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Biung-Ghi Ju
University of Kansas, Department of Economics

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Strategy-Proof Risk Sharing

Biung-Ghi Ju*

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Abstract

We consider risk sharing problems with a single good and finite number of states. Agents have a common prior and their preferences are represented in the expected utility form and are risk averse. We study *efficient* and *individually rational* risk sharing rules satisfying *strategy-proofness*, the requirement that no one can ever be benefited by misrepresenting his preference. When aggregate certainty holds, we show that "fixed price selections" from the Walrasian correspondence are the *only* rules satisfying *efficiency*, *individual rationality*, and *strategy-proofness*. However, when aggregate uncertainty holds, we show that there exists no rule satisfying the three requirements. Moreover, in the two agents case, we show that dictatorial rules are the only *efficient* and *strategy-proof* rules. Dropping the common prior assumption in the model, we show that this assumption is necessary and sufficient for the existence of rules satisfying the three main requirements in the two agents and aggregate certainty case.

Keywords: Risk sharing; strategy-proofness; efficiency; individual rationality.

*Department of Economics, University of Kansas, 1300 Sunnyside Avenue, Lawrence, KS 66045, USA. E-mail: `bgju@ku.edu`. Tel.:(785)864-2860. Fax.:(785)864-5270. This research was initiated by the suggestion of Michel Le Breton. I appreciate him for his encouragement and nice discussions. I thank William Thomson for many helpful comments, suggestions, and discussions. I also thank Larry Epstein for his comments on the first draft of this paper. I thank Matthew Jackson, Bettina Klaus, and seminar participants in Rice University, State University of New York at Stony Brook, University of Kansas, University of Nebraska at Lincoln, University of Rochester, the Midwest Theory Meeting at Northwestern University, the Conference of the Society for Economic Design at NYU, and the Sixth International Meeting of the Society for Social Choice and Welfare at Caltech. All remaining errors are mine.

1 Introduction

Economic transactions are often made prior to the realization of uncertain states. Agents typically face different types of risks in their endowments and have different attitudes toward risk. Thus, by transacting state-wise, they may share individual risks for their common benefit. In particular, when there is a single good, or money, such state-wise transactions are made purely for the purpose of risk sharing. We consider such simple risk sharing problems.

More precisely, there is a finite number S of states. Agents have individual endowments, or state-contingent commodity vectors in $\mathbb{R}^S_+$, and preferences over $\mathbb{R}^S_+$. An *economy* is characterized by a profile of preferences and individual endowments. An allocation designates how much income each agent receives at each state. A risk sharing rule, or simply, a *rule*, associates with each economy a *single* feasible allocation. We study rules satisfying the following requirements. *Efficiency* requires that the choice by a rule should always be such that no one can be made better off without anyone else being made worse off. *Individual rationality* requires that everyone should be at least as well off as in his individual endowment. *Strategy-proofness* (Gibbard, 1973, Satterthwaite, 1975) requires that no one can ever be benefited by misrepresenting his preference, independently of others' representations.

Without any additional restriction on preferences other than "monotonicity", "continuity", and "convexity", risk sharing problems are identical to allocation problems in exchange economies. Thus we can apply the earlier studies by Hurwicz (1972), Dasgupta et al. (1979), Hurwicz and Walker (1990), Zhou (1991), Barberà and Jackson (1995), Schummer (1997), Serizawa (2000), Serizawa and Weymark (2002) among others. There is no rule satisfying *efficiency, individual rationality,* and *strategy-proofness* as shown by Hurwicz (1972) and Serizawa (1998).[1] Moreover, in the two agents case, there is no *efficient* and *strategy-proof* rules satisfying the minimal equity criteria, "non-dictatorship" as shown by Dasgupta et al. (1979), Zhou (1991), and Schummer (1997). These impossibility results rely crucially on some "richness conditions" of the family of admissible preferences in exchange economies.[2]

In Decision Theory, various systems of behavioral axioms are introduced for preferences over state-contingent outcomes, or "acts". These systems characterize

[1] An even stronger impossibility result is established by Serizawa and Weymark (2002), replacing *individual rationality* with a much weaker axiom called "minimum consumption guarantee".

[2] Dasgupta et al. (1979), Zhou (1991), and Schummer (1997) rely on successively weaker form of "richness" condition.

interesting restricted families of preferences; in particular, the "expected utility preferences" by von Neumann and Morgenstern (1944) and Savage (1954) and the "maximin expected utility preferences" by Gilboa and Schmeidler (1989), etc. Over these restricted domains, we may hope for some possibility results.

In this paper, we consider expected utility preferences. We further assume that all agents have the same belief over states, or the "common prior" and they are "risk averse". We refer readers to Aumann (1976), Bacharach (1985), and McKelvey and Page (1986) for the motivation of the common prior assumption. These studies support the emergence of consensus in agents' beliefs in the environment with possible communications among agents and public devices for transforming private information structures.

Our results depend crucially on the nature of aggregate risk. *Aggregate certainty* holds when the aggregate income is constant across states. Otherwise *aggregate uncertainty* holds. We show that when aggregate certainty holds, there always exist Walrasian (equilibrium) allocations supported by the price equal to the common prior (the definition of Walrasian equilibrium is the same as in standard exchange economies). Hence "fixed price selections" from the Walrasian correspondence are well-defined when the price is equal to the common prior. We show that all fixed price selections from the Walrasian correspondence are *efficient, individually rational,* and *strategy-proof* rules and, moreover, they are the only such rules. However, when aggregate uncertainty holds, we show that there exists no *efficient, individually rational,* and *strategy-proof* rules. Therefore, aggregate certainty is a necessary and sufficient condition for the existence of rules satisfying our three requirements. In the two agents case, we establish a stronger impossibility result: dictatorial rules are the only *efficient* and *strategy-proof* rules. Dropping the common prior assumption in the model, we show that in the two agents and aggregate certainty case, this assumption is necessary and sufficient for the existence of rules satisfying the three main requirements. Finally, extending the model to allow for more than one good, we show that when there are two agents and *at least two goods*, dictatorial rules are the only *efficient* and *strategy-proof* rules (without regard to the nature of aggregate risk).

Risk sharing problems with expected utility preferences when agents' beliefs are not revealed, are considered by Ju (2001). He shows that in the two agents case, only dictatorial rules are *efficient* and *strategy-proof.* Billot et al. (2000) and Chateauneuf et al. (2000) consider risk sharing problems with non-expected utility preferences and aggregate certainty. Billot et al. (2000) investigate properties of *efficient* allocations for "maximin expected utility preferences" (Gilboa and

Schmeidler, 1989). Also Chateauneuf et al. (2000) study *efficient* risk sharing for the "Choquet-expected utility" case (Schmeidler, 1989). In the case of maximin expected utility preference, uncertainty is represented by a set of beliefs, instead of a single belief, and state-contingent bundles are compared in terms of the "minimum expected utility" over these multiple beliefs. Ju (2002) studies *efficient* and *strategy-proof* rules for maximin expected utility preferences. In particular, he shows that in the aggregate certainty case, more variety of rules are *efficient*, *individually rational*, and *strategy-proof* than in the current model with expected utility preferences. These rules include non-fixed price selections from the Walrasian correspondence as well as fixed price selections.

