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Regime-switching risk in the Term Structure of Interest Rates

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Abstract

This paper incorporates the systematic risk of regime shifts into a general equilibrium model of the term structure of interest rates. The regime-switching risk introduces a new source of time-variation in bond term premiums. A closed-form solution for the term structure of interest rates is obtained for an affine model under log-linear approximation. The model is estimated by Efficient Method of Moments. We find that the market price of the regime-switching risk is not only statistically significant, but also economically important, accounting for a significant portion of the term premiums for long-term bonds. Ignoring the regime-switching risk leads to underestimation of long-term interest rates and therefore flatter yield curves.

JEL Classification: G12, E43
Key Words: The Term Structure, General Equilibrium, Markov Regime Shifts

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1 Introduction

There is strong empirical evidence suggesting that the aggregate economy is characterized by periodic shifts between distinct regimes of the business cycle (e.g. Hamilton 1989, Filardo 1994 and Diebold and Rudebusch 1996). A number of papers have also successfully used Markov regime-switching models to fit the dynamics of the short-term interest rate (see, among others, Hamilton 1988, Garcia and Perron 1996, Gary 1996 and Ang and Bakeart 1998).[1] These results have motivated the recent studies of the impact of regime shifts on the entire yield curve using dynamic term structure models. A common approach, as in Naik and Lee (1997), Boudoukh et al.(1999), Evans (2001) and Bansal and Zhou (2002), is to incorporate Markov-switching into the stochastic processes of the pricing kernel and/or state variables.[2] The regime-dependence introduced by these studies implies richer dynamic behavior of the market prices of risk and therefore offers greater econometric flexibility for the term structure models to simultaneously account for the time series and cross-sectional properties of interest rates. However, as pointed out by Dai and Singleton (2003), the risk of regime shifts is not priced in these models, hence does not contribute independently to bond risk premiums.

The main objective of current paper is to extend this strand of literature by developing and estimating a fully fledged dynamic term structure model under the systematic risk of regime shifts in a general equilibrium setting similar to that in Cox, Ingersoll and Ross (1985a, 1985b). We show that the regime-switching risk can be priced in a similar way as in the case of jump risk (e.g. Ahn and Thompson, 1988). Our model implies that bond risk premiums include two components under regime shifts in general. One

[1] The expectation theory is usually invoked to relate long-term interest rates to the short rate in this literature, such as in Ang and Bakeart (1998).

[2] Other studies of the term structure of interest rates under hidden Markov chains include Bielecki and Rutkowski (2001), Elliott and Mamon (2001) among others.

is a regime-dependent risk premium due to diffusion risk as in the previous studies. This risk premium has added econometric flexibilities relative to those in single-regime models because of the Markov-shift of the underlying parameters. The other is a regime-switching risk premium that depends on the difference of bond prices across regimes as well as the Markov transition probabilities. Therefore the model introduces a new source of time-variation in bond risk premiums. This additional component of the term premiums is associated with the systematic risk of periodic shifts in bond prices due to regime changes. Given the empirical evidence from the previous studies that the yield curve exhibits significantly different properties across regimes, the model implies that the regime-switching risk is likely to be an important factor that affects bond returns.

To get a quantitative measure of the impact of the regime-switching risk on the yield curve, we obtain a closed-form solution of the term structure of interest rates for an affine-type model using the log-linear approximation similar to that in Bansal and Zhou (2002). The model is estimated by the Efficient Method of Moments (EMM) developed in Bansal et al. (1995) and Gallant and Tauchen (1996, 2001). We use the monthly data on 6-month treasury bill and 5-year treasury bond from 1964 to 2000 in the estimation. We find that the market price of the regime-switching risk is not only statistically significant, but also economically important. Empirical results suggest that the risk of regime shifts account for a significant portion of the term premiums, particularly for bonds of maturities longer than 5 years. Ignoring the regime-switching risk leads to underestimation of long-term interest rates and therefore flatter yield curves.

The rest of the paper is organized as follows. Section 2 specifies the underlying economy and lays out the asset pricing model under the systematic regime-switching risk. Section 3 obtains the closed-form solution for the term structure of interest rates. Section 4 discusses the empirical results from EMM estimation and section 5 concludes.

2 The Model

2.1 The Underlying Economy

Consider an economy with a single good and a large number of infinitely lived and identical consumers similarly to that in Cox, Ingersoll and Ross (1985a, 1985b) (henceforth CIR). We first describe the state variables and investment opportunities of the economy as well as the representative consumer's objective function below.

2.1.1 State Variables

We assume that the economy is driven by two state variables.[3] One of the state variable $x(t)$ has a continuous path and is determined by the stochastic differential equation below

$$dx = \mu_x dt + \sigma_x dB_t \tag{1}$$

where the drift term μ_x and the diffusion term σ_x are in general time-varying and regime-dependent. The other state variable is a continuous-time Markov chain $s(t)$ taking on values of $1, 2, \cdots, N$ if there are N distinct regimes. Following Landen (2000), we make use of the marked point process to get a convenient representation of $s(t)$. In particular, the mark space E is defined as

$$E = \{(i,j) : i \in \{1, ..., N\}, j \in \{1, 2, ..., N\}, i \neq j\}$$

with σ-algebra $\mathcal{E} = 2^E$. Denote $z = (i, j)$ as a generic point in E. A marked point process, $m(t, \cdot)$ is uniquely characterized by its stochastic intensity

[3]It is straight forward to extend the model to include more state variables. We keep the model as simple as possible for exposition purpose.

kernel,[4], which can be defined as

$$\gamma_m(dt, dz) = h(z, x(t-))\mathbf{I}\{s(t-) = i\}\epsilon_z(dz)dt, \tag{2}$$

where $h(z, x(t-))$ is the regime-shift (from regime i to j) intensity at $z = (i, j)$, $\mathbf{I}\{s(t-) = i\}$ is an indicator function, and $\epsilon_z(A)$ is the Dirac measure (on a subset A of E) at point z (defined by $\epsilon_z(A) = 1$ if $z \in A$ and 0, otherwise). Heuristically, for $z = (i, j)$, $\gamma_m(dt, dz)$ can be thought of as the conditional probability of shifting from Regime i to Regime j during $[t, t + dt)$ given $x(t-)$ and $s(t-) = i$.

Let A be a subset of E. Then $m(t, A)$ counts the cumulative number of regime shifts that belong to A during $(0, t]$. $m(t, A)$ has its compensator, $\gamma_m(t, A)$, given by

$$\gamma_m(t, A) = \int_0^t \int_A h(z, x(\tau-))\mathbf{I}\{s(\tau-) = i\}\epsilon_z(dz)d\tau. \tag{3}$$

This simply implies that $m(t, A) - \gamma_m(t, A)$ is a martingale.

Using the above notations, the evolution of the regime $s(t)$ can be conveniently represented as

$$ds = \int_E \zeta(z)m(dt, dz) \tag{4}$$

with the compensator given by

$$\gamma_s(t)dt = \int_E \zeta(z)\gamma_m(dt, dz), \text{ where } \zeta(z) = \zeta((i, j)) = j - i \tag{5}$$

For example, if there is a regime shift from i to j occurred at time t, equation (4) then implies $s_t = (j - i) + s_{t-} = j$. Note that $\int_E$ is equivalent to $\sum_{i \neq j}$.

[4] See Last and Brandt (1995) for detailed discussion of marked point process, stochastic intensity kernel and related results.

2.1.2 Investment Opportunities

We assume that, without loss of generality, the output is produced by a single technology which depends on both state variables $x(t)$ and $s(t)$ as described by the following stochastic differential equation

$$dy = y\mu_y dt + y\sigma_y dB_t + \int_E y\delta_y(z)m(dt, dz) \tag{6}$$

where both μ_y and σ_y can be functions of $x(t-)$ and $s(t-)$. And $\delta_y(z)$ is the discrete percentage change in y due to a regime shift, i.e. $\delta_y(z) = \frac{\Delta_s y}{y} = \frac{y(t,s(t))-y(t-,s(t-))}{y(t-,s(t-))}$.

As in CIR model, we assume that there is a competitive market for instantaneous borrowing and lending at the short-term interest rate $r(t)$. There is also a competitive market for default-free pure discount bonds whose prices are given by

$$dF = F\mu_F dt + F\sigma_F dB_t + \int_E F\delta_F(z)m(dt, dz) \tag{7}$$

Note that μ_F, σ_F and $\delta_F(z)$ are to be determined by the equilibrium conditions. And $\delta_F(z)$ is the discrete percentage change in the bond prices due to a regime shift. (7) can be alternatively written as

$$\begin{aligned} dF = F\mu_F dt &+ \int_E F\delta_F(z)\gamma_m(dt, dz) \\ &+ F\sigma_F dB_t + \int_E F\delta_F(z)(m(dt, dz) - \gamma_m(dt, dz)) \end{aligned} \tag{8}$$

The last two terms in the above equation are martingales. Therefore (7) implies that the instantaneous expected bond return is

$$E_{t-}\left(\frac{dF}{F}\right) = \mu_F dt + \int_E \delta_F(z)\gamma_m(dt, dz) \tag{9}$$

2.1.3 The Consumer's Objective Function

Given the initial wealth w_0, a representative consumer seeks to maximize her expected lifetime utility given by

$$E_0\left[\int_0^\infty e^{-\rho t}U(c(t))dt\right] \tag{10}$$

where $c(t)$ is the flow of consumption and $U(\cdot)$ is the instantaneous utility function. As usual, $U(\cdot)$ is assumed to be strictly concave, increasing and twice differentiable with $U(0) = 0$ and $U'(0) = \infty$.

At each instant, the representative consumer allocates her wealth among investment in the physical production, the discount bonds, the risk-free borrowing and lending and consumption. We assume that both physical investment and trading of the financial assets take place in continuous time without borrowing constraint, transaction cost and all other forms of market frictions. The consumer's budget constraint is therefore given by

$$dw = w\mu_w dt + w\sigma_w dB_t + \int_E w\delta_w(z)m(dt, dz) \tag{11}$$

where

$$w\mu_w = w[\phi_1(\mu_y - r) + \phi_2(\mu_F - r) + r] - c \tag{12}$$

$$w\sigma_w = w[\phi_1\sigma_y + \phi_2\sigma_F] \tag{13}$$

$$w\delta_w(z) = w[\phi_1\delta_y(z) + \phi_2\delta_F(z)] \tag{14}$$

In the above equations, $w(t)$ is the consumer's wealth at time t, ϕ_1 is the proportion of her wealth invested in the physical production, ϕ_2 is the proportion of her wealth invested in the discount bonds, and $c(t)$ is her flow of consumption.

2.2 The Equilibrium Bond Returns

In this section we state the main asset pricing results for the economy subject the systematic regime-switching risk. The detailed derivations are provided in the Appendix.

