a homogeneous multilateral representative agent, but first we shall need the following lemma.

Lemma 4: Under Assumptions 1, 2a, 2b, and 3, there exists g such that $g_k = g$ for all $k \in \{1,\ldots, K\}$ so that tastes for consumer goods are identical across countries.

Proof: By equation (10), it follows that

$$d \log [p_k^*(t) e_k(t)] = d \log \hat{P}\left[\mathbf{p}_k(t) e_k(t)\right]$$

for all $k \in \{1,\ldots, K\}$. Hence the same consumer goods price aggregator function $\hat{P}$ applies for all $k \in \{1,\ldots, K\}$. But the consumer goods quantity aggregator function, g_k, is dual to the consumer goods price aggregator function. Hence the consumer goods quantity aggregator functions g_k must also be independent of k. *Q.E.D.*

To move further towards the existence of a homogeneous multilateral representative consumer, we also need the following assumption, which is analogous to Assumption 3.

Assumption 4: Suppose that convergence over the euro area results in the existence of $\hat{\Pi}$ such that

$$\frac{d}{dt}[\log (\prod_k(t) e_k(t))] = \frac{d}{dt}[\log \hat{\Pi}(t)]$$

for all $k \in \{1,\ldots, K\}$ and all t.

Clearly under this assumption, it follows from Definition 5 that $\hat{\Pi}(t) = \Pi(t)$ for all t. The following lemma depends heavily upon Assumption 4.

Lemma 5: Under Assumptions 1, 2a, 2b, and 4, there exists u such that $u_k = u$ for all $k \in \{1,\ldots,K\}$, so that tastes for monetary services are identical across countries.

Proof: Analogous to the proof of Lemma 4. *Q.E.D.*

The form of Decision 1 now is as follows.

Decision 1a: Choose $(\mathbf{m}_1^*, \ldots, \mathbf{m}_K^*, \mathbf{x}_1, \ldots, \mathbf{x}_K)$ to

$$\text{maximize } U\{V[s_1 u(\mathbf{m}_1^*), \ldots, s_K u(\mathbf{m}_K^*)], G[s_1 g(\mathbf{x}_1), \ldots, s_K g(\mathbf{x}_K)]\}$$

$$\text{subject to } \sum_{k=1}^{K} s_k \mathbf{m}_k^{*\prime} \boldsymbol{\pi}_k e_k + \sum_{k=1}^{K} s_k \mathbf{x}_k' \mathbf{p}_k e_k = I.$$

Observation 2: Under Assumptions 1, 2a, 2b, 3, and 4, the solutions to Decisions 1 and 1a will be the same, as is evident from Lemmas 4 and 5. Because of the homogeneity of tastes across countries in Decision 1a, we have the following definition.

Definition 8: We define a homogeneous multilateral representative agent to be an economic agent who solves Decision 1a under Assumptions 1, 2a, 2b, 3, and 4.

Observe that despite the homogeneity of tastes across countries, the decision remains multilateral, as a result of the assumption of blockwise weak separability of tastes across countries. That separability assumption produces existence of within-country tastes, u,

independent of consumption in other countries. The fact that the tastes are identical for all euro area countries does not negate the existence of those tastes, u.

In econometric studies, there could be reason to investigate convergence of the general multilateral representative consumer towards the homogeneous multilateral representative agent. But for data construction purposes, we see no advantage to adopting the homogeneous multilateral representative agent model. We have shown that the general multilateral representative agent model can be used to construct aggregates recursively, first within countries and then across countries. When producing the aggregates within countries, there is not benefit to imposing uniformity of tastes across countries.

In the next section, we explore the unilateral representative agent model that would produce a large gain in data construction simplification, but only under a very strong assumption that is not likely to be reasonable within the near future, if ever.

5.2. Existence of a Unilateral Representative Agent over the Euro Area

A unilateral representative agent considers the same goods and assets to be perfect substitutes, regardless of the country within which the goods and assets are purchased and regardless of the country within which the purchaser resides. Under this assumption, our subscripts j and k will be irrelevant to the tastes of the unilateral representative agent. Only the subscript i will matter, since countries, and thereby country subscripts, will be irrelevant to the decision.