A number of authors have studied implementability of the Walrasian correspondence in Nash equilibrium: see Hurwicz (1979), Schmeidler (1980), Bennassy (1986), etc. Nash implementation is applicable under the strong informational requirement that agents know each others' strategies. Such a requirement is not needed for the powerful notion of "implementability in dominant strategy equilibrium", of which necessary and sufficient condition is *strategy-proofness.* However, in most economic environments studied earlier, including the exchange economy, no selection from the Walrasian correspondence is *strategy-proof.*[3] The risk sharing problem with aggregate certainty is one of few exceptions. In the linear production economy, Maniquet and Sprumont (1999) characterizes a unique *efficient* and *strategy-proof* rule that treats agents "anonymously". The common feature between this rule and our fixed price selections from the Walrasian correspondence is that there exists a "constant" opportunity set, or budget set, for each agent and the choice by these rules always maximizes each agent's utility over his budget set.

Sobel (1981, 1998) considers exchange economies with risk averse expected utility preferences. He studies the relation between Walrasian allocations in markets "after the resolution of uncertainty" and the equilibrium outcomes of "distortion quasi-games".[4] In distortion quasi-games, agents report their utility functions and an outcome correspondence associates with reported utility functions a set of allocations. The equilibrium concept is similar, in spirit, to Nash equilibrium in standard games. Sobel (1998) shows that every Walrasian allocation is an equilibrium outcome for the distortion quasi-game of which outcome

[3]As shown by the impossibility results in Hurwicz (1972), Satterthwaite and Sonnenschein (1981), Zhou (1991), etc

[4]More precisely, he considers the "constrained" Walrasian equilibrium, which is the same as Walrasian equilibrium except the boundary equilibrium allocation case. The terminology, "quai-game" is due to Thomson (1984).

correspondence is induced by the "relative utilitarian" bargaining solution. In the transferable utility and exchange economic environment, Thomson (1984) studies outcome correspondences selecting only *efficient* and *individually rational* allocations for reported preferences. Their studies are related with ours, since they connect the Walrasian correspondence with *efficiency*, *individual rationality*, and a kind of "realizability" through quasi-games.

Although, in the aggregate uncertainty case or the multiple goods case, we draw similar conclusions to the previous studies by Hurwicz (1972), Dasgupta et al. (1979), Zhou (1991), Schummer (1997), and Serizawa (2000) in exchange economies, their results are not applicable because of the expected utility and common prior restrictions on preferences. Many useful and common techniques in exchange economies, for example, the admissibility of "(strong Maskin) monotonic transformation", the admissibility of common monotonic transformation of two preferences, the existence of Pareto sets arbitrarily close to the boundary of the feasibility set, etc., are severely restricted.

This paper is organized as follows. In Section 2, we introduce our model and define basic concepts. In Section 3, we state the main results. Proofs are collected in Section 4. We conclude with a few remarks in Section 5.

2 The model and basic concepts

Consider a society consisting of $n \geq 2$ agents, $N \equiv \{1, \cdots, n\}$. There is a finite number S of uncertain ***states*** with $S \geq 2$ (for convenience, we also use S to denote the state space). At each state $s \in S$, a fixed amount Ω_s of a single good, or money, is available in the society, which is the sum of individual incomes. Let $\boldsymbol{\Omega} \equiv (\Omega_s)_{s \in S}$ be the ***aggregate endowment***. ***Aggregate certainty*** holds if the aggregate endowment is composed of a constant amount across states, that is, $\Omega_1 = \cdots = \Omega_S$. Otherwise, ***aggregate uncertainty*** holds. We admit a variety of "individual endowments" whose sum is equal to Ω. Let $\boldsymbol{\mathcal{W}} \equiv \{(\omega_i)_{i \in N} \in \mathbb{R}_+^{S \times N}$: for all $i \in N$, $\omega_i \in \mathbb{R}_+^S$ and $\sum_N \omega_i = \Omega\}$ be the set of all profiles of ***individual endowments***. For each profile $(\omega_i)_N \in \mathcal{W}$, the i-th component $\omega_i \equiv (\omega_{is})_{s \in S}$ indicates agent i's endowment. Note that although we admit variability of individual endowments, we assume that aggregate endowment is fixed. Such an ad hoc feature does not play a crucial role in our results except Theorem 2. All other results apply even if we assume that both individual and aggregate endowments are fixed.

We consider the problem of allocating the aggregate endowment among agents before the realization of a state. An ***allocation*** is a list $(z_i)_N \in \mathbb{R}_+^{N\times S}$ of state-contingent bundles indexed by agents. Each agent $i \in N$ receives the i-th component $z_i \in \mathbb{R}_+^S$. The allocation is ***feasible*** if the sum of individual bundles is equal to the aggregate endowment. Let $\boldsymbol{Z}$ be the set of all feasible allocations. We use $z \equiv (z_i)_N$ to denote allocations generically. We use x, y to denote bundles in $\mathbb{R}_+^S$. Let $\boldsymbol{Z_0} \equiv \{x \in \mathbb{R}_+^S : x \leqq \Omega\}$ be the set of all possible bundles at feasible allocations.[5] A ***full insurance bundle*** is composed of a constant amount of money across states. A ***full insurance allocation*** is an allocation consisting of only full insurance bundles. Clearly, there is a feasible full insurance allocation if and only if aggregate certainty holds.

Throughout the paper, we assume that all agents have the same belief over the state space, denoted by $\boldsymbol{\pi} \in \Delta^{S-1}$. We call π the ***common prior***. We refer readers to Aumann (1976), Bacharach (1985), and McKelvey and Page (1986) for the motivation of the common prior assumption.

Each agent has a preference that is a *continuous*, *strictly monotonic*, and *convex* weak ordering[6] and is represented in the following "expected utility form". An ***expected utility preference*** R_i is represented by a real valued function $u_i\colon \mathbb{R}_+ \to \mathbb{R}$ as follows: for all $x, x' \in \mathbb{R}_+^S$,

$$x \; R_i \; x' \Leftrightarrow \sum_{s\in S} \pi_s u_i(x_s) \leq \sum_{s\in S} \pi_s u_i(x'_s).$$

We call u_i a ***utility index for*** $\boldsymbol{R_i}$.[7] Preference R_i is ***risk averse*** if its utility index is concave, that is, for all $m, m' \in \mathbb{R}_+$ and all $\lambda \in [0,1]$, $u_i(\lambda m + (1-\lambda)m') \geq \lambda u_i(m) + (1-\lambda) u_i(m')$. It is ***strictly risk averse*** if the inequality is strict for all $m, m' \in \mathbb{R}_+$ and all $\lambda \in (0,1)$. It is ***risk neutral*** if the inequality holds with equality for all $m, m' \in \mathbb{R}_+$ and all $\lambda \in (0,1)$.