Let $J(w(t), s(t), x(t)) = \sup_{\{c,\phi_1,\phi_2\}} E_t \left[\int_t^\infty e^{-\rho(\tau-t)} U(c(\tau))d\tau\right]$. $J(w, s, x)$ is the indirect utility function. We use the following notations in our discussions below: $J_w = \frac{\partial J}{\partial w}$, $J_x = \frac{\partial J}{\partial x}$, $J_{ww} = \frac{\partial^2 J}{\partial w^2}$ and $J_{wx} = \frac{\partial^2 J}{\partial w \partial x}$. As in CIR, we also denote $Var(w^c) = (w\sigma_w)^2$, $Var(x) = \sigma_x^2$, and $Cov(w^c, x) = (w\sigma_w)\sigma_x$. To further simplify notations, we define $\Delta_s f = f(s(t)) - f(s(t-))$ for any function $f(\cdot)$ that depends on $s(t)$. $\Delta_s f$ is therefore the difference in $f(\cdot)$ due to a regime shift at time t.

The following two propositions give the equilibrium instantaneous short-term interest rate and the expected excess rate of return on a bond respectively.

Proposition 1 *The equilibrium short-term interest rate is given by*

$$\begin{aligned} r = \mu_y^* - \left(-\frac{J_{ww}}{J_w}\right)\frac{Var(w^c)}{w} - \left(-\frac{J_{wx}}{J_x}\right)\frac{Cov(w^c, x)}{w} \\ - \int_E \left(-\frac{\Delta_s J_w}{J_w}\right)\frac{\Delta_s w}{w}\gamma_m(dz) \end{aligned} \tag{15}$$

where

$$\mu_y^* = \mu_y + \int_E \delta_y(z)\gamma_m(dz) \tag{16}$$

and

$$\gamma_m(dz) = h(z, x(t-))\mathbf{I}\{s(t-) = i\}\epsilon_z(dz) \tag{17}$$

Note that $\mu_y^* dt$ is in fact the expect rate of return of the production technology $E_{t-}\left(\frac{dy(t)}{y(t-)}\right)$. And $\frac{\Delta_s w}{w}$ and $\frac{\Delta_s J_w}{J_w}$ in (15) are the discrete percentage changes in w and J_w respectively due to a regime shift, i.e. $\frac{\Delta_s w}{w} = \delta_w(z) =$

$\phi_1\delta_y(z) + \phi_2\delta_F(z)$, and

$$\frac{\Delta_s J_w}{J_w} = \frac{J_w(w(1+\delta_w(z)), s+\zeta(z), x) - J_w(w,s,x)}{J_w(w,s,x)}$$

Proposition 1 implies that the instantaneous short-term interest rate $r(t)$ is a function of both state variables $x(t)$ and $s(t)$ because of the regime-dependence of μ_y^*, $Var(w^c)$, $Cov(w^c, x)$ and the marginal utility J_w. If the regime shifts is not a systematic risk as assumed in the previous literature, $\frac{\Delta_s J_w}{J_w}$ would be equal to zero. Otherwise the short-term interest rate will also be affected by the last term. Since the indirect value function $J(w, s, x)$ is concave in w, if it is also separable in w and s (as in the case of log utility function), we will have

$$\left(-\frac{\Delta_s J_w}{J_w}\right)\frac{\Delta_s w}{w} > 0$$

Therefore the impact of the systematic regime-switching risk is to lower the equilibrium short-term interest rate, as the local risk-free borrowing and lending opportunity offers a hedge against such risk. This result is similar to the impact of a systematic jump risk on the interest rate as shown in Ahn and Thompson (1988).

Proposition 2 *Let* $\mu_F^* dt = E_{t-}\left(\frac{dF(t)}{F(t-)}\right)$*, the instantaneous expected rate of return of a discount bond. At equilibrium, we have*

$$\begin{aligned}\mu_F^* - r = &\left[\left(-\frac{J_{ww}}{J_w}\right)Var(w^c) + \left(-\frac{J_{wx}}{J_w}\right)Cov(w^c,x)\right]\frac{F_w}{F}\\ &+\left[\left(-\frac{J_{ww}}{J_w}\right)Cov(w^c,x) + \left(-\frac{J_{wx}}{J_w}\right)Var(x)\right]\frac{F_x}{F}\\ &+\int_E\left(-\frac{\Delta_s J_w}{J_w}\right)\frac{\Delta_s F}{F}h(z,x)\mathbf{I}\{s=i\}\epsilon_z(dz)\end{aligned} \tag{18}$$

where $F_w = \frac{\partial F}{\partial w}$, $F_x = \frac{\partial F}{\partial x}$, and $\frac{\Delta_s F}{F} = \delta_F(z)$, the discrete percentage change in the bond price due to a regime shift. $Cov(w^c, x)$, $Var(w^c)$ and $Var(x)$ are defined at the beginning of this section.

We can clearly see from the above proposition that, when there is only one regime, $\frac{\Delta_s F}{F} = 0$, $\frac{\Delta_s J_w}{J_w} = 0$ and $\frac{\Delta_s w}{w} = 0$, (18) is reduced to the standard result in CIR. On the other hand, in the term structure models with regime shifts such as those in Bansal and Zhou (2002) and Evans (2001) among others, $Var(w^c)$, $Var(x)$ and $Cov(w^c, x)$ are assumed to be regime-dependent. Therefore these models have additional econometric flexibility and are shown to have better empirical performance than models that don't include regime shifts. One implicit assumption maintained by these models, however, is that $\frac{\Delta_s J_w}{J_w} = 0$.

To further illustrate the role played by $\frac{\Delta_s J_w}{J_w}$ in determining bond returns, let's assume that $J_{wx} = 0$ and $F_w = 0$.[5] We can obtain from (18) that

$$\mu_F^* - r = -\frac{J_{ww}}{J_w} w\sigma_w \frac{\sigma_x F_x}{F} - \int_E \frac{\Delta_s J_w}{J_w} \frac{\Delta_s F}{F} h(z, x)\mathbf{I}\{s = i\}\epsilon_z(dz) \qquad (19)$$

The first term in (19) is the instantaneous diffusion risk premium. $\frac{\sigma_x F_x}{F}$ is the volatility of the bond return due to diffusions in $x(t)$. $\left(-\frac{J_{ww}}{J_w}\right) w\sigma_w$ measures the extra rate of return per unit of such volatility and is commonly referred to as the market price of risk in the literature. The second term can be thought of as the instantaneous regime-switching risk premium, where $\frac{\Delta_s F}{F} h(z, x)\mathbf{I}\{s = i\}\epsilon_z(dz)$ is the expected discrete percentage change in the bond price due to regime shifts and $\left(-\frac{\Delta_s J_w}{J_w}\right)$ measures the excess bond return per unit of such expected changes. Hence $\left(-\frac{\Delta_s J_w}{J_w}\right)$ can be analogously defined as the market price of regime-switching risk. And we can see from (19), unless J_w or the marginal utility does not depend on regime $s(t)$, investors will price the regime-switching risk into the bond returns.

[5]For example, both $J_{wx} = 0$ and $F_w = 0$ hold under log utility function for discount bonds as shown in CIR.

Note that, by Ito's formula, the diffusion term of the bond price is given by $F\sigma_F = (w\sigma_w)F_w + \sigma_x F_x$. Hence more insight can be obtained by rewriting (18) as

$$\mu_F^* - r = -Cov\left(\frac{dJ_w^c}{J_w}, \frac{dF^c}{F}\right) - \int_E \frac{\Delta_s J_w}{J_w}\frac{\Delta_s F}{F} h(z,x)\mathbf{I}\{s=i\}\epsilon_z(dz) \quad (20)$$

where

$$Cov\left(\frac{dJ_w^c}{J_w}, \frac{dF^c}{F}\right) = \left(\frac{J_{ww}w\sigma_w + J_{wx}\sigma_x}{J_w}\right)\frac{\sigma_F F}{F}$$

(20) implies that the expected excess bond return includes two components under regime shifts. The first term depends on the covariance of the continuous part of the bond return and continuous part of the rate of change in the marginal utility of wealth. The second term depends on the covariation between the discrete percentage change in the marginal utility and the discrete percentage change in the bond price under regime shifts. The higher the covariation, the greater the payoff the bond provides when marginal utility is higher. Hence consumers are willing to accept a lower the expected rate of return on the asset.

In the previous regime-switching term structure models, greater flexibilities are obtained in fitting the time-varying excess bond returns by making $Cov\left(\frac{dJ_w^c}{J_w}, \frac{dF^c}{F}\right)$ regime-dependent. Equation (20) shows that allowing investors to price the regime-switching risk (i.e. $\frac{\Delta_s J_w}{J_w} \neq 0$) introduces an additional source of time-variation in the expected excess bond returns. This new component in the bond returns is associated with the potentially large shifts in bond prices across different regimes $\frac{\Delta_s F}{F}$. It also depends on the regime-shift intensity $h(z,x)$ as well as the market price of regime-switching risk $-\frac{\Delta_s J_w}{J_w}$. Holding $\frac{\Delta_s F}{F}$ and $-\frac{\Delta_s J_w}{J_w}$ constant, the higher the regime-shift intensity $h(z,x)$, the larger the risk premium.[6] On the other hand, given

[6]Boudoukh et al (1999) found that business turning points are usually characterized by highly volatile and strongly time-varying term premium.

$\left(-\frac{\Delta_s J_w}{J_w}\right)$ and $h(z,x)$, the regime-switching risk premium depends on the magnitude $\frac{\Delta_s F}{F}$. The bigger the difference in the bond price across different regimes, the more important the risk premium due to regime shifts. Empirical results from the previous studies (e.g. Bansal and Zhou, 2002) imply sizeable $\frac{\Delta_s F}{F}$, hence suggesting that the regime-switching risk premium is unlikely a negligible component of bond returns.

3 The Term Structure of Interest Rates

In this section we obtain a closed form solution for the term structure of interest rates. We assume that $U(c) = log(c)$ as in CIR model. It can be shown that the prices of default-free pure discount bonds are given by the following proposition (see the Appendix for details).

Proposition 3 *The price at time t of a default-free pure discount bond $F(t, x(t), s(t), T)$ that matures at time T satisfies the following system of partial differential equations*

$$\begin{aligned} &F_t + (\mu_x - \sigma_y \sigma_x) F_x + \frac{1}{2}\sigma_x^2 F_{xx} \\ &\quad + \int_E (1 - \lambda_s(z)) \Delta_s F h(z,x) \mathbf{I}\{s = i\} \epsilon_z(dz) = rF \end{aligned} \tag{21}$$

with the boundary condition: $f(T, x, s, T) = 1$.

In the above equation, $F_t = \frac{\partial F}{\partial t}$, $F_x = \frac{\partial F}{\partial x}$, $F_{xx} = \frac{\partial^2 F}{\partial x^2}$, $\Delta_s F = F(t, x(t), s(t-) + \zeta(z), T) - F(t-, x(t-), s(t-), T)$, and $\lambda_s(z) = \frac{\delta_y(z)}{1+\delta_y(z)}$. Note that (21) holds for each regime of $s(t)$, it therefore defines a system of N partial equations if there are N distinct regimes. Moreover, under the log utility function, the equilibrium short-term interest rate $r(t)$ can be obtained from Proposition 1 as

$$r = \mu_y - \sigma_y^2 + \int_E \lambda_s(z) h(z,x) \mathbf{I}\{s = i\} \epsilon_z(dz) \tag{22}$$

In general the system (21) does not admit a closed form solution to the bond price. Hence we consider the following affine specification which is known to offer a tractable model of the term structure of interest rate.[7] In particular, we assume

$$\mu_x = a_0(s) + a_1(s)\, x \tag{23}$$

$$\sigma_x = \sqrt{\sigma(s)\, x} \tag{24}$$

$$h(z,x) = e^{\eta_s(z)} \tag{25}$$

$$\sigma_y = \theta_x(s)\sqrt{\sigma(s)\, x} \tag{26}$$

$$\mu_y = x + \theta_x^2(s)\sigma(s)\, x - \int_E \lambda_s(z)\gamma_m(dz) \tag{27}$$

$$\lambda_s(z) = 1 - e^{\theta_s(z)} \tag{28}$$

where $\gamma_m(dz) = h(z,x)\mathbf{I}\{s=i\}\epsilon_z(dz)$.