We no longer can accept Assumption 2b, but instead will have to make a much stronger, but nonnested, assumption. Assumption 2b assumed weak separability among countries of residence of consumers. But a unilateral representative agent neither recognizes the country of residence of a consumer nor the country within which a good or asset was acquired.[37] Hence tastes specific to a country no longer exist. It is important to recognize the fundamental difference between the homogeneous multilateral representative consumer and the unilateral representative consumer. The former imputes identical tastes to each country's residents, but does recognize the existence of different countries and the existence of the identical tastes, u, within each country. But the unilateral representative consumer does not impute existence of weakly separable tastes to the residents of any euro area country.

Since we no longer can assume weak separability among countries, we shall have to rewrite Decision 1 as:

[37] Under the weak separability assumption in Assumption 2b, the marginal rate of substitution among assets (or goods) by residents of a country is independent of consumption of the services of the same assets (or goods) by residents of another country. But a unilateral representative agent recognizes neither the country of residence of a consumer nor the country within which an asset or good was acquired. Hence the functions u and g cannot exist.

Decision 1b: Choose ($\mathbf{m}_1^*, \ldots, \mathbf{m}_K^*, \mathbf{x}_1, \ldots, \mathbf{x}_K$) to

$$\text{maximize } U[\breve{V}(s_1\mathbf{m}_1^*,\ldots., s_K\mathbf{m}_K^*), \breve{G}(s_1\mathbf{x}_1,\ldots., s_K\mathbf{x}_K)]$$

$$\text{subject to } \sum_{k=1}^{K} s_k \mathbf{m}_k^{*\prime}\boldsymbol{\pi}_k e_k + \sum_{k=1}^{K} s_k \mathbf{x}_k{}'\mathbf{p}_k e_k = I..$$

Hence we now replace Assumption 2b with the following much stronger assumption, which is neither necessary nor sufficient for Assumption 2b.

Assumption 5: Let $\mathbf{m}^* = \sum_{k=1}^{K}\sum_{j=1}^{K+Z} s_k \mathbf{m}_{kj}^*$, and $\mathbf{x} = \sum_{k=1}^{K} s_k \mathbf{x}_k$. Suppose there exists linearly homogeneous $\hat{V}$ such that $\hat{V}(\mathbf{m}^*) = \breve{V}(s_1\mathbf{m}_1^*, \ldots, s_K\mathbf{m}_K^*)$, where $\breve{V}$ is as defined in equation (5a) . Then for any i, all monetary assets of that type are perfect substitutes, regardless of the country within which they are located or the country in which the owner resides. Analogously for consumer goods, assume there exists $\hat{G}$ such that $\hat{G}(\mathbf{x}) = \breve{G}(s_1\mathbf{x}_1,\ldots.,s_K\mathbf{x}_K)$, where $\breve{G}$ is as defined in equation (5a).[38] Hence for any i, all consumer goods of that type are perfect substitutes, regardless of the country within which they are located or the country in which the owner resides. Further assume that there exist $\boldsymbol{\pi}(t)$ and $\mathbf{p}(t)$ such that $\boldsymbol{\pi}_{kj}(t)e_k(t) = \boldsymbol{\pi}(t)$ and $\mathbf{p}_k(t)e_k(t) = \mathbf{p}(t)$ for all $k \in \{1,\ldots.,K\}$, $j \in \{1,\ldots.,K+Z\}$, and all t.

The assumptions $\boldsymbol{\pi}_{kj}(t)e_k(t) = \boldsymbol{\pi}(t)$ and $\mathbf{p}_k(t)e_k(t) = \mathbf{p}(t)$ are needed to avoid corner solutions allocating no consumption to residents of some countries. Otherwise, with perfect substitutability across countries of residence, all consumption of each good by the unilateral representative agent would be allocated to residents of the country having the lowest price of that good. Under Assumption 5, Decision 1b now becomes Decision 1c, defined as follows.

Decision 1c: Choose ($\mathbf{m}^*$, $\mathbf{x}$) to

[38] The more rigorous notation used for monetary assets is implicit in our less formal notation for consumer goods.

$$\text{maximize } U[\hat{V}(\mathbf{m}^*), \hat{G}(\mathbf{x})]$$

$$\text{subject to } \mathbf{m}^{*\prime}\boldsymbol{\pi} + \mathbf{x}'\mathbf{p} = I.$$

The following theorem demonstrates that Decision 1c is the decision of a unilateral representative consumer for the euro area.