Let $\boldsymbol{\mathcal{R}}$ be the family of all risk averse expected utility preferences and $\boldsymbol{\mathcal{R}^N}$ the family of profiles of preferences in $\mathcal{R}$. An ***economy*** is characterized by agents' preferences and individual endowments. Let $\boldsymbol{\mathcal{E}} \equiv \mathcal{R}^N \times \mathcal{W}$ be the family of all economies. A "risk sharing rule", or briefly, a ***rule*** is a function $\varphi\colon \mathcal{E} \to Z$

[5] Throughout the paper, we use the following notation for vector inequality. For all $x, x' \in \mathbb{R}^S$, $x \geqq x'$ if for all $s \in S$, $x_s \geq x'_s$. We denote $x \geq x'$ if for all $s \in S$, $x_s \geq x'_s$ and for some $r \in S$, $x_r > x'_r$. We denote $x > x'$ if for all $s \in S$, $x_s > x'_s$.

[6] A preference is *continuous* if for each bundle x, both the set of all bundles weakly preferred to x and the set of all bundles to which x is weakly preferred are closed. A preference is *monotonic* if for all x, y, x is weakly preferred to y whenever $x \geqq y$ and x is preferred to y whenever $x > y$. It is *strictly monotonic* if for all x, y, x is preferred to y whenever $x \geq y$. Finally, a preference is *convex* if for all x, the set of all bundles preferred to x is convex.

[7] Utility indices are unique up to "affine transformations".

associating with each economy a feasible allocation.

We are interested in rules satisfying the following requirements. A rule φ is ***efficient*** if it always recommends an "efficient allocation", formally, for all $(R,\omega) \in \mathcal{E}$, there exists no $z \in Z$ such that $z_i \ R_i \ \varphi_i(R,\omega)$ for all $i \in N$ and $z_j \ P_j \ \varphi_j(R,\omega)$ for some $j \in N$. It is ***individually rational*** if all agents are at least as well off as in their individual endowments. It is ***strategy-proof*** if no one can ever be benefited by misrepresenting his preference, independently of others' representations; formally, for all $(R,\omega) \in \mathcal{E}$, all $i \in N$, and all $R'_i \in \mathcal{R}$, $\varphi_i(R,\omega) \ R_i \ \varphi_i((R'_i, R_{-i}), \omega)$.

The following rules are crucial in our work. We first define rules that always select from the following well-known notion of market outcome. A ***Walrasian*** (equilibrium) ***allocation for*** $(\boldsymbol{R},\boldsymbol{\omega})$ is an allocation $z \in Z$ that has a price vector $p \in \Delta^{S-1}$ such that for all $i \in N$ and all $z'_i \in \mathbb{R}^S_+$ with $p \cdot z'_i \leq p \cdot \omega_i$, $z_i \ R_i \ z'_i$. We call p an *equilibrium price.* For each economy (R,ω), let $W(R,\omega)$ be the set of Walrasian allocations. We refer to the set valued function $W\colon \mathcal{E} \twoheadrightarrow Z$ as the ***Walrasian correspondence***. Since preferences are continuous, monotonic, and convex, it is non-empty valued.[8] A ***selection from the Walrasian correspondence*** is a rule associating with each economy a "single" Walrasian allocation. It is well-known that any selection from the Walrasian correspondence is *efficient*.[9] Since individual endowments are always available in the "Walrasian budget sets", it is also *individually rational*. We will show later that in the aggregate certainty case, some selections from the Walrasian correspondence are *strategy-proof* and that in the aggregate uncertainty case, there is no *strategy-proof* selection.

A rule φ is ***dictatorial*** if for each profile of individual endowments $\omega \in \mathcal{W}$, there exists an agent $i \in N$, the "dictator", who always receives the entire aggregate endowment Ω independently of preferences, that is, for all $R \in \mathcal{R}^N$, $\varphi_i(R,\omega) = \Omega$. Note that the dictator may vary across profiles of individual endowments. By strict monotonicity of preferences, any dictatorial rule is *efficient*. Since for each $\omega \in \mathcal{W}$, dictatorial rules are constant across economies with ω, they are *strategy-proof*. Since, at least one agent, say j, receives 0, by strict monotonicity of preferences, no dictatorial rule is *individually rational*.

[8] See Mas-Colell, Whinston, and Green (1995).

[9] See, for instance, Mas-Colell, Whinston, and Green (1995), p. 326.

3 The results

Our results depend crucially on aggregate risk: aggregate certainty or aggregate uncertainty.

The Aggregate Certainty Case

In the aggregate certainty case, there exist rules satisfying *efficiency*, *individual rationality*, and *strategy-proofness*. To show this, note first that by aggregate certainty, for any profile ω of individual endowments, the full insurance allocation where each agent receives the expected value of his endowment $\pi \cdot \omega_i$ for sure, is feasible. By risk aversion, this allocation is best for each agent over his budget set associated with the "common prior price" π. So this full insurance allocation is a Walrasian allocation supported by equilibrium price π. Now define the rule that always recommends such full insurance allocation. Then it is *efficient* and *individually rational*. Since the choice does not depend on preferences, it is *strategy-proof*.

The above rule is an example of the following important family of rules. A rule is a ***fixed price selection from the Walrasian correspondence*** if there is a price p such that the rule maps each economy into a Walrasian allocation supported by the equilibrium price p. Note that there are economies in which the common prior is the only equilibrium price up to normalization.[10] Thus π is the only price that can be used for a fixed price selection.

Since there can be multiple Walrasian allocations supported by π, there exist a variety of fixed price selections. However, such a variety is inessential because all Walrasian allocations supported by π are always indifferent for all agents.[11] Since there can be Walrasian allocations supported by prices not parallel to π, "non-fixed price selections" also exist.

Now we are ready to state our first result.

Theorem 1. *In the aggregate certainty case, a rule is efficient, individually rational, and strategy-proof if and only if it is a fixed price selection from the Walrasian correspondence.*

The independence of the three requirements can be established by the following three examples. Any dictatorial rule satisfies *efficiency* and *strategy-proofness*,

[10]In fact, we can show that for every economy with smooth preferences, the common prior is the unique equilibrium price up to normalization.

[11]Suppose that z is a Walrasian allocation supported by equilibrium price π. Then by risk aversion, for all $i \in N$, z_i is indifferent to the certain amount $\pi \cdot \omega_i$.

but violates *individual rationality*. Consider, next, the "no trade rule" that recommends the profile of individual endowments for each economy. Clearly, this rule is *individually rational* and *strategy-proof*. However, it is not *efficient* since there are economies in which the profile of individual endowments is not *efficient*. Finally, consider the rule selecting a best allocation for agent 1 among *efficient* and *individually rational* allocations. It is easy to show that this rule satisfies *efficiency* and *individual rationality*, but violates *strategy-proofness*.