Under (23) and (24), the state variable $x(t)$ follows a conventional mean-reverting square-root process with regime-dependent drift and diffusion terms, that is

$$dx = (a_0(s) + a_1(s)\, x)\, dt + \sqrt{\sigma(s)\, x}\, dB_t \tag{29}$$

(26) and (27) are assumptions about the drift term and the diffusion term of the physical production process $y(t)$. Under the log utility function, they imply that in equilibrium the market price of the diffusion risk is given by

$$\lambda_x(s) = \theta_x(s)\sqrt{\sigma(s)\, x} \tag{30}$$

(25) assumes that the Markov chain $s(t)$ has constant transition probabilities given by $e^{\eta_s(z)}$. Equation (28) parameterizes the market price of the regime-switching risk $\lambda_s(z)$ as $1 - e^{\theta_s(z)}$. Moreover (22), (26) and (27) together

[7] Duffie and Kan (1996) and Dai and Singleton (2000) offer detailed discussions of affine term structure models under diffusions. Bansal and Zhou (2002), Evans (2001) and Landen (2000) also use affine specifications in their regime-switching models.

imply that $r(t) = x(t)$. Therefore

$$dr = \kappa(s)(\bar{r}(s) - r)\,dt + \sqrt{\sigma(s)\,r}\,dB_t \tag{31}$$

where $\kappa(s) = -a_1(s)$, $\bar{r}(s) = \frac{a_0(s)}{-a_1(s)}$.

The Appendix shows that, using a log-linear approximation similar to that in Bansal and Zhou (2002), the term structure of interest rates can be solved as follows

Proposition 4 *Under the assumptions* (23) – (28)*, the price at time t of a default-free pure discount bond with maturity τ is given by $F(t,\tau) = e^{A(\tau,s(t))+B(\tau,s(t))r(t)}$ and the τ-period interest rate is given by $R(t,\tau) = -\frac{A(\tau,s(t))}{\tau} - \frac{B(\tau,s(t))r(t)}{\tau}$, where $A(\tau,s)$ and $B(\tau,s)$ are determined by the following system of differential equations*

$$\begin{aligned} -\frac{\partial B(\tau,s)}{\partial\tau} &+ [a_1(s) - \theta_x(s)\sigma(s)]B(\tau,s) + \frac{1}{2}\sigma(s)B^2(\tau,s) \\ &+ \int_E \left(e^{\Delta_s A}\Delta_s B\right) e^{\eta_s(z)+\theta_s(z)}\mathbf{1}(s=i)\epsilon_z(dz) = 1 \end{aligned} \tag{32}$$

and

$$-\frac{\partial A(\tau,s)}{\partial\tau} + a_0(s)B(\tau,s) + \int_E \left(e^{\Delta_s A} - 1\right) e^{\eta_s(z)+\theta_s(z)}\mathbf{1}(s=i)\epsilon_z(dz) = 0 \tag{33}$$

with boundary conditions $A(0,s) = 0$ and $B(0,s) = 0$, where $\Delta_s A = A(\tau, s+\zeta(z)) - A(\tau,s)$ and $\Delta_s B = B(\tau, s+\zeta(z)) - B(\tau,s)$

Proposition 4 nests the models in Bansal and Zhou (2002) and Landen (2000). Without using the log-linear approximation, Landen(200) only considers models where $\Delta_s B = 0$ and is silent on the market prices of risk $\theta_x(s)$ and $\theta_s(z)$. In the case of Bansal and Zhou (2002), the risk of regime shifts is not priced, namely $\theta_s(z)$ is assumed to be zero in their model. The model in Proposition 4 is in fact a special case of that in Dai and Singleton

(2003), which proposes a general dynamic term structure model where the risk of regime shifts is priced. The main difference between the current paper and Dai and Singleton (2003) is that we derived our model from a general equilibrium framework with economic interpretations of the market price of the regime-switching risk. Another difference is that we also provided an explicit solution for the term structure of interest rates using log-linear approximation. This allows us to assess quantitatively the impact of the regime-switching risk on the yield curve.

4 Empirical Results

4.1 Data and Summary Statistics

The data used in this study are monthly interest rates from June 1964 to November 2000 obtained from the Center for Research in Security Prices (CRSP). There are eight interest rates with maturities ranging from 1 month to 5 years. Table 1 contains the summary statistics of the interest rates. We can see that the yield curve is on average upward-sloping. The large skewness and kurtosis suggest significant departure from Gaussian distribution.

We also report in Table 2 the results from regressions of excess bond returns on forward interest rates and a business cycle dummy variable. We use NBER dates of business cycles to distinguish between expansions and recessions. As in the previous literature we find significant coefficients on forward rates in the regression, suggesting that forward rates contain information about the state variables driving the interest rates. More importantly, the coefficients on the business cycle dummy variable are all negative and significant, suggesting important regime-dependent property of the bond returns. Note that the sign of the coefficient on the business cycle dummy variable is consistent with the counter-cyclical behavior of risk premiums as documented in Fama and French (1989).

4.2 Econometric Methodology

We use Efficient Method of Moments (EMM) in this paper to estimate the term structure model in Proposition 4. We assume that there are two distinct regimes ($N = 2$) for $s(t)$. Therefore (32) and (33) define a system of 4 differential equations that must be solved simultaneously. The model has a total of 12 parameters. Bansal et al (1995) and Gallant and Tauchen (1996, 2001) contain detailed discussion of EMM. As in Bansal and Zhou (2002), we fit the model to the data on the 6-month and the 5-year rates.

Under EMM procedure, the empirical conditional density of the observed interest rates is first estimated by an auxiliary model that is a close approximation to the true data generating process. Gallant and Tauchen (2001) suggests a semi-nonparametric (SNP) series expansion as a convenient general purpose auxiliary model. As pointed out by Bansal and Zhou (2002), one advantage of using the semi-nonparametric specification for the auxiliary model is that it can asymptotically converge to any smooth distributions (Gallant and Tauchen, 1998), including Markov regime-switching models. The dimension of this auxiliary model can be selected by, for example, the Schwarz's Bayesian Information Criterion (BIC). The score function of the auxiliary model are then used as moment conditions to compute a chi-square criterion function, which can be evaluated through simulations given the term structure model under consideration. A nonlinear optimizer is used to find the parameter setting that minimizes the criterion function. Gallant and Tauchen (1996) shows that such estimation procedure yields fully efficient estimators if the score function of the auxiliary model encompasses the score functions of the model under consideration. Bansal and Zhou (2002) is an excellent example of applying EMM to estimate the term structure model under regime shifts. Dai and Singleton (2000) also provides extensive discussions of estimating affine term structure models using EMM procedure.

4.3 Results

Table 3 and Table 4 report the results from SNP estimations. In searching for the preferred specification, we follow closely Bansal and Zhou (2002) mainly because we use the same interest rate data (with longer sample period). Different choices of SNP density and their corresponding BIC values are reported in Table 3. Consistent with Bansal and Zhou (2002), we found that the SNP specification with 1 lag ($L_\mu = 1$) in the VAR-based conditional mean, 5 lags in ARCH term ($L_r = 5$) and a polynoinal of order 4 ($K_z = 4$) in the standardized residual z has the overall best fit based on BIC. Estimates of the coefficients in the preferred SNP density are reported in Table 4.

Given the SNP density, the parameters of the term structure model can be estimated through simulations. We estimate three versions of CIR model. One is the standard one-factor CIR model. The second is a one-factor CIR model with regime shifts, but the risk of regime shifts is not priced. We allow the parameters in the diffusion process of the instantaneous short-term interest rate $r(t)$ to be regime-dependent. We also assume different market prices of the diffusion risk across regimes. The third model is the one-factor CIR developed in Proposition 4 where the risk of regime shifts is priced. The results are reported in Table 5.

In the standard 1-factor CIR model, the parameter estimates imply a highly persistent short-term interest rate process with a long-run average level of 6.4% $[\bar{r} = a_0/(-a_1)]$ and a speed of adjustment of only 0.0907 ($\kappa = -a_1$). This roughly corresponds an AR(1) process with the coefficient on the lagged interest rate being around 0.91, which is consistent with the results from the previous empirical studies of the interest rate. The estimates also imply stochastic volatility is an important property of the interest rate process with a conditional standard deviation of 1.27% on average $\left(\sqrt{\sigma\bar{r}}\right)$.

Model 2 introduces Markov regime shifts into the standard CIR model without pricing such risk. Consistent with the previous studies, we find that the interest rate process is characterized by two distinct regimes. In one regime (Regime 2) the short rate $r(t)$ is highly persistent ($\kappa = -a_1 = 0.0735$) with a long-run mean of 11.7% [$\bar{r} = a_0/(-a_1)$]. The interest rate $r(t)$ in the other regime (Regime 1), however, is less persistent ($\kappa = -a_1 = 0.1501$) with a much lower long-run mean of 1.8%. Given the average short-term interest rate of 5% - 6% in the sample (see Table 1), this implies that the interest rate is usually rising in Regime 2 and declining in Regime 1. This empirical regularity is consistent with features of the business cycle where expansions are usually characterized by rising interest rates and recessions tend to witness declining interest rates. Moreover, the estimated coefficients on the conditional volatility and the market price of (diffusion) risk all are different across regimes, further suggesting that the yield curve exhibits strong regime-dependence properties.[8] The regime-shift intensity is parameterized as $e^{\eta_s(z)}$ in the model. Table 5 reports that $\eta_s(1,2) = -1.2040$ and $\eta_s(2,1) = -1.4706$. This implies that Regime 2 is more persistent than Regime 1 (namely smaller transition probability from Regime 2 to Regime 1). For the monthly data used in the paper, the estimates of $\eta_s(z)$ suggest that the probability of switching from Regime 1 to Regime 2 is approximately 0.0250 while the probability of switching from Regime 2 to Regime 1 is approximately 0.0191. These are consistent with the results from the previous studies based on discreet time models (e.g. Bansal and Zhou, 2002).