Theorem 8: Let $(\mathbf{m}_1^*, \ldots, \mathbf{m}_K^*, \mathbf{x}_1, \ldots, \mathbf{x}_K)$ solve Decision 1b, and let $\mathbf{m}^*$ and $\mathbf{x}$ be as defined in Assumption 5. Under Assumptions 1, 2a, and 5, it follows that $\mathbf{m}^*$ and $\mathbf{x}$ will solve Decision 1c.

Proof: Observe that there is no need to include Assumptions 3 or 4 in this theorem, since Assumption 5 implies Assumptions 3 and 4. The result follows directly from the theorem's assumptions and the definitions of $\mathbf{m}^*$ and $\mathbf{x}$. *Q. E. D.*

We thereby are led to the following definition.

Definition 9: Under Assumptions 1, 2a, and 5, we define a unilateral representative consumer to be an economic agent who solves Decision 1c.

Note that a unilateral economic agent recognizes no differences in tastes among countries, either for the owner's country of residence or for the country within which the asset or good is located or purchased. But in a more fundamental sense, observe that in general it is impossible to factor out of $\hat{V}(\mathbf{m}^*)$ or $\hat{G}(\mathbf{x})$ the consumption or asset holdings of residents of any country. Hence country specific separable subfunction u_k or g_k, do not exist, and hence separable tastes of residents of a country do not exist.[39] In fact for any

[39] An exception is the case in which the functions, $\hat{V}$ and $\hat{G}$, in the representative agent's utility function are linear. Since separability is an ordinal property, it is thereby invariant to monotonic transformations. Hence this special case could be weakened slightly to permitting $\hat{V}$ and $\hat{G}$ to be monotonically increasing (isotonic) transformations of linear functions. Then every asset is completely strongly separable from every other asset, and every good is completely strongly separable from every other good, regardless of the good or asset, i, country of residence of the purchaser, k, or country within which the good was acquired or held, j. Hence all possible blockings of goods, assets, and countries are both weakly and strongly separable within $\hat{V}$ and $\hat{G}$. Then the unilateral representative agent can be treated as a special case of the multilateral representative agent. In addition, in this special case every asset is a perfect substitute for

solution for (**m***,**x**) to Decision 1c, the allocation of asset holdings and consumption expenditure to countries is indeterminate. Assumptions 3 and 4 have been omitted from Theorem 8, because of redundancy with Assumption 5. But Assumption 2b, which also has been omitted, is not redundant, but rather is omitted since it contradicts Assumption 5. The unilateral representative agent exists under much stronger assumptions than the multilateral representative agent. But the unilateral representative agent is not a nested special case of the multilateral representative agent, whether in its general or homogeneous form.

Decision 1c is the representative agent model previously used in some studies to aggregate within the euro area. But the required convergence conditions, Assumptions 1, 2a, and 5 and the implied Assumptions 3 and 4, are clearly unreasonable, since they imply decision independence of the country of residence of purchasers and of the country of location of the purchase. Rather than requiring identical tastes of consumers among all countries in the euro area, as in the homogeneous multilateral representative agent case, the unilateral representative agent case implies nonexistence of separable tastes for any country through irrelevancy of the location of the purchaser or of the purchased good or asset. That cultural consequence is not likely to materialize within the EMU in the near future.[40] Even with the existence of the euro common currency within the euro area, the assumptions needed to produce the unilateral representative consumer are very strong.

The multilateral representative agent model of Decision 1 is far more reasonable, requiring only Assumptions 1, 2a, and 2b. But we see from Theorem 6 that our heterogeneous agents approach would produce the same results as the multilateral representative agent theory, if the necessary conditions for existence of a multilateral representative agent were satisfied.

6. Interest Rate Aggregation

Since interest rates play important roles in policy, it could be useful to compute the interest rate aggregate that is dual to the Divisia monetary quantity index. We show that the correct interest rate aggregate is not the one in common use by central banks, and we view the commonly used interest rate aggregates to be unacceptable. In particular we provide the correct formula for aggregating interest rates jointly over monetary assets and over countries.