The Aggregate Uncertainty Case

We now turn to the aggregate uncertainty case. Note that in the aggregate certainty case, the set of feasible full insurance allocations is nonempty and is always included in the "Pareto set" (the set of efficient allocations), independently of preferences. Such a "partial invariance" of the Pareto set plays an important role for the existence result in Theorem 1. In the aggregate uncertainty case, no full insurance allocation is feasible and there is no "focal set" of allocations that is always included in the Pareto set. The Pareto set varies more substantially across economies than in the aggregate certainty case. This makes it harder to design *strategy-proof* selections from the Pareto set.

We establish two impossibility results similar to the results in exchange economies, established by Hurwicz (1972), Dasgupta et al. (1979), Zhou (1991), Schummer (1997), and Serizawa (1998). The next result corresponds to the results by Hurwicz (1972) and Serizawa (1998).

Theorem 2. *In the aggregate uncertainty case, there exists no rule satisfying efficiency, individual rationality, and strategy-proofness.*

The proof of Theorem 2 relies crucially on the admissibility of a variety of profiles of individual endowments in $\mathcal{W}$.

It follows from Theorems 1 and 2 that aggregate certainty is a necessary and sufficient condition for the existence of rules satisfying the three requirements. Moreover, we obtain

Corollary 1. *The following three statements are equivalent.*

(i) Aggregate certainty holds.

(ii) There exists a strategy-proof selection from the Walrasian correspondence.

(iii) There exists an efficient, individually rational, and strategy-proof rule.

The impossibility extends even further in the two agents case. We show that the non-existence result continues to hold, after replacing *individual rationality* with the minimal equity criterion, *non-dictatorship*. Formally:

Theorem 3. *In the aggregate uncertainty and two agents case, a rule is efficient and strategy-proof if and only if it is dictatorial.*

When there are more than two agents, there exist non-dictatorial rules satisfying *efficiency* and *strategy-proofness.* For example, consider the rule that selects $(\Omega, 0, 0)$, whenever agent 3's preference satisfies a certain fixed property (e.g. the ratio of the first and second partial derivatives at Ω is less than or equal to 1) and selects $(0, \Omega, 0)$, otherwise (the definition is due to Satterthwaite and Sonnenschein, 1981 and Zhou, 1991). Although this rule is non-dictatorial, it is quite close to dictatorship since its range is composed of only most unequal allocations in which an agent receives the entire aggregate endowment. In exchange economies, a well-known conjecture by Zhou (1991) and Kato and Ohseto (2001) is that no other rule satisfies *efficiency* and *strategy-proofness.*

Common Prior and the Existence of Efficient, Individually Rational and Strategy-Proof Rules

In the aggregate certainty case, we showed that the common prior assumption in our model is sufficient for the existence of *efficient, individual rational,* and *strategy-proof* rules. Is the common prior assumption also necessary? In order to address this question, we drop the common prior assumption and consider a more general model with fixed beliefs. Our next result shows that the common prior assumption is also necessary, in the two agents case.

In what follows, we assume that $N \equiv \{1, 2\}$ and each agent $i \in N$ has a *fixed* and *non-degenerate* belief $\pi^i \in \Delta^{S-1}$.[12] For each $i \in N$, let $\mathcal{R}_{\pi^i}$ be the family of risk averse expected utility preferences associated with π^i. Let $\mathcal{D}(\pi^1, \pi^2) \equiv \mathcal{R}_{\pi^1} \times \mathcal{R}_{\pi^2} \times \mathcal{W}$ be the domain of two agents economies with fixed beliefs π^1 and π^2 and risk averse expected utility preferences.

Theorem 4. *In the aggregate certainty case, there exists an efficient, individually rational, and strategy-proof rule over the two agents domain $\mathcal{D}(\pi^1, \pi^2)$ with fixed beliefs π^1 and π^2 if and only if the common prior condition holds, that is, $\pi^1 = \pi^2$.*

Multiple Goods and an Impossibility Result

In our model, we assumed that there is only one good, or money, at each state. Thus, "ex post efficiency" simply means complete usage of aggregate endowment.

[12] When beliefs are variable (or unrevealed), the same impossibility result as Theorem 3 holds as shown by Ju (2001).

When there are more than one goods, *ex post efficiency* is more complicated and so variability of Pareto set increases. Will our positive result, Theorem 1, continue to hold in the multiple goods case? The answer is negative as implied by the next result.

At each state $s \in S$, there are l goods, $l \geq 2$, and the state-s endowment, denoted by $\mathbf{\Omega}_s$, is a vector in $\mathbb{R}^l_{++}$. Let $\mathbf{\Omega} \equiv (\mathbf{\Omega}_s)_{s \in S}$ be the aggregate endowment. Let $\mathcal{U}$ be the family of utility index functions over $\mathbb{R}^l_+$, which represent monotonic, continuous, and convex (ex post) preferences over $\mathbb{R}^l_+$. Let $\mathcal{R}_{l\text{-goods}}$ be the family of preferences represented by a utility index in $\mathcal{U}$ and the common prior π in the expected utility form. Let $\mathcal{E}_{l\text{-goods}}$ be the family of all economies with preferences in $\mathcal{R}_{l\text{-goods}}$.

Theorem 5. *When there are two agents, a rule over the multiple goods domain $\mathcal{E}_{l\text{-goods}}$ with $l \geq 2$ is efficient and strategy-proof if and only if it is dictatorial.*

4 Proofs

We use the following concepts and notation throughout Section 4. An allocation z is ***efficient*** if $z \in Z$ and there exists no $z' \in Z$ such that for all $i \in N$, $z'_i \; R_i \; z_i$ and for some $j \in N$, $z'_j \; P_j \; z_j$. For each profile $R \in \mathcal{R}^N$, let $\boldsymbol{P(R)}$ be the set of all *efficient* allocations for R. Note that this set does not depend on individual endowments but only on the aggregate endowment Ω, which is fixed; so we do not need the extra argument ω. For each $i \in N$, let $\boldsymbol{P_i(R)} \equiv \{z_i : z \in P(R)\}$. An allocation z is ***individually rational at*** $\boldsymbol{(R, \omega)}$ if for all $i \in N$, $z_i \; R_i \; \omega_i$.