In Model 3, the risk of regime shifts is priced. The estimates of the model parameters (a_0, a_1, σ, θ_x and η_s) are similar to those obtained in Model 2, confirming that the periodic shifts across distinct regimes is an important empirical property of the interest rate dynamics. In Model 3, the implied long-run mean of $r(t)$ becomes 11.1% in Regime 2 and 2.4% in Regime 1 respectively. This again suggests that the interest rate tends

[8] The conditional volatility of the short term interest rate $r(t)$ is given by $\sqrt{\sigma r(t)}$, and the market price of diffusion risk is given by $\theta_x\sqrt{\sigma r(t)}$ in the model.

to increase in Regime 2 and tends to decrease in Regime 1. The estimates of a_1 and σ indicate that $r(t)$ is more persistent with a larger conditional variance in Regime 2 ($\kappa = -a_1 = 0.0916$, $\sigma = 0.0034$) than in Regime 1 ($\kappa = -a_1 = 0.1491$, $\sigma = 0.0025$). As in Model 2, we also find that the market price of the diffusion risk varies across regimes. The estimate of θ_x is -15.54 in Regime 1 and -17.00 in Regime 2 respectively. Moreover the estimates of regime-switching intensities confirm that Regime 2 is more persistent than Regime 1. In particular, the estimates of $\eta_s(z)$ imply that, at monthly frequency, the Markov chain $s(t)$ switches from Regime 1 to Regime 2 with probability 0.0260 and switches from Regime 2 to Regime 1 with probability 0.0169.

In Figure 1, we plot the estimated average yield curve in Regime 1 together with that in Regime 2 by fixing the short-term interest rate at the sample average of 5.6% using Model 3. The differences in the yield curves are obvious. In Regime 2 the yield curve is higher and steeper compared to that in Regime 1 due to the fact that interest rates tend to rise in Regime 2. The average yield curve in Regime 1 not only has a lower level, but also has a different shape. It initially slopes downward and then slopes upward. This is because that the short-term interest rate declines in Regime 1 on average. Since the term premiums are small for bonds of short maturities, the interest rates on these bonds are mainly determined by the expectation of the short-term interest rate in the near future, therefore resulting in a negative slope in the yield curve. However, as maturities of the bonds increase, the term premiums start to play a more important role in determining the interest rates. Moreover, since Regime 1 is not as persistent as Regime 2, the short-term interest rate is also expected to increase over a long horizon as $s(t)$ switches from Regime 1 to Regime 2 in the future. Hence the slope of the yield curve becomes positive as maturities increase.

Figure 1 clearly shows that the yield curve alternates between two distinct regimes as $s(t)$ evolves over time. Whether or not these regime shifts pose a significant risk to investors depends on the estimate of $\theta_s(z)$, the parameter that determines the market price of the regime-switching risk.[9] Table 5 shows that the estimates of $\theta_s(z)$ are highly significant in both regimes. In particular, $\theta_s(z)$ is estimated to be 0.1438 with a standard error of 0.0337 in Regime 1 and -0.0789 with a standard error of 0.0109 in Regime 2 respectively. Moreover there is a big improvement in the goodness-of-fit of the regime-switching model. The χ^2 statistics decreases from 109.70 with 18 degree of freedom in Model 2 to 60.38 with 16 degree of freedom after the regime-switching risk is priced in Model 3. These results suggest that the regime-switching risk is likely to be an important factor that determines bond returns.

4.3.1 Decomposition of the Term Premiums

To assess the quantitative impact of the regime-switching risk, we decompose the term premium on a bond into two parts. One is a term premium due to the diffusion risk and the other component is associated with the risk of regime shifts. In particular, ignoring the Jensen's inequality term, one can obtain that,

$$
\begin{aligned}
&R(t_0,\tau) - \frac{1}{\tau}E_{t_0}\left[\int_{t_0}^{t_0+\tau} r_t dt\right] \approx \\
&\frac{1}{\tau}E_{t_0}\left[\int_{t_0}^{t_0+\tau} \theta_x(s_t)\sigma(s_t)x_t B(t_0+\tau-t, s_t)dt\right] \\
&\frac{1}{\tau}E_{t_0}\left[\int_{t_0}^{t_0+\tau}\int_E \lambda_s(z)\left(e^{\Delta_S A(t_0+\tau-t,s_t)+\Delta_s B(t_0+\tau-t,s_t)x_t} - 1\right)\gamma_m(dt,dz)\right]
\end{aligned} \tag{34}
$$

where $R(t_0,\tau)$ is the interest rate on a default-free bond of maturity τ at time t_0, r_t is the instantaneous short-term interest rate. So the left-hand

[9]The market price of the regime switching risk $\lambda_s(z)$ is parameterized as $1-e^{\theta_s}$ in the model, see equation (28).

side of (34) is the difference between the interest rate on the long-term bond of maturity τ and the average of the expected future short-term interest rate. The Expectation Hypothesis of the term structure maintains that the difference is zero because long-term interest rates are solely determined by the expected future short rate. However, as it is made clear in (34), long-term interest rates can deviate substantially from the levels implied by the Expectation Hypothesis. The first term on the right-hand of (34) is the excess return demanded by investors due to the diffusion risk and the second term is the excess return due to the regime-switching risk, where the market prices of risk are given by $\theta_x(s_t)\sqrt{\sigma(s_t)x_t}$ and $\lambda_s(z)$ respectively. Note that $A(\tau, s)$ and $B(\tau, s)$ are determined in Proposition 4 and $\gamma_m(dt, dz)$ is given in (2). Using the estimated parameters, the average values of these two components of the term premium can be obtained through Monte Carlo simulations.[10] In Figure 2, we report the average total term premiums and the two components for bonds of various maturities ranging from 3 months to 30 years. The lower part of each graph measures the diffusion risk premiums and the upper part of each graph measures the regime-switching risk premiums. For bonds with short maturities (less than 3-year), we can see that not only the term premiums are small, but also most of the term premiums are due to the diffusion risk. As maturities increase, both components of the term premiums start to increase, and the regime-switching risk premiums become a significant part of the total term premiums. The lower panel of Figure 2 indicates that the regime-switching risk can account for more than 10% of the term premiums for bonds with maturities longer than 6 years, and up to 15% for a 30-year bond.

[10]More specifically, we simulate the sample paths of $x(t)$ and $s(t)$ 5000 times given $x(t_0)$ and $s(t_0)$. To get ride of the impact of the initial values of $x(t_0)$ and $s(t_0)$, in each simulation the first one thousand sample points of $x(t)$ and $s(t)$ are ignored. We then take the average over the 5000 sample paths of $x(t)$ and $s(t)$.

4.3.2 How Does the Regime-shift Risk Affect the Yield Curve?

The term structure model developed in Proposition 4 allows us to compute the prices of bonds of different maturities in the presence of the regime-switching risk. In this section we examine the impact of such risk on the yield curve. Specifically, we compare the estimated bond prices in Model 3 with the bond prices obtained using the same model and the same parameter values except holding the market price of the regime-switching risk $\theta_s(z)$ at zero.[11] When computing the bond prices, we set the short-term interest rate at the sample average of the 1-month rate of 5.6%. We report the price differential in both regimes in Figure 3. The corresponding yield curves are plotted in Figure 4. We find that when fixing $\theta_s(z)$ at zero, the prices of long-term bonds become significantly higher. Figure 3 shows that, for example, the price of a 30-year bond would be about 15% higher if $\theta_s(z)$ is set to zero instead of its estimated value of 0.1438 in Regime 1 or -0.0789 in Regime 2. On the other hand, the regime-switching risk has almost no effect on short-term bonds. For bonds with maturities of less than 3 years, we get almost the same price. This implies that the regime-shift risk affects mainly the long end of the yield curve. Ignoring the regime-shift risk would lead to underestimation of long-term interest rate, and therefore flatter yield curves (see Figure 4).

Another way to examine the impact of the regime-shift risk on the yield curve is to compare the bond prices estimated by Model 2 and Model 3 respectively. The two models are the same 1-factor CIR model subject to Markov regime changes except that Model 2 does not price the risk of such regime shifts. To compute the bond prices, we again fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%. We report in Figure 5 the bond price differential in both regimes. The

[11] In Model 3, the estimates of $\theta_s(z)$ are 0.1438 with a standard error of 0.0337 in Regime 1 and -0.0789 with a standard error of 0.0109 in Regime 2 respectively.

corresponding yield curves are plotted in Figure 6. The results are similar to those obtained above. In particular, for bonds of short maturities (less than 3 years), the two term structure models give almost the same result, suggesting that the risk of regime shifts is not important for short-term bonds. However, as bond maturities increase, the price differential becomes significantly bigger. In fact, Model 2 implies much higher prices (up to 40% higher in Regime 1 and 35% higher in Regime 2) than Model 3 does for bonds of long maturities. The corresponding difference in the yield curves can be clearly seen in Figure 6. Without pricing the risk of regime shifts, Model 2 obtains much flatter yield curves in both regimes than Model 3 does.

4.3.3 Are the Regime Shifts a Systematic Risk?

Ignoring the risk of regime shifts, the previous studies have essentially treated the regime shifts as an idiosyncratic risk that can be diversified away. However, these studies have also shown that the regimes are intimately related to the business cycle, suggesting a close link between the regime shifts and aggregate uncertainties. Hence it is more likely that the Markov regime shifts represent a systematic risk which should be priced in the term structure models.

We plot in Figure 7 the implied regimes together with the business cycle expansions and recessions identified by NBER. To find the implied regimes, we follow the suggestion of Bansal and Zhou (2002).[12] Figure 7 confirms the results from the previous studies that the distinct regimes underlying the dynamics of the term structure of interest rates are closely related to the fluctuations of the aggregate economy. Therefore it is important that

[12] Specifically, the estimated term structure model allows us to compute interest rates of different maturities conditional on the regime $\hat{R}(t,\tau|s_t)$. An estimate of s_t can be obtained by choosing the regime that minimizes the average difference between the actually observed interest rate $R(t,\tau)$ and $\hat{R}(t,\tau|s_t)$, i.e. $\hat{s}_t = \arg\min \sum_\tau |R(t,\tau) - \hat{R}(t,\tau|s_t)|$

these periodic shifts of regimes are treated as a systematic risk to investors in the term structure models. It is also interesting to note from Figure 7 that the regime shifts tend to precede business cycle turning points.[13] This is consistent with the empirical finding that the yield curve has significant predictive power for the business cycle, see Estrella and Mishkin (1995) and Chauvet and Potter (2003) among others.

5 Concluding Remarks

Previous studies have provided strong empirical evidence that the joint movements of interest rates of different maturities can be well described by dynamic term structure models that incorporate regime shifts. Moreover these studies also show that there is a close link between the regime shifts and the business cycle fluctuations. Therefore it is very likely that such large periodic shifts of interest rates across distinct regimes present a systematic risk to investors. This paper develops and estimates an affine-type term structure model where such risk is priced. We show that the regime-switching risk introduces a new source of time-variation in bond risk premiums. The model offers an additional econometric flexibility to account for the joint movements of interest rates with different maturities. Moreover, we find that the regime-switching risk is not a negligible factor determining bond prices and has a major impact on the term structure of interest rates. Empirical results suggest that regime shifts can account for a significant portion of the term premiums for long-term bonds. Ignoring the regime-shift risk will lead to underestimation of long-term interest rate and result in flatter yield curves. Such systematic risk of regime shifts is also likely to have important implications for pricing bond derivatives (e.g. Singleton and Umantsev, 2002) as well as for investors' optimal portfolio choice problem

[13]Note that the regime shifts near the end of the sample period precede the most recent recession starting from March 2001 as classified by NBER.