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every other asset, and every good is a perfect substitute for every other good, regardless of the good or asset, i, country of residence of the purchaser, k, or country within which the good was acquired or held. But this special case is much too unreasonable to be considered seriously.

[40] The assumptions could be weakened somewhat to permit imperfect substitutability of the assets located in the Z countries that are outside the euro area. But this minor change, while complicating the notation, would not weaken the implications regarding assets owned and located within the euro area.

Let $\bar{r}_k$ be the dual aggregate interest rate for country k. It follows from Definition 1 and the definition of the vector of component user costs prices, $\boldsymbol{\pi}_k^*$, that $R_k - \bar{r}_k = \Pi_k^*$, where $\Pi_k^* = \Pi_k^*(\boldsymbol{\pi}_k^*)$. Hence $\bar{r}_k$ easily can be computed from $\bar{r}_k = R_k - \Pi_k^*$. In discrete time when $\pi_{ki}^* = (R_k - r_{ki})/(1+R_k)$, it follows that $(R_k - \bar{r}_k)/(1+R_k) = \Pi_k^*$, with $\bar{r}_k$ being computed by solving that equation.

After aggregating over countries, the interest rate that is dual to M* is similarly easy to compute, if the same benchmark rate applies to all countries. In that case, which we believe not to be applicable prior to the introduction of the euro, our heterogeneous agents approach to aggregating over countries becomes mathematically equivalent to our multilateral representative agent approach.

Let $R = R(t)$ be the common benchmark rate applying to all countries in the EMU, and let $\bar{r} = \bar{r}(t)$ be the interest rate aggregate dual to M*. In continuous time, it follows that $R - \bar{r} = \Pi^*$, where $\Pi^* = \Pi^*(t) = \Pi^*(\Pi_1^*, ..., \Pi_K^*)$. Hence $\bar{r}$ easily can be computed from $\bar{r} = R - \Pi^*$. Analogously in discrete time, it follows that $(R - \bar{r})/(1 + R) = \Pi^*$, with $\bar{r}$ being computed by solving that equation.[41]

Note that our aggregation-theoretic interest-rate aggregates are not the interest-rate weighted averages often used in this literature.

7. Divisia Second Moments

Our use of the stochastic approach to aggregation lends itself naturally to the computation of Divisia second moments, although in the above sections we have provided only the Divisia first moments. In this tradition, the "Divisia index" is synonymous with the Divisia growth rate mean. We believe that the Divisia growth rate variance could be especially useful for exploring distribution effects of policy within the euro area and progress towards convergence. We propose below some potentially useful Divisia growth rate variances and covariances. Conversion of our continuous time formulas to their discrete time version is analogous to that available for the within-country Divisia quantity and user cost growth rate variances in Barnett and Serletis (2000, p. 172, eqs. 4 and 7).

We believe that the Divisia growth rate variances could be especially useful when computed about the Divisia means of the following growth rates: (a) the monetary quantity growth rates, d log M and d log M*, in Definition 4, (b) the Divisia means of the user cost price growth rates, d log Π and d log Π*, in Definition 5, and (c) the inflation

[41] In the heterogeneous agents approach, there does not exist a common benchmark rate that can be imputed to all countries. Under those circumstances, the aggregation theoretic method of producing the interest rate aggregate, , can be found in Barnett (2000, p. 278, equation 5).

growth rate, d log p*, in equation 1 or the inflation growth rate, d log P*, in equation 6. Repeating those Divisia mean formulas and producing the analogous Divisia variances, we have the following formulas.

The Divisia growth rate means are:

$$d \log M = \sum_{k=1}^{K} W_k \, d \log (s_k M_k e_k)$$

$$d \log M^* = \sum_{k=1}^{K} W_k \, d \log (s_k M_k^*),$$

$$d \log \Pi = \sum_{k=1}^{K} W_k \, d \log (\Pi_k e_k),$$

$$d \log \Pi^* = \sum_{k=1}^{K} W_k \, d \log (\Pi_k^*),$$

$$d \log p^* = \sum_{k=1}^{K} W_k \, d \log \, p_k^* e_k \, ,$$

and

$$d \log P^* = \sum_{k=1}^{K} B_k \, d \log \, p_k^* e_k \, ,$$

where

$$B_k = \frac{X_k p_k^* e_k}{\sum_{\kappa=1}^{K} X_\kappa p_\kappa^* e_\kappa} \, .$$