Let $\boldsymbol{FI_0}$ be the set of all full insurance bundles. Let $\boldsymbol{FI}$ be the set of all full insurance allocations. For all $p \in \mathbb{R}^S_+$ and all $\omega_i \in \mathbb{R}^S_+$, let $\boldsymbol{B(p, \omega_i)} \equiv \{y \in \mathbb{R}^S_+ : p \cdot y \leq p \cdot \omega_i\}$ be the Walrasian budget set with price p and individual endowment ω_i. For all $X \subseteq \mathbb{R}^S_+$ and all $R_i \in \mathcal{R}$, let $\boldsymbol{Max[R_i, X]}$ be the set of all best bundles for R_i in X. For all $R_i \in \mathcal{R}$ and all $x \in \mathbb{R}^S_+$, let $\boldsymbol{UC(R_i, x)} \equiv \{y \in \mathbb{R}^S_+ : y \; R_i \; x\}$, $\boldsymbol{SUC(R_i, x)} \equiv \{y \in \mathbb{R}^S_+ : y \; P_i \; x\}$, and $\boldsymbol{LC(R_i, x)} \equiv \{y \in \mathbb{R}^S_+ : x \; R_i \; y\}$ be the "upper contour set", "strict upper contour set", and "lower contour set" at x, respectively. For all $x \in \mathbb{R}^S_+$ and all $p \in \mathbb{R}^S_{++}$, ***p supports R_i at x*** if for all $y \in UC(R_i, x)$, $p \cdot y \geq p \cdot x$. For all $z \equiv (z_i)_N \in \mathbb{R}^{N \times S}$, ***$p$ supports R at z*** if for all $i \in N$, p supports R_i at z_i.

4.1 Proof of Theorem 1

By convexity of preferences, in the two states case, if an indifference curve is steeper at a bundle x than the constant expected value line through x, then x should be above the full insurance path, or the 45-degree line. If the indifference curve is flatter, then x should be below the full insurance path. The following lemma is a generalization of this fact in more than two states case.

Lemma 1. *If a risk averse expected utility preference with belief π is supported by $p \in \Delta^{S-1}$ at $x \in \mathbb{R}^S_+$. Then for all $r, s \in S$, if $\pi_r/\pi_s < p_r/p_s$, $x_r \leq x_s$.*

Proof. Let R_0 be an expected utility preference with prior π. Let p support R_0 at x. Suppose to the contrary that for some $r, s \in S$, $\pi_r/\pi_s < p_r/p_s$ and $x_r > x_s$. Let $\bar{x} \in \mathbb{R}^S_+$ be such that $\pi \cdot \bar{x} = \pi \cdot x$, $\bar{x}_r = \bar{x}_s$, and for all $q \neq r, s$, $\bar{x}_q \equiv x_q$. Thus, $\pi_r \bar{x}_r + \pi_s \bar{x}_s = (\pi_r + \pi_s)\bar{x}_r = \pi_r x_r + \pi_s x_s$. Clearly, $p \cdot \bar{x} < p \cdot x$. Let u_0 be the utility index for R_0. Then since u_0 is concave,

$$\begin{aligned}
&\pi_r u_0(\bar{x}_r) + \pi_s u_0(\bar{x}_s) + \textstyle\sum_{q \neq r,s} u_0(\bar{x}_q) \\
&= (\pi_r + \pi_s) u_0(\bar{x}_r) + \textstyle\sum_{q \neq r,s} u_0(x_q) \\
&\geq (\pi_r + \pi_s)(\tfrac{\pi_r}{\pi_r + \pi_s} u_0(x_r) + \tfrac{\pi_s}{\pi_r + \pi_s} u_0(x_s)) + \textstyle\sum_{q \neq r,s} u_0(x_q) \\
&= \pi_r u_0(x_r) + \pi_s u_0(x_s) + \textstyle\sum_{q \neq r,s} u_0(x_q).
\end{aligned}$$

Hence, $\bar{x}\ R_0\ x$. Since $\bar{x}$ is strictly below the hyperplane through x with normal vector p, this contradicts to the assumption that p supports R_0 at x. ∎

Lemma 2. *In the aggregate certainty case, if $p \in \mathbb{R}^S_{++}$ supports $R \in \mathcal{R}^N$ at $z \in Z$, then for all $r, s \in S$ and all $i \in N$, either $z_{ir} = z_{is}$ or $p_r/p_s = \pi_r/\pi_s$.*

Proof. Suppose $z_{ir} > z_{is}$. Then by aggregate certainty, there exists $j \in N$ such that $z_{jr} < z_{js}$. Since $z_{ir} > z_{is}$, then by Lemma 1, $p_r/p_s \leq \pi_r/\pi_s$. Similarly, since $z_{jr} < z_{js}$, then by Lemma 1, $p_s/p_r \leq \pi_s/\pi_r$. So $p_r/p_s \geq \pi_r/\pi_s$. ∎

In the aggregate certainty case, we establish the following simple characterization of *efficient* allocations.

Lemma 3. *In the aggregate certainty case, a feasible allocation $z \in Z$ is efficient for $R \in \mathcal{R}^N$ if and only if the common prior π supports R at z.*

Proof. Clearly, if the common prior π supports R at z, then z is *efficient*. In order to prove the converse, suppose that z is *efficient* for R. Since preferences are represented in the expected utility form associated with π and are risk averse, if z is a full insurance allocation, then π supports R at z. Suppose that z is not a full insurance allocation. Let $i \in N$ be such that $z_i \notin FI_0$. Let $s \in S$ be such that $z_{is} \neq z_{i1}$. Let $p \in \mathbb{R}^S_{++}$ be a vector supporting R at z.[13] Then by Lemma 2, $p_s/p_1 = \pi_s/\pi_1$. Let $r \in S$. Then either $z_{ir} = z_{i1}$ or $z_{ir} \neq z_{i1}$. We show, in both cases, $p_r/p_1 = \pi_r/\pi_1$. When $z_{ir} = z_{i1}$, $z_{ir} \neq z_{is}$. By Lemma 2, $p_r/p_s = \pi_r/\pi_s$. Since $p_s/p_1 = \pi_s/\pi_1$, $p_r/p_1 = \pi_r/\pi_1$. When $z_{ir} \neq z_{i1}$, by Lemma 2, $p_r/p_1 = \pi_r/\pi_1$. Therefore, p is parallel to the common prior π and so the common prior supports R at z. ∎

The following lemma is used to prove the uniqueness part of Theorem 1.

Lemma 4. *For all $x, y \in \mathbb{R}^S_+$ and all risk averse expected utility preferences R_0 associated with belief π, if R_0 is supported by π at x and $\pi \cdot y > \pi \cdot x$, then there exists a risk averse expected utility preference R'_0 associated with belief π such that (i) $y\ P'_0\ x$ and (ii) for all $x' \in LC(R_0, x) \cap UC(R'_0, x)$, either $x'\ I'_0\ x$ or R'_0 is not supported by π at x'.*

Before proving Lemma 4, we consider a special case when x is a full insurance bundle. Since $\pi \cdot y > \pi \cdot x$, there is a sufficiently less risk averse, yet, strictly risk averse preference R'_0 for which y is preferred to x; so part (i) of the lemma is met (see Figure 1). Note that x is the only full insurance bundle in $LC(R_0, x) \cap UC(R'_0, x)$. Hence by strict risk aversion of R'_0, x is the only bundle in $LC(R_0, x) \cap UC(R'_0, x)$, where R'_0 is supported by π (see Figure 1). So part (ii) of the lemma is met. A difficulty arises when x is not a full insurance bundle.