(e.g. Campbell and Viceira, 2001). Another extension is to estimate the term structure model jointly with macroeconomic variables under regime shifts. This would provide more direct evidence regarding the nature of the regime-switching risk. These extensions are left for future research.

A Proof of Proposition 1

Let $J(w(t), s(t), x(t)) = \sup E_t \left[\int_t^\infty e^{-\rho(\tau-t)} U(c(\tau)) d\tau\right]$. We assume that a solution to the consumer's problem exists and indirect utility function $J(w(t), s(t), x(t))$ as well as the optimal consumption and portfolio choice satisfy the Bellman equation (see the result in Section 3.1.6 of Kushner and Dupuis, 2001)

$$\sup_{(\phi_1,\phi_2,c)} \mathcal{D}J(w,s,x) - \rho J(w,s,x) + U(c) = 0 \tag{35}$$

where

$$\mathcal{D}J(w,s,x) = (w\mu_w)J_w + \mu_x J_x + \frac{1}{2}(w\sigma_w)^2 J_{ww} + (w\sigma_w)\sigma_x J_{wx} + \frac{1}{2}\sigma_x^2 J_{xx} + \int_E \Delta_s J \gamma_m(dz)$$

and

$$\Delta_s J = J(w(1+\delta_w(z)), s+\zeta(z), x) - J(w,s,x)$$

.

In the above equations, $J_w = \frac{\partial J}{\partial w}$, $J_x = \frac{\partial J}{\partial x}$, $J_{ww} = \frac{\partial^2 J}{\partial w^2}$, $J_{wx} = \frac{\partial^2 J}{\partial w \partial x}$. μ_w, σ_w and δ_w are given in (12), (13) and (14) respectively. $\gamma_m(dz)$ is defined in (17). μ_x and σ_x are the drift and diffusion terms of the state variable $x(t)$ respectively as defined in (1).

The first order conditions (35) are

$$U'(c) - J_w = 0 \tag{36}$$

$$\begin{aligned} w(\mu_y - r)J_w &+ (w\sigma_w)(w\sigma_y)J_{ww} + (w\sigma_y)\sigma_w J_{wx} \\ &+ \int_E w\delta_y(z) J_w(w(1+\delta_w(z)), s+\zeta(z), x)\gamma_m(dz) = 0 \end{aligned} \tag{37}$$

$$\begin{aligned} w(\mu_F - r)J_w &+ (w\sigma_w)(w\sigma_F)J_{ww} + (w\sigma_F)\sigma_x J_{wx} \\ &+ \int_E w\delta_F(z) J_w(w(1+\delta_w(z)), s+\zeta(z), x)\gamma_m(dz) = 0 \end{aligned} \tag{38}$$

Note that at equilibrium $\phi_1 = 1$ and $\phi_2 = 0$, hence Proposition 1 follows from (37).

B Proof of Proposition 2

From (38) above, we have

$$\mu_F + \int_E \delta_F(z)\gamma_m(dz) - r = \left(-\frac{J_{ww}}{J_w}\right)\frac{(w\sigma_w)(w\sigma_F)}{w} + \left(-\frac{J_{wx}}{J_w}\right)\frac{(w\sigma_F)\sigma_x}{w} + \int_E \left(-\frac{\Delta_s J_w}{J_w}\right)\frac{\Delta_s F}{F}\gamma_m(dz) \tag{39}$$

Apply Ito's formula to $F(t, w, s, x)$, we have[14]

$$dF = \left[F_t + (w\mu_w)F_w + \mu_x F_x + \frac{1}{2}(w\sigma_w)^2 F_{ww} + (w\sigma_w)\sigma_x F_{wx} + \frac{1}{2}\sigma_x^2 F_{xx}\right] dt + \left[(w\sigma_w)F_w + \sigma_x F_x\right] dB(t) + \int_E \Delta_s F m(dt, dz) \tag{40}$$

Compare (40) with (7), we have

$$F\mu_F = F_t + (w\mu_w)F_w + \mu_x F_x + \frac{1}{2}(w\sigma_w)^2 F_{ww} + (w\sigma_w)\sigma_x F_{wx} + \frac{1}{2}\sigma_x^2 F_{xx} \tag{41}$$

$$F\sigma_F = (w\sigma_w)F_w + \sigma_x F_x \tag{42}$$

and

$$\delta_F(z) = \frac{\Delta_s F}{F} \tag{43}$$

Proposition 2 follows after substituting (42) and (43) into (39) and defining $\mu_F^* = \mu_F + \int_E \delta_F(z)\gamma_m(dz)$.

[14]See Protter (1990) for the generalized Ito's formula for semi-martingales.

C Proof of Proposition 3

Proposition 2 implies that

$$\begin{aligned}
&\mu_F + \int_E \delta_F(z)\gamma_m(dz) - r \\
&= \left[\left(-\frac{J_{ww}}{J_w}\right) Var(w) + \left(-\frac{J_{wx}}{J_w}\right) Cov(w,x)\right] \frac{F_w}{F} \\
&+ \left[\left(-\frac{J_{ww}}{J_w}\right) Cov(w,x) + \left(-\frac{J_{wx}}{J_w}\right) Var(x)\right] \frac{F_x}{F} \\
&+ \int_E \left(-\frac{\Delta_s J_w}{J_w}\right) \frac{\Delta_s F}{F}\gamma_m(dz)
\end{aligned} \tag{44}$$

Using (41), we have

$$\begin{aligned}
&F_t + \frac{1}{2}(w\sigma_w)^2 F_{ww} + (w\sigma_w)\sigma_x F_{wx} + \frac{1}{2}\sigma_x^2 F_{xx} \\
&+ \left[\mu_x - \left(-\frac{J_{ww}}{J_w}\right)(w\sigma_w)\sigma_x - \left(-\frac{J_{wx}}{J_w}\right)\sigma_x^2\right] F_x \\
&+ \left[(w\mu_w) - \left(-\frac{J_{ww}}{J_w}\right)(w\sigma_w)^2 - \left(-\frac{J_{wx}}{J_w}\right)(w\sigma_w)\sigma_x\right] F_w \\
&+ \int_E \left(1 + \frac{\Delta_s J_w}{J_w}\right) \Delta_s F \gamma_m(dz) = rF
\end{aligned} \tag{45}$$

Note again that at equilibrium, $\phi_1 = 1$ and $\phi_2 = 0$. Hence (12), (13) and (14) imply that

$$w\mu_w = w\mu_y - c \tag{46}$$

$$w\sigma_w = w\sigma_y \tag{47}$$

$$w\delta_w(z) = w\delta_y(z) \tag{48}$$

Moreover, Proposition 1 implies

$$
\begin{aligned}
wr = w\mu_y - \left(-\frac{J_{ww}}{J_w}\right)(w\sigma_y)^2 - \left(-\frac{J_{wx}}{J_x}\right)(w\sigma_y)\sigma_x \\
+ \int_E \left(1 + \frac{\Delta_s J_w}{J_w}\right) w\delta_y(z)\gamma_m(dz)
\end{aligned}
\tag{49}
$$

Combining (45) – (49), We have the following fundamental partial differential equations of asset pricing as in CIR

$$
\begin{aligned}
F_t + \frac{1}{2}(w\sigma_y)^2 F_{ww} + (w\sigma_y)\sigma_x F_{wx} + \frac{1}{2}\sigma_x^2 F_{xx} \\
+ \left[\mu_x - \left(-\frac{J_{ww}}{J_w}\right)(w\sigma_y)\sigma_x - \left(-\frac{J_{wx}}{J_w}\right)\sigma_x^2\right] F_x \\
+ \left[wr - c - \int_E \left(1 + \frac{\Delta_s J_w}{J_w}\right) w\delta_y(z)\gamma_m(dz)\right] F_w \\
+ \int_E \left(1 + \frac{\Delta_s J_w}{J_w}\right) \Delta_s F \gamma_m(dz) = rF
\end{aligned}
\tag{50}
$$

Under logarithm utility function $U(c(t)) = \log c(t)$, it is well known that the indirect utility function is separable in $w(t)$ and $x(t)$ and $s(t)$, i.e. $J(w,s,x)$ can be written as $\frac{1}{\rho}\log w + f(s,x)$ where $f(s,x)$ solve the system of differential equation after substituting $J(w,s,x)$ and the optimal choice of consumption (c^*) and portfolio (ϕ_1^*, ϕ_2^*) into the Bellman equation (35). This separability implies that $J_{wx} = 0$.

Moreover, for default-free discount bonds , $F_w = 0$, $F_{ww} = 0$ and $F_{wx} = 0$. Therefore equation (50) can simplified as

$$
F_t + \frac{1}{2}\sigma_x^2 F_{xx} + (\mu_x - \sigma_y\sigma_x)F_x + \int_E \left(1 + \frac{\Delta J_w}{J_w}\right) \Delta_s F \gamma_m(dz) = rF \tag{51}
$$

Using the fact that $J_w = \frac{1}{\rho w}$ and (48), it can be easily shown

$$
1 + \frac{\Delta_s J_w}{J_w} = 1 - \lambda_s(z) \tag{52}
$$

where $\lambda_s(z) = \frac{\delta_y(z)}{1+\delta_y(z)}$.

Proposition 3 can be obtained by substituting the above equation into (51). Note that (51) defines a system of partial differential equations.

D Proof of Proposition 4

First note that under assumptions (26) and (27), (22) implies that $r(t) = x(t)$.

Without loss of generality, let the price at time t of a pure-discount bond that will mature at T be given as

$$F(t, s(t), x(t), T) = e^{A(\tau, s(t)) + B(\tau, s(t)) r(t)}$$

where $\tau = T - t$ and $A(0, s) = 0$, $B(0, s) = 0$.

Proposition 3 then implies

$$\begin{aligned} r = & -\frac{\partial A(\tau, s)}{\partial \tau} - \frac{\partial B(\tau, s)}{\partial \tau} r \\ & + [a_0(s) + (a_1(s) - \theta_x(s)\sigma(s))r]B(\tau, s) + \frac{1}{2}[\sigma(s)r]B^2(\tau, s) \\ & + \int_E \left(e^{\Delta_s A + \Delta_s B r} - 1\right) e^{\eta_s(z) + \theta_s(z)} \mathbf{1}(s = i)\epsilon_z(dz) \end{aligned} \tag{53}$$

where $\Delta_s A = A(\tau, s + \zeta(z)) - A(\tau, s)$ and $\Delta_s B = B(\tau, s + \zeta(z)) - B(\tau, s)$

Using the log-linear approximation

$$e^{\Delta_s B r} \approx 1 + \Delta_s B r$$

Proposition 4 follows by substituting the above equation into (53) and match the coefficients on r on both side of the equation.

References

[1] Ahn, C. M. and H. E. Thompson (1988), "Jump-Diffusion Processes and the Term Structure of Interest Rates", *The Journal of Finance* 43, 155-174.

[2] Ang, A. and G. Bekaert (1998), "Regime Switches in Interest Rates", *NBER working paper* 6508.