The analogous Divisia growth rate variances are:

$$K = \sum_{k=1}^{K} W_k \, [d \log (s_k M_k e_k) - d \log M]^2 ,$$

$$K^* = \sum_{k=1}^{K} W_k \, [d \log (s_k M_k^*) - d \log M^*]^2,$$

$$J = \sum_{k=1}^{K} W_k \, [d \log (\Pi_k e_k) - d \log \Pi]^2,$$

$$J^* = \sum_{k=1}^{K} W_k \, [d \log (\Pi_k^*) - d \log \Pi^*]^2,$$

$$G_M = \sum_{k=1}^{K} W_k \, [d \log p_k^* e_k - d \log p^*]^2,$$

and

$$G = \sum_{k=1}^{K} B_k \, [d \log p_k^* e_k - d \log P^*]^2.$$

An additional potentially useful Divisia growth rate variance is that of the monetary expenditure share growth rates:

$$\Psi = \sum_{k=1}^{K} W_k \, [d \log W_k - d \log W]^2,$$

where

$$d \log W = \sum_{k=1}^{K} W_k \, d \log W_k \, .$$

The Divisia monetary services growth rate variances, K and K*, and the Divisia monetary-services expenditure-share growth-rate variance, Ψ, are measures of the dispersion of monetary service growth rates across countries in nominal and real terms, respectively, while the Divisia inflation rate variances, G and G_M, are measures of the dispersion of inflation rates across countries. Increasing values of K, K*, Ψ, and G over time are indications of growth in the distribution effects of monetary policy over the countries of the euro area. Decreases in K, K*, Ψ, and G over time are indications of convergence towards more uniform effects of policy over the euro area. If variations in K, K*, and Ψ tend to precede those of G, then there is an implication of causality. The converse could indicate that policy is accommodating other causal factors. The Divisia growth rate variances, J and J*, are measures of the progress of harmonization of

financial markets over countries and hence are less directly connected with monetary policy and more directly connected with structural progress in the unification of money markets over the euro area.

Since policy often operates through interest rates, it can be useful to explore the strength of the connection between user cost growth rates and monetary service growth rates as indications of the effectiveness of transmission mechanisms that operate through interest rates. Further into the transmission mechanism, it is useful to explore the strength of the connection between monetary service growth rates or user cost growth rates and inflation growth rates. For those purposes, the following Divisia growth rate covariances can be computed:

$$\Gamma(M^*,\Pi^*) = \sum_{k=1}^{K} W_k \, [d \log (s_k M_k^*) - d \log M^*][d \log (\Pi_k^*) - d \log \Pi^*],$$

$$\Gamma(M,\Pi) = \sum_{k=1}^{K} W_k \, [d \log (s_k M_k e_k) - d \log M][d \log (\Pi_k e_k) - d \log \Pi],$$

$$\Gamma(M^*,p^*) = \sum_{k=1}^{K} W_k \, [d \log (s_k M_k^*) - d \log M^*][d \log (p_k^* e_k) - d \log p^*],$$

and

$$\Gamma(\Pi^*,p^*) = \sum_{k=1}^{K} W_k \, [d \log (s_k \Pi_k^*) - d \log \Pi^*][d \log (p_k^* e_k) - d \log p^*].$$

The following result connects together the nominal $(K,J,\Gamma(M,\Pi))$ and real $(K^*,J^*,\Gamma(M^*,p^*))$ growth rate variances and covariances.

Theorem 9: Using the above definitions, we have

$$K=K^* + 2\Gamma(M^*,p^*) + G_M,$$

$$J=J^* + 2\Gamma(\Pi^*,p^*) + G_M,$$

and

$$\Gamma(M,\Pi) = \Gamma(M^*,\Pi^*) + \Gamma(\Pi^*,p^*) + \Gamma(M^*,p^*) + G_M.$$

Proof: Follows from Lemma 1, Theorem 1, and equation 1. *Q.E.D.*

The corresponding Divisia standard deviations are produced by taking the square roots of the cumulated Divisia variance growth rates. The corresponding coefficients of variation are acquired by dividing the Divisia standard deviation level by the level of the cumulated Divisia mean index. The purpose of the coefficients of variation is to disconnect the linkage that normally exists over time between trends in standard deviation and mean. But there usually is little advantage to computing the Divisia standard deviation or coefficient of variation levels, since the Divisia growth rate variances and covariances are scale invariant and not inherently trended in a predictable direction.