Proof. The following claim is useful to construct a preference satisfying (i) and (ii).

Claim 1. *For all concave index functions $u_0 \colon \mathbb{R}_+ \to \mathbb{R}$, all non-degenerate probability vectors $p \in \mathbb{R}^S_{++}$, and all $x \in \mathbb{R}^S_+$, if $\sum_{s \in S} p_s u_0(x_s) = u_0\left(\sum_{s \in S} p_s x_s\right)$, then there exist $m_*, m^* \geq 0$ such that $x_1, \cdots, x_S \in [m_*, m^*]$ and u is linear over $[m_*, m^*]$, that is, for all $\lambda \in [0,1]$, $u_0(\lambda m_* + (1-\lambda) m^*) = \lambda u_0(m_*) + (1-\lambda) u_0(m^*)$* (see Figure 2).

Proof. Let u_0 be a concave index function. We prove the following statement S(k) with respect to $k \in \{1, \cdots, S\}$, using an induction argument with respect to k.

S(k): *For all non-degenerate probability vectors $p \in \mathbb{R}^S_{++}$ and all $x \in \mathbb{R}^S_+$ with $|\{x_1, \cdots, x_S\}| \leq k$ and $\sum_{s \in S} p_s u_0(x_s) = u_0\left(\sum_{s \in S} p_s x_s\right)$, there exist $m_*, m^* \geq 0$*

[13] Under our assumptions on preferences, for every *efficient* allocation z for R, there exists a vector $p \in \mathbb{R}^S_{++}$ consisting of positive numbers, which supports R at z.

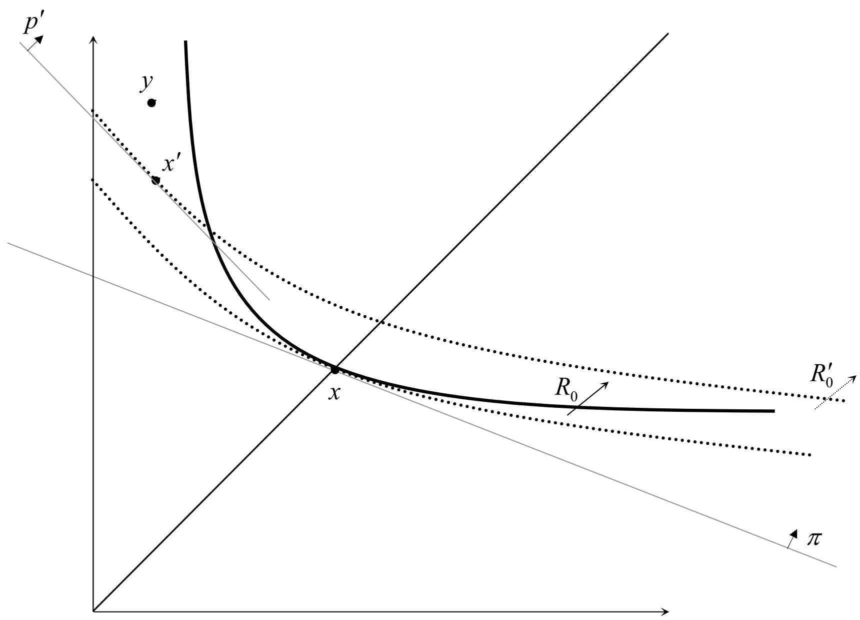

Figure 1: When x is a full insurance bundle, find a *strictly* risk averse preference R'_0 that is sufficiently close to the risk neutral preference. Then $y \; P'_0 \; x$ and $LC\left(R_0, x\right) \cap UC\left(R'_0, x\right) = \{x\}$. Note that at each $x' \in LC\left(R_0, x\right) \cap UC\left(R'_0, x\right)$, R'_0 is supported by a vector p' that is not parallel to π.

such that $x_1, \cdots, x_S \in [m_*, m^*]$ *and* u_0 *is linear over* $[m_*, m^*]$, *that is, for all* $\lambda \in [0,1]\,,\; u_0\left(\lambda m_* + (1-\lambda) m^*\right) = \lambda u_0\left(m_*\right) + (1-\lambda) u_0\left(m^*\right).$

S(1) holds trivially. Let $k \in \{2, \cdots, S\}$. Suppose S($k-1$).

In order to show S(k), consider $p \in \mathbb{R}^S_{++}$ and $x \in \mathbb{R}^S_+$ with $|\{x_1, \cdots, x_S\}| = k$. Assume $\sum_{s\in S} p_s u_0\left(x_s\right) = u_0\left(\sum_{s\in S} p_s x_s\right)$. Without loss of generality, assume $x_1 \le x_2 \le \cdots \le x_S$. Let $r_1, \cdots, r_k \in \{1, \cdots, S\}$ be such that $r_1 = 1 < \cdots < r_k$; $x_{r_1} < \cdots < x_{r_k}$; for all $l \in \{1, \cdots, k\}$ and all $s \in S$, if $r_l \le s < r_{l+1}$, $x_s = x_{r_l}$. Let $P_1 \equiv p_{r_1} + p_{r_1+1} + \cdots + p_{r_2-1},\; P_2 \equiv p_{r_2} + \cdots + p_{r_3-1},\; \cdots,\; P_k \equiv p_{r_k} + \cdots + p_S$. Let $\tau \equiv \frac{1}{P_1+P_2}\left(P_1 x_{r_1} + P_2 x_{r_2}\right)$. Let $x' \equiv \left(\tau, \cdots, \tau, x_{r_3}, \cdots, x_S\right)$. Note that $\sum_{s\in S} p_s x_s = \sum_{s\in S} p_s x'_s$. Then by risk aversion, $u_0\left(\sum_{s\in S} p_s x_s\right) \ge \sum_{s\in S} p_s u_0\left(x'_s\right) \ge \sum_{s\in S} p_s u_0\left(x_s\right)$. Since $\sum_{s\in S} p_s u_0\left(x_s\right) = u_0\left(\sum_{s\in S} p_s x_s\right)$,

$$u_0\left(\sum_{s\in S} p_s x'_s\right) = \sum_{s\in S} p_s u_0\left(x'_s\right) = \sum_{s\in S} p_s u_0\left(x_s\right).$$

Since $|\{x'_1, \cdots, x'_S\}| = k-1$, then by the first equality and the induction hypothesis, u_0 is linear over $[\tau, x_S]$. On the other hand, from the second equality,

$u_0(\tau) = \frac{1}{P_1+P_2}(P_1 u_0(x_{r_1}) + P_2 u_0(x_{r_2}))$. Hence by concavity of u_0, u_0 is linear over $[x_{r_1}, x_{r_2}] = [x_1, x_{r_2}]$. Since $\tau < x_{r_2}$, u_0 is linear over $[x_1, x_S]$. Therefore, if we let $m_* \equiv x_1$ and $m^* \equiv x_S$, then S(k) holds. □