[3] Bansal, R. and H. Zhou (2002), "Term Structure of Interest Rates with Regime Shifts", forthcoming, *Journal of Finance.*

[4] Bansal, R., R. Gallant and G. Tauchen (1995), "Nonparametric Estimation of Structural Models for High-frequency Currency Market Data", *Journal of Econometrics* 66, 251-287.

[5] Bielecki, T. and M. Rutkowski (2001), "Modeling of the Defaultable Term Structure: Conditional Markov Approach", Working paper, The Northeastern Illinois University.

[6] Boudoukh, Jacob et al. (1999), "Regime Shifts and Bond Returns", Working paper, New York University.

[7] Campbell, J. Y. and L. M. Viceira (2001), "Who Should Buy Long-Term Bonds", *The American Economic Review* 91, 99-127.

[8] Chauvet, M. and S. Potter (2003), "Forecasting Recessions Using the Yield Curve", *Journal of Forecasting*, forthcoming.

[9] Cox, J., J. Ingersoll and S. Ross (1985a), "An Interemporal General Equilibrium Model of Asset Prices", *Econometrica* 53, 363-384.

[10] Cox, J., J. Ingersoll and S. Ross (1985b), "A Theory of the Term Structure of Interest Rates", *Econometrica* 53, 385-407.

[11] Dai Q¿ and K. Singleton (2000), "Specification Analysis of Affine Term Structure Models", *Journal of Finance* LV, 1943-1978.

[12] Dai, Q. and K. Singleton (2003), "Term Structure Dynamics in Theory and Reality", *The Review of Financial Studies* 16, 631-678

[13] Diebold, F. and G. Rudebusch (1996), "Measuring Business Cycles: A Modern Perspective", *Review of Economics and Statistics 78*, 67-77.

[14] Duffie, D. and R. Kan (1996), “A Yield-Factor Model of Interest Rates”, *Mathematical Finance 6*, 379-406.

[15] Duffie, D., J, Pan and K. Singleton (2000), “Transform Analysis and Asset Pricing for Affine Jump-Diffusions”, *Econometrica 68*, 1343-1376.

[16] Elliott, R. J. and R. S. Mamon, “Term structure of a Vasicek model with a Markovian mean reverting level”, Working paper, University of Calgary.

[17] Estrella, Arturo and Federic S. Mishkin (1995), “The Term Structure of Interest Rates and Its Role in Monetary Policy for the European Central Bank”, *NBER working paper*, No.5279.

[18] Evans, M. (2001), “Real Risk, Inflation Risk, and the Term Structure”, Working paper, Georgetown University.

[19] Filardo, A.J. (1994), “Business Cycle Phases and Their Transitional Dynamics”, *Journal of Business and Economic Statistics* 12, 299-308.

[20] Garcia, R. and P. Perron (1996), “An Analysis of the Real Interest Rate Under regime Shifts”, *Review of Economics and Statistics 78*, 111-125.

[21] Gallant, R. and G. Tauchen (1996), “Which moment to match?”, *Econometric Theory* 12, 657-681.

[22] Gallant, R. and G. Tauchen (1998), “Reprojecting Partially Observed System with Application to Interest Rate Diffusions”, *Journal of the American Statistical Association* 93, 10-24.

[23] Gallant, R. and G. Tauchen (2001), “Efficient Method of Moments”, manscript.

[24] Gray, S. (1996), “Modeling the Conditional Distribution of Interest Rates as a Regime-Switching Process”, *Journal of Financial Economics 42*, 27-62.

[25] Hamilton, J. (1988), “Rational Expectations Econometric Analysis of Changes in Regimes: An Investigation of the Term Structure of Interest Rates”, *Journal of Economic Dynamics and Control 12*, 385-423.

[26] Hamilton, J. (1989), “A New Approach to the Economic Analysis of Nonstationary Time Series and the Busines Cycle”, *Econometrica 57*, 357-384.

[27] Kushner, H. J. and P. Dupuis (2001), "Numerical Methods for Stochastic Control Problems in Continuous Time", 2nd edition, Springer, New York.

[28] Landen, C. (2000), "Bond Pricing in a Hidden Markov Modle of the Short Rate", *Finance and Stochastics 4*, 371-389.

[29] Last, G. and A. Brandt (1995), "Marked Point Processes on the Real Line", New York: Springer.

[30] Naik, V. and M. H. Lee (1997), "Yield Curve Dynamics with Discrete Shifts in Economic Regimes: Theory and Estimation", Working paper, University of British Columbia.

[31] Protter, P. (1990), "Stochastic Intergration and Differential Equations", Berling: Springer Verlag.

[32] Sigleton, K. J. and L. Umantsev (2002), "Pricing Coupon-Bond Options and Swaptions in Affine Term Structure Models", *Mathematical Finance* 12, 427-446.

Table 1 Summary Statistics of the term structure of interest rates 1964 - 2000 (the first column indicates the maturities of the interest rates)

	mean	std	corr	skew	kurt	max	min
1-month	0.05608	0.025452	1	1.28697	5.17923	0.161376	0.013176
3-month	0.060275	0.026706	0.985712	1.26099	4.91011	0.160332	0.017112
6-month	0.062443	0.026954	0.978383	1.21604	4.69365	0.165168	0.017808
1-year	0.06455	0.026314	0.964593	1.07663	4.21845	0.15812	0.01956
2-year	0.066561	0.025774	0.938704	1.01143	3.93613	0.15639	0.02368
3-year	0.068159	0.025069	0.916046	0.998881	3.84814	0.15557	0.0296
4-year	0.069354	0.024795	0.897531	0.965216	3.74017	0.15824	0.03118
5-year	0.069981	0.024589	0.883406	0.921516	3.57469	0.15001	0.03346

Note: mean = average interest rate; std = standard deviation; corr = correlation coefficient between long rates and the 1-month interest rate; skew = skewness; kurt = kurtosis; max = maximum interest rate; min=minimum interest rate

Table 2: Using forward rates to predict excess bond returns.

	3-month	6-month	1-year	2-year	3-year	4-year	5-year
constant	0.0030	0.0116	0.0188	0.0224	0.0236	0.0187	0.0161
	(0.0022)	(0.0054)	(0.0124)	(0.0227)	(0.0321)	(0.0321)	(0.0455)
Bus-cycle	-0.0074	-0.0167	-0.0313	-0.0495	-0.0619	-0.0694	-0.0773
	(0.0021)	(0.0050)	(0.0111)	(0.0199)	(0.0272)	(0.0272)	(0.0371)
f1	-0.2145	-0.5015	-0.1386	-0.8860	-1.4536	-1.9372	-2.2324
	(0.0979)	(0.2362)	(0.5121)	(0.8805)	(1.1746)	(1.1746)	(1.7405)
f6	0.1363	0.3895	-0.2358	-0.0988	-0.0620	-0.3291	-0.6753
	(0.1039)	(0.1963)	(0.4121)	(0.7022)	(0.9602)	(0.9602)	(1.4695)
f60	0.1935	0.2128	0.5717	1.2403	1.8126	2.6277	3.3306
	(0.0694)	(0.1458)	(0.3320)	(0.6012)	(0.8451)	(0.8451)	(1.2642)

Note: The first row indicates maturities of the bonds. The first column includes the explanatory variables in the regression. Bus-cycle is a business cycle dummy variable according to NBER business dates. Bus-cycle=1: expansion; Bus-cycle=0: recession. f1 = 1-month rate , f6 = 6-month forward rate, f60 = 5-year forward rate. Numbers in parentheses are Newy-West standard errors.

Table 3: SNP specifications

L_μ	L_r	L_p	K_z	I_z	K_x	I_x	l_θ	$s_n(\hat{\theta})$	BIC
1	0	1	0	0	0	0	9	-.82698	-.76277
2	0	1	0	0	0	0	13	-.85153	-.75897
3	0	1	0	0	0	0	17	-.85784	-.73656
1	1	1	0	0	0	0	11	-1.02082	-.94234
1	2	1	0	0	0	0	13	-1.05923	-.96648
1	3	1	0	0	0	0	15	-1.10226	-.99525
1	4	1	0	0	0	0	17	-1.15356	-1.03228
1	5	1	0	0	0	0	19	-1.17153	-1.03598
1	6	1	0	0	0	0	21	-1.17394	-1.02412
1	7	1	0	0	0	0	23	-1.18247	-1.01839
1	5	1	2	1	0	0	23	-1.18124	-1.01716
1	5	1	2	0	0	0	24	-1.22077	-1.04956
1	5	1	3	2	0	0	25	-1.18248	-1.00413
1	5	1	3	1	0	0	26	-1.22941	-1.04393
1	5	1	3	0	0	0	28	-1.23006	-1.03030
1	5	1	4	3	0	0	27	-1.22163	-1.02901
1	**5**	**1**	**4**	**2**	**0**	**0**	**28**	**-1.25705**	**-1.05729**
1	5	1	4	1	0	0	30	-1.26133	-1.04703
1	5	1	4	0	0	0	33	-1.28814	-1.05272
1	5	1	5	4	0	0	29	-1.22625	-1.01936
1	5	1	5	3	0	0	30	-1.25837	-1.04435
1	5	1	5	2	0	0	32	-1.26270	-1.03441
1	5	1	5	1	0	0	35	-1.29353	-1.04383
1	5	1	5	0	0	0	39	-1.29058	-1.01235
1	5	1	4	2	1	0	48	-1.32073	-.97829
1	5	1	4	2	2	1	68	-1.36002	-.87490
1	5	1	4	2	2	0	78	-1.37679	-.82033

Note: L_μ is the number of lags in VAR conditional mean. L_r is number of lags in ARCH conditional standard deviation. K_z is the degree of the square Hermite polynomial that captures the deviation of the standardized innovation z from conditional Gaussian distribution. The interaction polynomial term above the I_z degree is suppressed as zero. The degree of x-polynomial K_x is fixed at 0, and by convention L_p is set to be 1. l_θ is the number of coefficients in the SNP model. $s_n(\hat{\theta})$ is the negative sample mean log-likelihood. BIC is the Bayesian Information Criterion. According to BIC, the preferred SNP specification is 10514200.