It can be shown in general that

$$K^* = \Psi - J^* - 2\Gamma(M^*,\Pi^*).$$

See Barnett and Serletis (2000, p. 173, equation 11).

8. Extensions and Variations

8.1. Conversion from Continuous to Discrete Time and from Per Capita to Total Values

We provide our results in continuous time in this paper. The conversion to discrete time for use with data is straightforward. All differentials, d log z(t), are replaced by finite changes, $\log z_t - \log z_{t-1}$. The Törnqvist approximation to the continuous time Divisia index then is the Simpson's rule approximation produced by replacing the share weights by their two-period moving average.[42] For example, $W_k(t)$ is replaced by $(W_{kt} + W_{k,t-1})/2$. Conversion from Divisia indexes in continuous time to Törnqvist indexes introduces a remainder term that is third order in the changes. This remainder term is usually less than the round-off error in the component data, and hence is truly negligible.[43]

[42] The Divisia growth rate mean formula is a differential equation. Its solution for the level of the index is a line integral. The Törnqvist index is the Simpson's rule approximation to the line integral.

[43] The existence of that third-order remainder term for the Törnqvist index proves that it is a superlative index, as defined by Diewert (1976). A third-order remainder term also appears in the factor reversal equations, such as in Theorems 2 and equation (7), so that the Törnqvist index is not exactly self dual. But the Divisia index in continuous time is exactly self dual (i.e., the Divisia user cost is the exact dual to the Divisia quantity index, and the user cost and quantity indexes exactly satisfy factor reversal).

We present all results in per capita form to connect with our representative agent theory. In addition for most purposes the per capital form is most useful, especially in modeling the demand for money and exploring the causes for inflation. Nevertheless, for some policy purposes it can be useful to compute total monetary service flows. To do so, M_k and M_k^* need only be multiplied by H_k, while M and M* need only be multiplied by $\sum_{k=1}^{K} H_k$.

8.2. Introduction of New Countries into the Union

When new countries join the EMU, the data for the new entry should be introduced in a manner that will not produce a discrete jump in the per capita data. One approach would be to use the actual growth rate in the entering country's per-capita monetary assets, converted to euros, during the entry period. A more difficult procedure would be to set the entering country's monetary asset quantities to zero during the period prior to entry and use the existing procedure for introducing new goods.[44] In this case, the data should be total, not per capita. The result would be a discrete jump in the total level of the quantity aggregate, but the jump would be smoothed once converted to per capita values after the introduction of the new member country.

If this latter procedure were used with per capita data, a misleading jump in per capita quantities would result. A choice between the two procedures will have to be made for the entry of Greece in 2001 into the EMU, two years after the launch of the euro. Whichever choice is made for that case should be continued into the future as other countries enter the monetary union.

8.3. Demand for Monetary Assets by Firms

It has been shown in general equilibrium theory that if money has positive value in equilibrium, there exists a derived utility function that contains money.[45] Analogously if money has positive value in equilibrium, there exists a derived production function that contains money.[46] These two result are independent of the motivation for holding money by consumers or firms. The ability to explain the motivation for holding money is lost when money is put into utility functions and production functions, since the inverse mapping from the derived utility functions and derived production functions to the underlying motive is non-unique. But the unknown motivation is irrelevant to our aggregation theory.

[44] For the new goods introduction procedure, see Barnett and Serletis (2000, p. 77, footnote 25) and Anderson, Jones, and Nesmith (1997a, pp. 77-78), who in turn referred to Diewert (1980, pp. 498-501).
[45] See, e.g., Arrow and Hahn (1971), Feenstra (1986), and Samuelson and Sato (1984).
[46] See, e.g., Fischer (1974).