Let $x, y \in \mathbb{R}^S_+$. Let R_0 be a risk averse expected utility preference with prior π such that $\pi \cdot y > \pi \cdot x$ and π supports R_0 at x. We now construct a risk averse expected utility preference R_0' satisfying (i) and (ii) below. Let u_0 be the concave utility index for R_0. Let $\mu_x \equiv \pi \cdot x$. Since π supports R_0 at x, then by risk aversion, $x \; I_0 \; (\mu_x, \cdots, \mu_x)$, that is, $\sum_{s \in S} \pi_s u_0(x_s) = u_0(\pi \cdot x)$. Then by Claim 1, there exist $m_*, m^* \geq 0$ such that $x_1, \cdots, x_S \in [m_*, m^*]$ and for all $\lambda \in [0,1]$, $u_0(\lambda m_* + (1-\lambda) m^*) = \lambda u_0(m_*) + (1-\lambda) u_0(m^*)$. Let $u_0^* \colon \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function such that $\frac{du_0^*}{dm}$ is constant over $[m_*, m^*]$ and strictly decreasing over $\mathbb{R}_+ \backslash [m_*, m^*]$ (see Figure 2). Let u_0^{neut} be a utility index for the risk neutral preference. For each $\gamma \in [0,1]$, let $u_0^\gamma \equiv \gamma u_0^* + (1-\gamma) u_0^{neut}$. Let R_0^γ be the preference represented by u_0^γ in the expected utility form. Since both u_0^* and u_0^{neut} are concave, then u_0^γ is also concave. So R_0^γ is risk averse.

As γ converges to 0, u_0^γ converges to u_0^{neut} and R_0^γ converges to the risk neutral preference R_0^{neut}. Since $\pi \cdot y > \pi \cdot x$, then there exists $\gamma^* > 0$ such that $y \; P_0^{\gamma^*} \; x$ (see Figure 2). Clearly, $R_0^{\gamma^*}$ satisfies (i). We only have to show that $R_0^{\gamma^*}$ also satisfies (ii). Let $x' \in LC(R_0, x) \cap UC(R_0^{\gamma^*}, x)$. There are two cases.

Case 1. *For all $s \in S$, $x_s' \in [m_*, m^*]$.* Then since both $u_0^{\gamma^*}$ and u_0 are linear over $[m_*, m^*]$, $x_1, \cdots, x_S \in [m_*, m^*]$, and $x_1', \cdots, x_S' \in [m_*, m^*]$, then both $R_0^{\gamma^*}$ and R_0 order x and x' in the same way. Therefore, since $x' \in LC(R_0, x) \cap UC(R_0^{\gamma^*}, x)$, $x' \; I_0^{\gamma^*} \; x$.

Case 2. *For some $s \in S$, $x_s' \notin [m_*, m^*]$.* Let $s \in S$ be such that $x_s' \notin [m_*, m^*]$. Since $x' \in LC(R_0, x) \cap UC(R_0^{\gamma^*}, x)$, there exists $r \in S$ such that $x_r' \neq x_s'$. Without loss of generality, suppose $x_r' < x_s'$. Then since $du_0^{\gamma^*}/dm$ is strictly decreasing over $\mathbb{R}_+ \backslash [m_*, m^*]$, $\frac{\pi_r \cdot du_0^{\gamma^*}(x_r')/dm}{\pi_s \cdot du_0^{\gamma^*}(x_s')/dm} > \frac{\pi_r}{\pi_s}$. Hence $R_0^{\gamma^*}$ is not supported by the common prior π at x'. ∎

We next show that every *efficient, individually rational,* and *strategy-proof* rule always gives each agent a bundle with the same expected value as his endowment.

Lemma 5. *In the aggregate certainty case, if a rule φ is efficient, individually rational, and strategy-proof, then for all $(R, \omega) \in \mathcal{E}$ and all $i \in N$, $\pi \cdot \varphi_i(R, \omega) = \pi \cdot \omega_i$.*

Proof. Let $(R, \omega) \in \mathcal{E}$ and $z \equiv \varphi(R, \omega)$. Suppose by contradiction that there