Table 4: Parameter Estimates of SNP density

Parameter	Estimates	Standard Error
a(0,0)	1.00000	0.00000
a(0,1)	-0.05179	0.06782
a(1,0)	0.07111	0.06689
a(0,2)	-0.10675	0.05547
a(1,1)	0.22368	0.06511
a(2,0)	-0.23111	0.08141
a(0,3)	0.01787	0.01269
a(3,0)	-0.01448	0.01384
a(0,4)	0.01470	0.00595
a(4,0)	0.02222	0.00684
μ(2,0)	-0.00079	0.00704
μ(1,0)	-0.00531	0.01195
μ(2,2)	0.98033	0.01447
μ(2,1)	0.00659	0.01593
μ(1,2)	0.01049	0.01401
μ(1,1)	0.97699	0.01507
R(1,0)	0.04068	0.00901
R(2,0)	0.04830	0.00820
R(3)	0.08870	0.01251
R(1,1)	0.33931	0.07707
R(2,1)	0.18582	0.05988
R(1,2)	0.22569	0.07642
R(2,2)	0.11341	0.05650
R(1,3)	0.16133	0.06670
R(2,3)	0.25376	0.07700
R(1,4)	0.12867	0.05434
R(2,4)	0.03708	0.05448
R(1,5)	0.09542	0.05642
R(2,5)	0.00238	0.06357

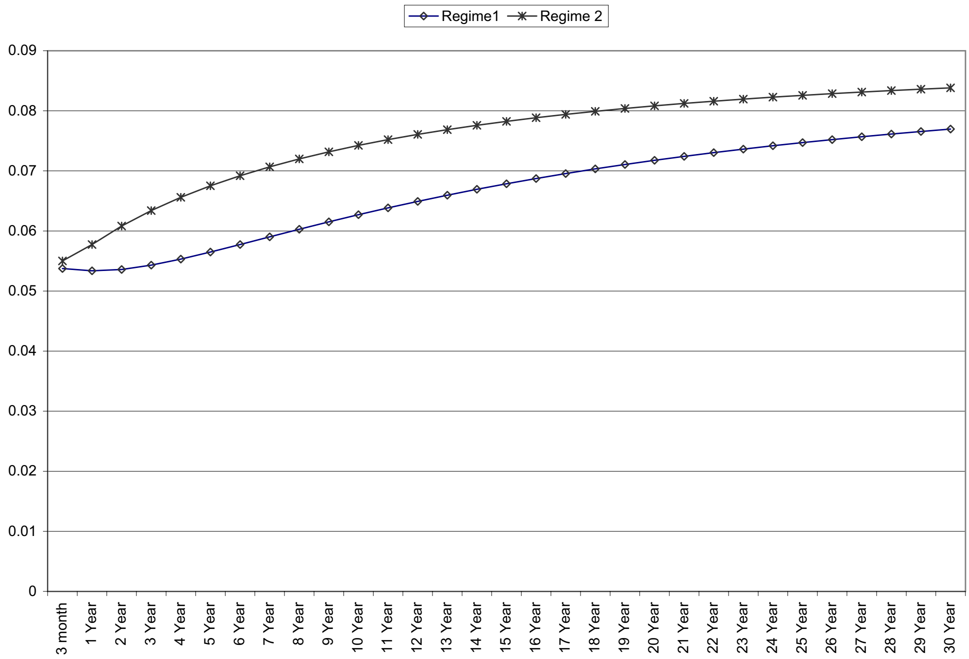
This table reports point estimates as well as their standard errors of the parameters in the SNP model (10514200). $a(i,j)$ are parameters of the Hermit polynomial function. $\mu(i,j)$ are parameters of the VAR conditional mean. $R(i,j)$ are parameters of the ARCH standard deviation of the innovation z. See Gallant and Tauchen (2001) or Bansal and Zhou (2002) for more detailed interpretations of these parameters.

Table 5: Parameter Estimates of the Term Structure Models

		Model 1	Model 2	Model 3
Regime 1	a_0	0.0058	0.0027	0.0036
		(0.0007)	(0.0002)	(0.0003)
	a_1	-0.0907	-0.1501	-0.1488
		(0.0096)	(0.0036)	(0.0006)
	σ	0.0025	0.0024	0.0025
		(0.0002)	(2.6e-5)	(0.0003)
	θ_x	-10.7774	-15.7689	-15.5444
		(1.7709)	(0.0070)	(0.0053)
	θ_s			0.1438
				(0.0337)
Regime 2	a_0		0.0086	0.0102
			(0.0004)	(0.0004)
	a_1		-0.0735	-0.0916
			(0.0022)	(0.0005)
	σ		0.0030	0.0034
			(1.4e-5)	(1.4e-5)
	θ_x		-17.6468	-16.9962
			(0.4841)	(0.0975)
	θ_s			-0.0789
				(0.0109)
Transition	$\eta_s(1,2)$		-1.2040	-1.1655
Intensity			(0.0097)	(0.0263)
	$\eta_s(2,1)$		-1.4706	-1.4457
			(0.0505)	(0.0130)
	χ^2	213.34	109.70	60.38
	Z-value	27.33	15.28	7.85
	d.o.f.	24	18	16

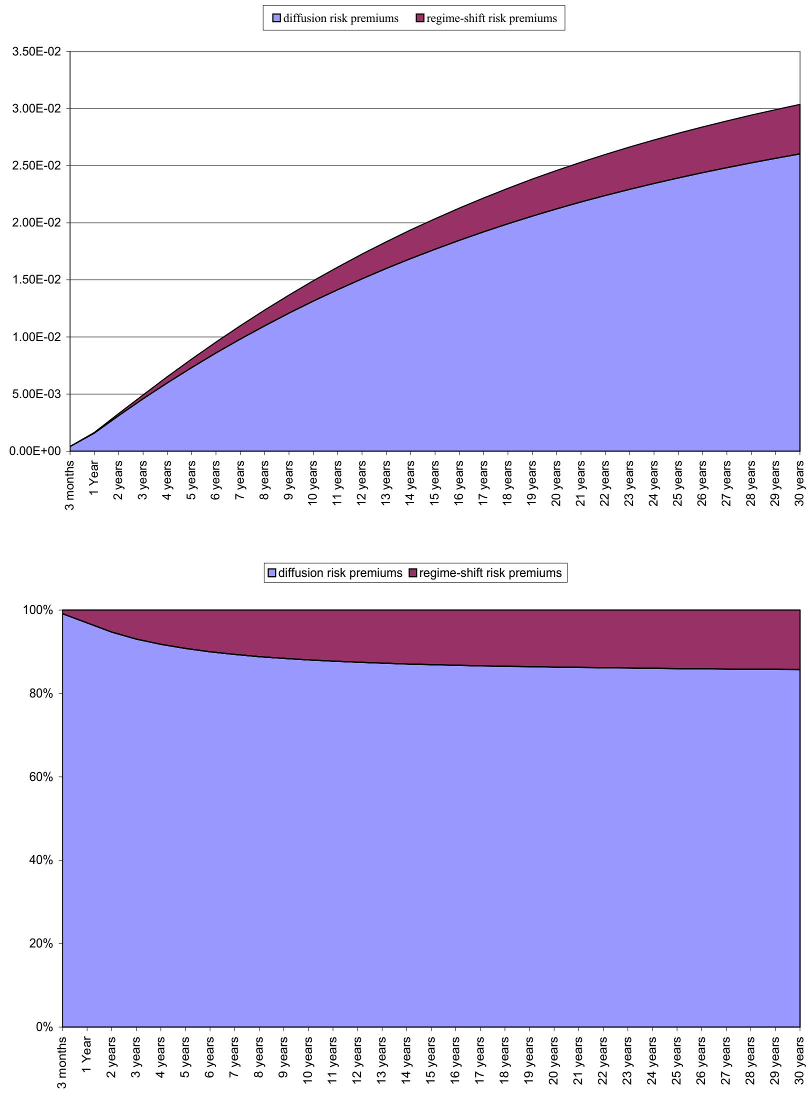
This table reports EMM estimates of the term structure models. Model 1 refers to the standard 1-factor CIR without regime shifts. Model 2 refers to the 1-factor CIR model with regime shifts, but the risk of regime shifts is not priced. Model 3 is the 1-factor CIR model developed in Proposition 4 where the risk of regime shifts is priced. a_0, a_1 and σ_0 are the coefficients in the diffusion process of $x(t)$: $dx = (a_0 + a_1 x)dt + \sqrt{\sigma x}dw$. θ_x is coefficient on the market price of diffusion risk, which is given by $\theta_x\sqrt{\sigma x}$. θ_s is the coefficient that determines the market price of regime-switching risk $\lambda_s(z)$, which is parameterized as $1 - e^{\theta_s(z)}$ in the model. The transition intensity of the Markov chain is given by $e^{\eta_s(z)}$. Numbers in parentheses are the standard errors. The table also reports the χ^2 statistics and its degree of freedom (dof) from the EMM estimation.

Figure 1: Different Regimes of the Yield Curves



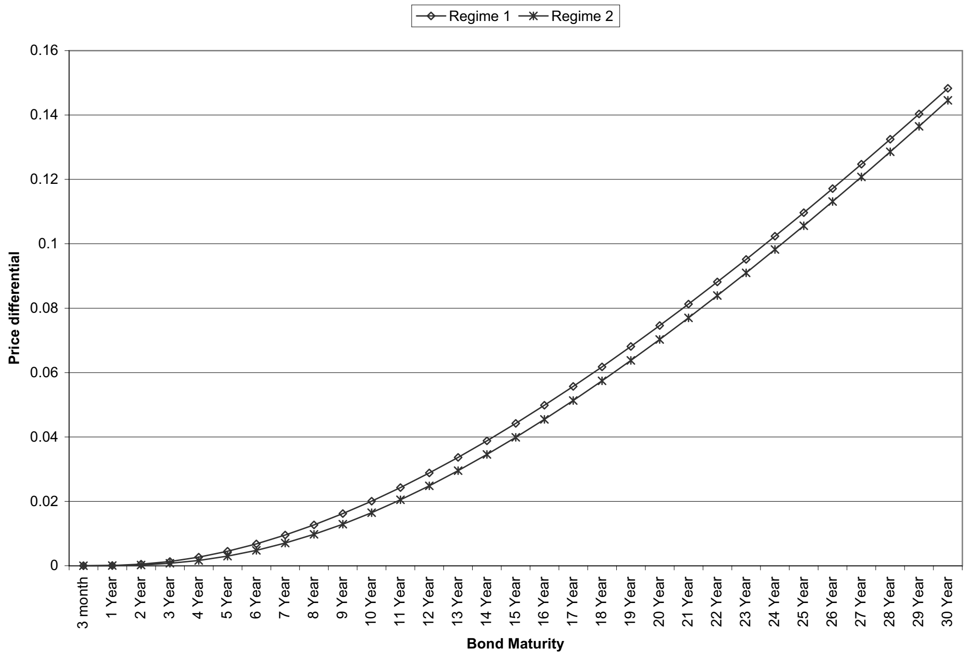
The figure plots the estimated average yield curves (Model 3) in Regime 1 ($s(t) = 1$ and Regime 2 ($s(2) = 2$)). When computing the yield curves, we fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%.

Figure 2: Decompositions of the Term Premiums



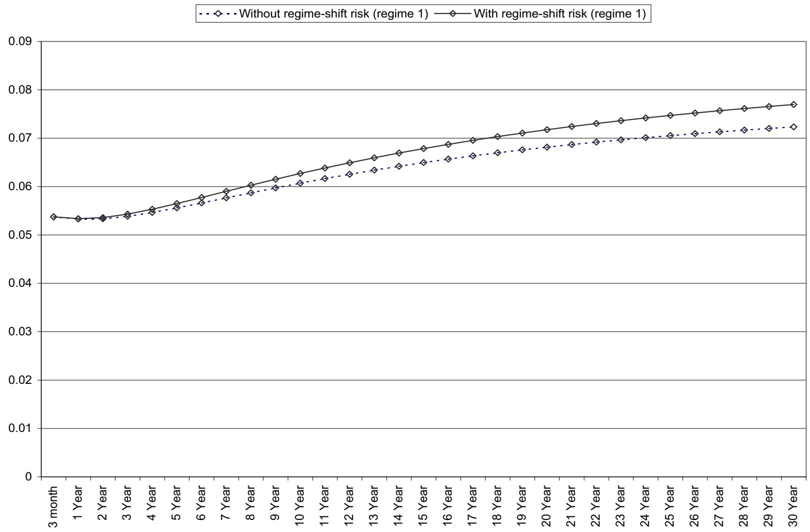
The above two figures report the decompositions of the term premiums for bonds of various maturities (from 3-month to 30-year). The upper panel plots the bond term premiums as the sum of two components due to the diffusion risk and the regime-shift risk respectively. The lower panel plots these two risk components as percentages of the total term premiums.