This paper treats all decisions as consumer decisions, despite the fact that some monetary assets are held in large quantities by firms. But Barnett (2000, p. 63, equations 40 and 41) proved that it makes no difference for aggregation theory whether monetary asset demand is by consumers or by firms, or by a combination of both. In this paper, we have produced our results using the derived utility function. Barnett's results using the derived production function for firms demonstrates that the conditional decisions that produce monetary aggregates are identical for firms and consumers. The fact that the conditional decision is nested in a constrained utility maximization decision for consumers and a profit maximization decision for firms has no effect on the aggregation theory.

The issues for aggregation over economic agents is no more or less difficult, if some of the economic agents are consumer and some are firms, all are consumers, or all are firms. A possible exception regards the measurement of the inflation rate for consumers versus firms. If aggregate markets are not cleared and incentive compatibility fails for firms, the inflation rate for firms can differ from that for consumers.

Consider the aggregate technology of a country. Assume that the technology is derived to be vertically integrated so that intermediate products are not among the outputs. The produced goods will be those consumed by consumers. If an exact aggregate over those goods exists for consumer and for firms and if the market in the aggregate is cleared, then the aggregate quantity consumed will equal the aggregate quantity produced. Hence the price duals also will be equal. Under these circumstances, P* will be the price dual both for the firm and for the consumer.[47]

Alternatively suppose markets might not be cleared or a regulatory wedge might exist. But assume incentive compatibility in the sense that managers make decisions that are in the best interests of owners. Since owners seek to maximize their utility and P* is their true cost of living index based upon their tastes, it again follows that output will be evaluated by the firm at price P*. This conclusion is easily acquired from the equivalent centralized decision in which the owner personally manages the firm and treats the firm's technology as a constraint in the owner's profit maximization decision.[48]

While these two results may be comforting, the assumptions required in either case are far from trivial. Fisher and Shell (1972, 1988) have considered in depth the potential consequences of violations of the assumptions that produces P* as the price aggregate on both sides of the market. Extension of our results to the case of separate P* for firms and consumers could be useful for some purposes.

But as we have seen, the price index, p*, used to deflate nominal to real money balances will not be the usual consumer price index, P*, and will not be affected by problems

[47] The assumptions required for this conclusion includes clearing of all component markets and lack of a regulatory wedge produced by different taxation on the two sides of the market. A more subtle problem can be produced from nonhomotheticity.

[48] The equivalence to the decentralized decision under incentive compatibility is acquired in the usual manner by the use of a separating hyperplane.

regarding market clearing and incentive compatibility. In fact our result on factor reversal that produced p* for consumers is equally as applicable to firms. Since p* is of far more importance for our results than P*, we currently view extension to separate P* for firms and consumer to be a relatively low priority in this research. That difficult issue is likely to be of more importance to the construction of an inflation target than to the measurement of real and nominal monetary service flows and user cost prices. For research relevant to the potential selection of an inflation target, see Diewert (2002).

8.4. The Benchmark Yield Proxy and Extension to Risk Aversion

Since risk in exchange rates can be considerable, the extension of our results to the case of risk aversion could be explored in later research, using the approach in Barnett and Serletis (2000, chapter 12). Improvements to this approach are possible using the newest methodology on capital asset pricing in the finance literature. Regarding the recommended procedure for producing a benchmark rate proxy, see the Appendix.

9. Conclusions

We advocate use of Barnett's (1980) representative agent approach to Divisia aggregation within euro area countries and then our heterogeneous countries approach to aggregation over countries. Our stochastic approach to aggregation over countries lends itself naturally to computation of Divisia second moments. We advocate computation of Divisia variance growth rates about the Divisia means across countries. Those Divisia second moments could provide useful information about the distribution effects of policy and about progess towards convergence over the euro area.

We introduce a new method of computation of the benchmark yield. Our approach is based upon summing premia extracted from the rate structure. Prior to introduction of the euro, our proposed procedure for computing the benchmark yield would produce a different benchmark rate for each country. At some time after the introduction of the euro, our procedure would equate the benchmark yields across euro area countries. The special case of a common benchmark yield across countries equates our heterogeneous-countries stochastic approach to another approach that we introduce: the multilateral representative agent approach. Hence either interpretation of our formulas can be applied, when the benchmark yield becomes the same for all countries in the euro area. But prior to the establishment of a common benchmark yield across euro area countries, our more general stochastic heterogeneous countries approach should be preferred to the multilateral representative agent approach.

We define and produce the theory relevant to a third very restrictive case, which we call the unilateral representative agent approach. This approach, which we show to be implied by some earlier studies of euro-area monetary aggregation, is not recommended for use either before or after the introduction of the euro.

With the heterogeneous countries or multilateral representative agent approach, we find the need for two different consumer price indexes: one for use in deflating nominal to real monetary balances after aggregation over countries, and one for deflating nominal to real consumer goods expenditure. Only under the unreasonably restrictive homogeneous multilateral representative agent assumptions or the even more unreasonable unilateral representative agent assumptions, do the growth rates of the two consumer price indexes become equal.

While this result may seem surprising, we believe that it is represents the usual case, rather than an exceptional case. We find that the source of the wedge between the two needed price indexes is the existence of different true-cost-of living indexes for different countries. But in fact it is well known that true-cost-of-living indexes are not only different for different countries and different regions of countries, but also for different consumers. The true cost of living index depends upon tastes, and hence is different for different consumers, even if the law of one price holds so that goods prices are the same for all consumers.[49]

[49] Consider two consumers faced with a rising price of rice, when one consumer likes rice and the other does not. The consumer who likes rice will experience an increase in cost of living and the other will not. In addition, for nontraded goods, there is no reason to believe that even the price of the same good will be the same for consumers in different locations. The fact that consumers are price takers provides no reason to believe that they face the same true cost of living index, and hence the usual view that consumer goods and monetary assets should be deflated by the same price index cannot be supported by theory.

Appendix: The Benchmark Rate

The benchmark rate of return is the expected rate of return received on a pure investment providing no services other than its yield. Hence the market subtracts no liquidity premia, denomination premia, or other service premia from the benchmark yield. It is unlikely that any country has ever had a securities market for the benchmark asset. In fact in principal the benchmark asset cannot have a well organized secondary market, since the existence of such a market would itself be a liquidity service excluded by the definition of the benchmark asset. Rates of return are not easily available on assets having low liquidity, such as human capital stock or small firms having no publicly traded securities. The benchmark rate must be at least as high as the upper envelope over all of the monetary aggregates' component yield-curve-adjusted rates of return. But that envelope must be raised further to include premia already removed by the market from those assets' returns by the existence of markets for the assets.[50]

To get from the upper envelope over the component yield-curve-adjusted rates of return to the benchmark rate, it is necessary to add to the upper envelope a rate structure premium representing the premium for giving up the liquidity of the assets within the envelope. Such a rate structure premium must be a rate differential extracted from the rate structure. We recommend using the difference between a corporate bond rate of moderate quality and the Treasury security rate of the same maturity. Within the euro area, that corporate bond would need to be selected from a country having a good market for that bond within the relevant sample period and a corresponding Treasury security of the same maturity.[51]

In theory the benchmark rate is an expected rate of return, not an ex-post rate of return. Since the rate differential between a bond rate and the corresponding Treasury security yield is likely to be much more volatile than the expected value of that rate differential, we advocate smoothing or forecasting that rate differential (e.g., by time series or moving average methods) before adding it to the upper envelope.

Earlier approaches to approximating the premium over the envelope usually involved either adding the full level of a bond rate to the envelope, or including a bond rate within the assets used in generating the envelope. Those approaches are not relevant to extracting premia from the rate structure, and therefore are not recommended.

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[50] At some date following the introduction of the euro, a single benchmark yield could be imputed to all countries in the euro area. Prior to introduction of the euro, investment in a totally illiquid asset in a foreign country was probably uncommon, except by investors planning to move to the foreign country. Since introduction of the euro greatly facilitates cross border investment, and removes the exchange rate risk associated with investing in a totally illiquid asset in another country, the upper envelope should be over the yield-curve-adjusted rates of return on all component assets in all euro area countries at some point following the introduction of the euro. This convergence of the benchmark rate to a common benchmark rate for the euro area would imply full integration in retail banking and financial markets in the euro, such that all rates available within the euro area become available to all residents of the euro area.

[51] Although the upper envelope can differ across countries prior to the convergence of those envelopes, the same rate structure premium over that envelope could be used for all countries within the EMU, even before the appearance of the euro, since the rate structure premium is inherently an imputed proxy, rather than a direct measurement.

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