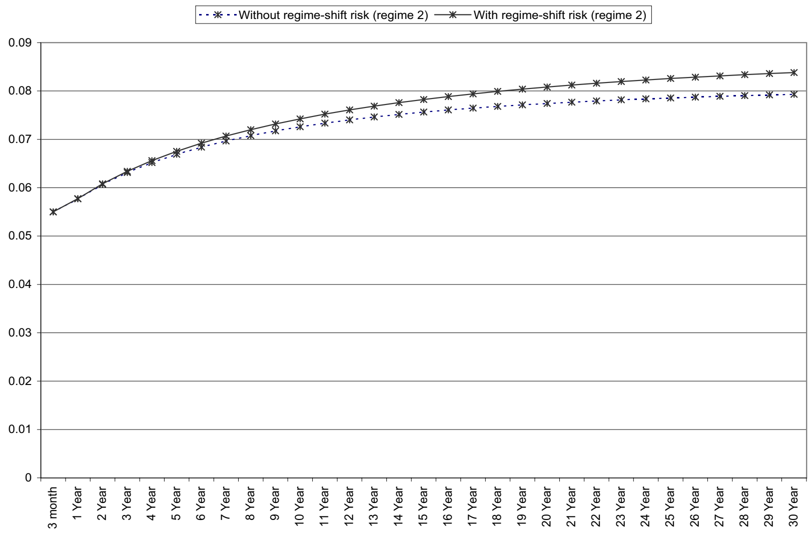
Figure 3: The Impact of Regime-shift Risk on Bond Prices



The figure reports the impact of regime-shift risk on bond prices. The price differential is obtained as $\frac{P^*(\tau)-P(\tau)}{P(\tau)}$, where $P(\tau)$ is the estimated price of the bond of maturity τ in Model 3. $P^*(\tau)$ is the price of the same bond obtained using the same model and the same parameter values except holding the market price of regime-shift risk at zero ($\theta_s(z) = 0$). In calculating the bond prices, we fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%. The line with diamonds is the bond price differential in regime 1. The line with crosses is the bond price differential in regime 2.

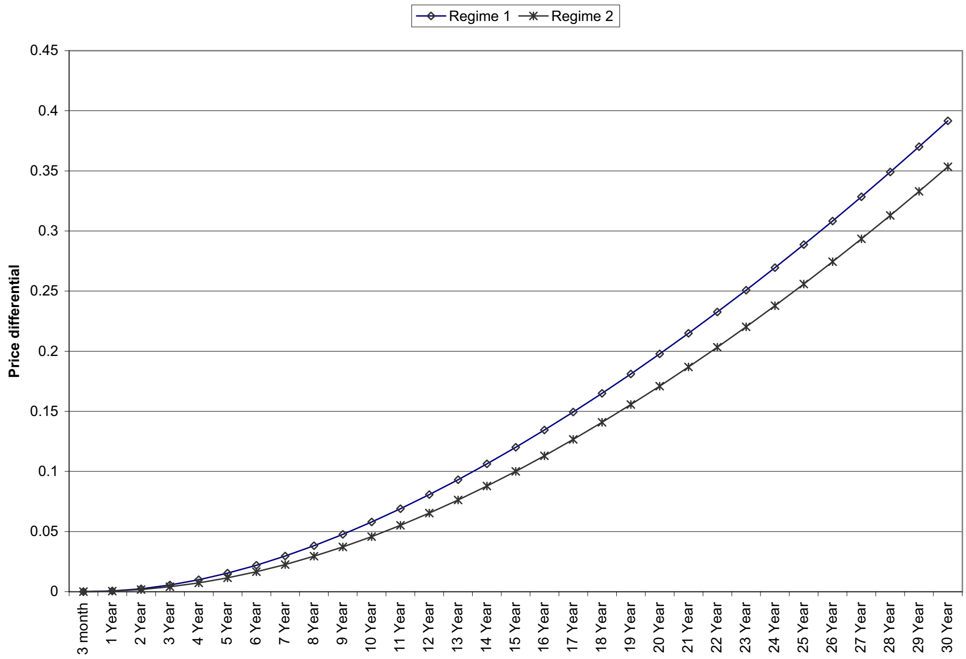
Figure 4: The Impact of Regime-shift Risk on the Yield Curve





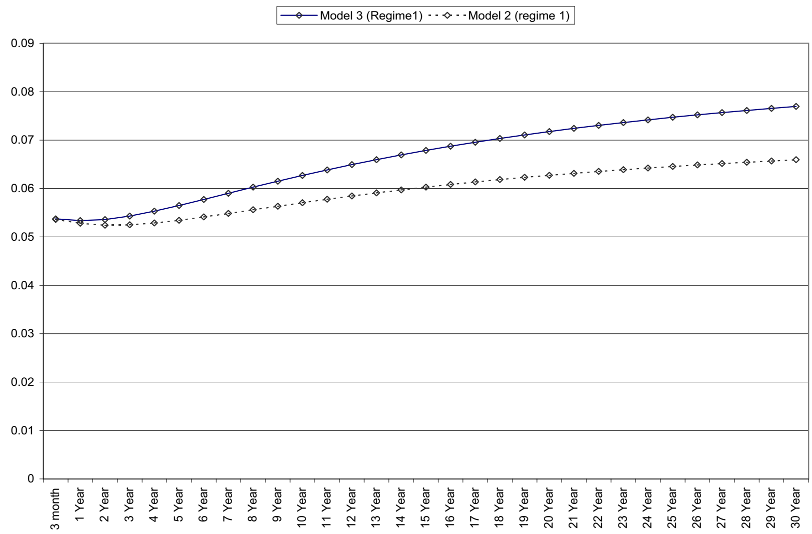
These two figures illustrate the impact of regime-shift risk on the yield curve in Regime 1 (the upper panel) and Regime 2 (the lower panel) respectively. The solid line is the estimated yield curve from Model 3. The dashed line is the yield curve obtained using the same model and the same parameter values except holding the market price of regime-shift risk at zero ($\theta_s(z) = 0$). In calculating the yields, we fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%.

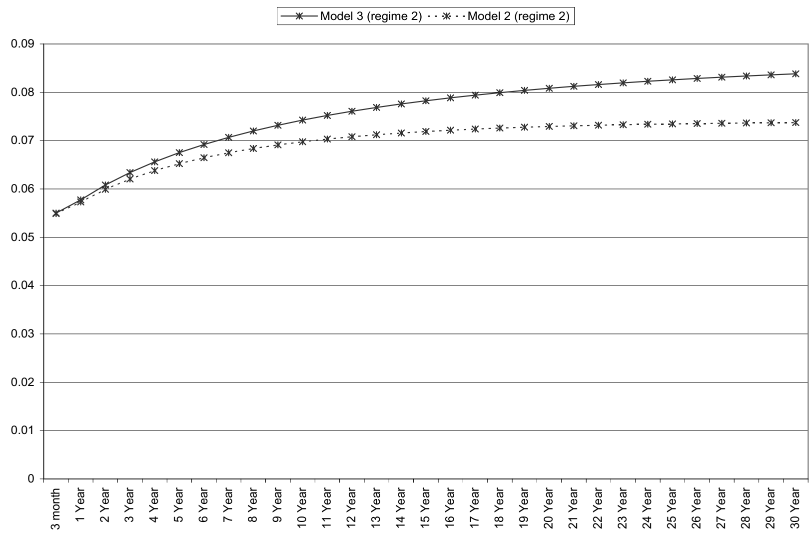
Figure 5: Bond Price Differentials



The price differential is obtained as $\frac{P(Model2)-P(Model3)}{P(Model3)}$, where $P(Model3)$ is the estimated bond price using Model 3. $P(Model2)$ is the estimated price of the same bond using Model 2. The difference between the two models is that in Model 2 the risk of regime shifts is not priced. In calculating the bond prices, we fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%. The line with diamonds is the bond price differential in regime 1. The line with crosses is the bond price differential in regime 2.

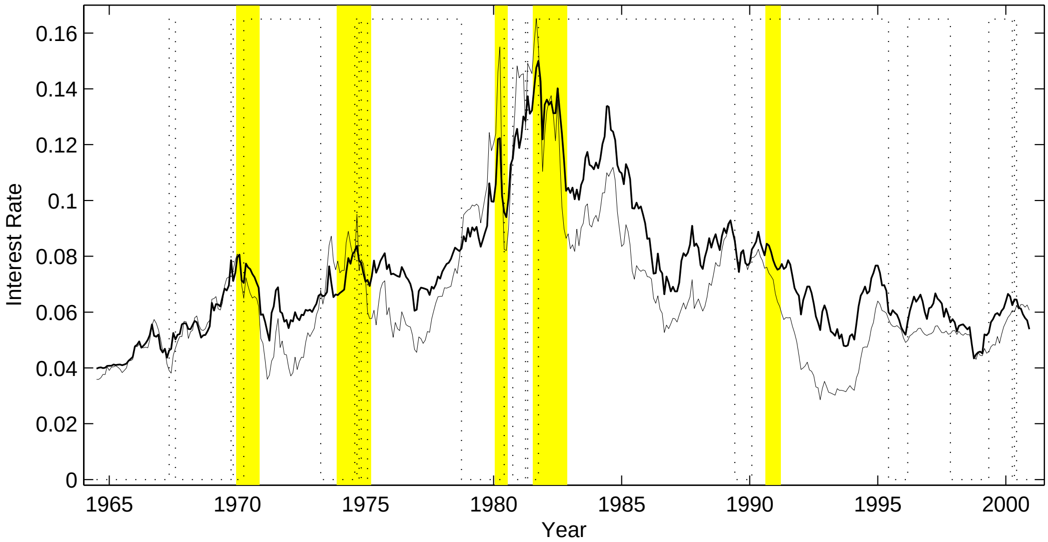
Figure 6: Differences in the Yield Curves





These two figures report the average yield curves estimated by Model 2 and Model 3 respectively in Regime 1 (the upper panel) and in Regime 2 (the lower panel). The difference between the two models is that Model 2 doesn't price the risk of regime shifts. The solid line is the estimated yield curve from Model 3. The dashed line is the estimated yield curve from Model 2. When calculating the yields, we fix the short-term interest rate $r(t)$ at the sample average of the 1-month rate of 5.6%.

Figure 7: Business cycle and the interest rate regimes



The figure plots the implied regimes by the term structure model (Model 3 in table 5) together with the business cycle. NBER business cycle recessions are indicated the shaded area. The dashed line indicates the regimes implied by the interest rates. We also plot the 6-month (the thinner solid line) and 5-year interest rate (the thicker solid line) in the graph.