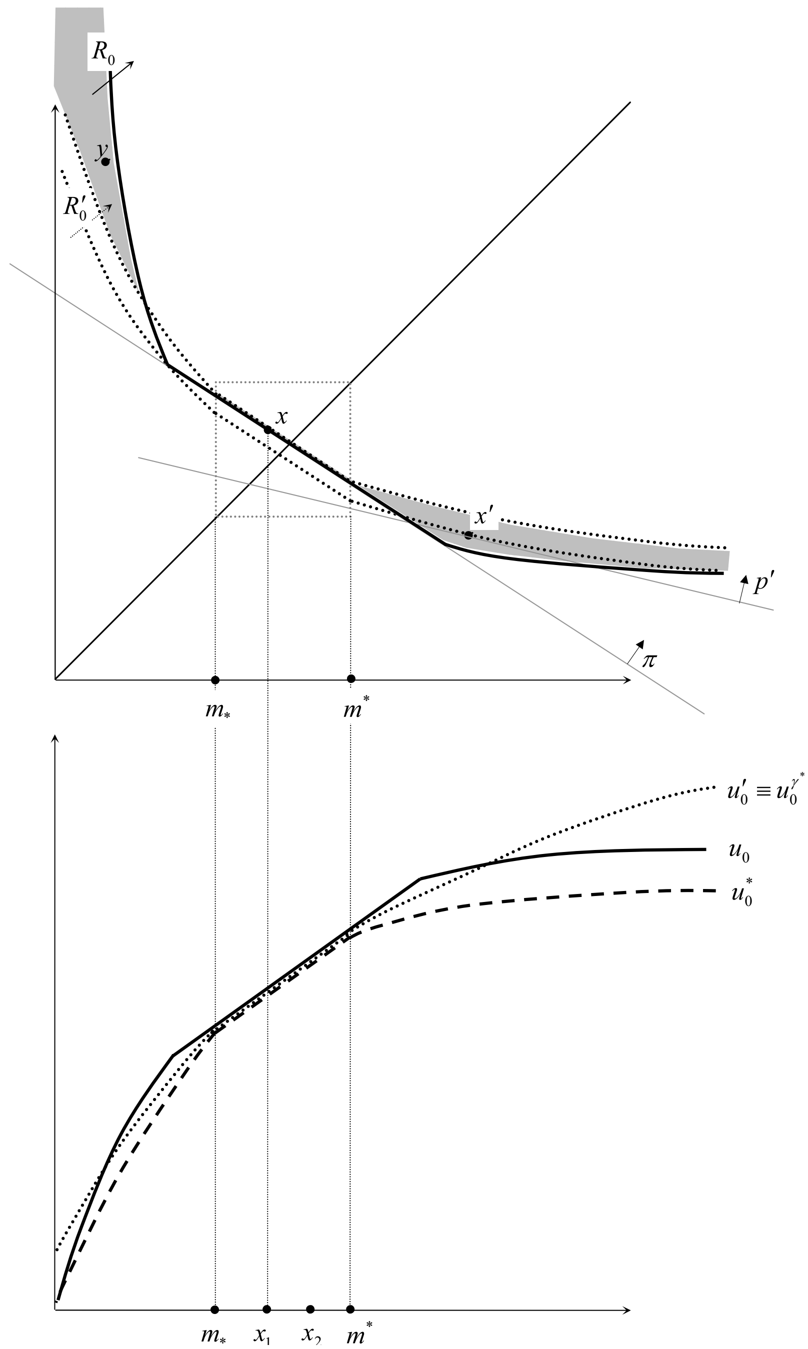

Figure 2: Proof of Lemma 4. Construction of R_0' satisfying (i) $y\ P_0'\ x$ and (ii) for all $x' \in LC(R_0, x) \cap UC(R_0', x)$, either $x'\ I_0'\ x$ or R_0' is not supported by π at x'. Utility index of R_0', denoted by u_0', is constructed in the bottom figure.

exists $i \in N$ such that $\pi \cdot z_i < \pi \cdot \omega_i$. By Lemma 3, π supports R at z. Then by Lemma 4, there exists $R'_i \in \mathcal{R}$ such that (i) $\omega_i \; P'_i \; z_i$ and (ii) for all $x \in LC(R_i, z_i) \cap UC(R'_i, z_i)$, either $x \; I'_i \; z_i$ or R'_i is not supported by π at x. Let $z' \equiv \varphi((R'_i, R_{-i}), \omega)$. By *strategy-proofness*, $\varphi_i((R'_i, R_{-i}), \omega) \in LC(R_i, z_i) \cap UC(R'_i, z_i)$. On the other hand, by Lemma 3, π supports R'_i at z'_i. Then by (ii), $z'_i \; I'_i \; z_i$. Hence by (i), $\omega_i \; P'_i \; z'_i$, contradicting *individual rationality.* ∎

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let φ^π be a fixed price selection from the Walrasian correspondence. Clearly, φ^π is *efficient.* Since for all $(R, \omega) \in \mathcal{E}$ and all $i \in N$, $\varphi^\pi_i(R) \in Max[R_i, B(\pi, \omega_i)]$, then $\varphi^\pi_i(R) \; R_i \; \omega_i$. So φ^π is *individually rational.* Note that under φ^π, each agent i can attain only bundles in $B(\pi, \omega_i)$ by misrepresenting his preference and that he attains a best bundle over $B(\pi, \omega_i)$ by truth-telling. Therefore, φ^π is *strategy-proof.*

In order to show the converse, let φ be a rule satisfying the three requirements. Let $(R, \omega) \in \mathcal{E}$ and $i \in N$. By Lemma 5, $\pi \cdot \varphi_i(R, \omega) = \pi \cdot \omega_i$. On the other hand, by Lemma 3, π supports R_i at $\varphi_i(R, \omega)$. So $\varphi_i(R, \omega) \in Max[R_i, B(\pi, \omega_i)]$. Therefore, $\varphi(R, \omega)$ is a Walrasian allocation with equilibrium price π. Since this holds for all $(R, \omega) \in \mathcal{E}$, φ is a fixed price selection from the Walrasian correspondence. ∎

Remark 1. Proof of Theorem 1 relies only on the richness property of $\mathcal{R}$, which is established in Lemma 4. Hence the same characterization result as in Theorem 1 applies in any subdomain of $\mathcal{R}$ satisfying this property.

4.2 Proofs of Theorems 2 and 3

Throughout this section, we assume aggregate uncertainty, that is, for some $r, s \in \{1, \cdots, S\}$, $\Omega_r \neq \Omega_s$. Without loss of generality, we assume $\Omega_1 \geq \cdots \geq \Omega_s \geq \cdots \geq \Omega_S$.

The following two lemmas are useful to prove Theorem 2. We first show that under an *efficient* and *individually rational* rule, if there is an agent i with sufficiently large individual endowment, then whenever agent i is risk neutral, any other risk averse agent is always fully insured.

Lemma 6. *Let $\omega \in \mathcal{W}$ be such that for an agent $i \in N$, all $z_i \in Z_0$ with $\pi \cdot z_i \geq \pi \cdot \omega_i$ are in the interior, that is, $z_i \in \mathbb{R}^S_{++}$. Then whenever agent i is risk neutral, all efficient and individually rational rules fully insure all other strictly risk averse agents.*

Proof. Let $i \in N$ and $\omega \in \mathcal{W}$ be given as above. Let φ be *efficient* and *individually rational.* Suppose that at $R \in \mathcal{R}$, agent i is risk neutral and agent j is strictly risk averse. Let $z \equiv \varphi(R, \omega)$. Since R_i is risk neutral, then by *individual rationality,* $\pi \cdot z_i \geq \pi \cdot \omega_i$. So, by the assumption on ω, z_i is in the interior. Suppose by contradiction that z_j is not fully insured. Then there exists a full insurance bundle z_0 such that $\pi \cdot z_j = \pi \cdot z_0$. By strict risk aversion, $z_0 \ P_j \ z_j$. Since z_i is in the interior, there exists $\lambda^* \in (0,1)$ sufficiently close to 1 such that $z_i - (1-\lambda^*)(z_0 - z_j)$ is positive. Let $z_i^* \equiv z_i - (1-\lambda^*)(z_0 - z_j)$, $z_j^* \equiv \lambda z_j + (1-\lambda) z_0$, and for each $h \neq i, j$, $z_h^* \equiv z_h$. Clearly, $z^* \in Z$. By risk neutrality of R_i, $z_i^* \ I_i \ z_i$ and by strict risk aversion of R_j, $z_j^* \ P_j \ z_j$. Since z^* is feasible, this contradicts *efficiency.* ∎

We next show that under any *efficient, individually rational,* and *strategy-proof* rule, whenever an agent i with sufficiently large individual endowment is risk neutral and all others are strictly risk averse, the rule chooses an Walrasian allocation.

Lemma 7. *Let $\omega \in \mathcal{W}$ be such that for an agent $i \in N$, all $z_i \in Z_0$ with $\pi \cdot z_i \geq \pi \cdot \omega_i$ are in the interior, that is, $z_i \in \mathbb{R}^S_{++}$. Consider an efficient, individually rational, and strategy-proof rule φ. If agent i is risk neutral and all others are strictly risk averse at $R \in \mathcal{R}$, then for all $h \in N$, $\pi \cdot \varphi_h(R, \omega) = \pi \cdot \omega_h$ and, moreover, $\varphi(R, \omega)$ is an Walrasian allocation supported by the common prior π.*

Proof. Let $i \in N$, $\omega \in \mathcal{W}$, φ, and $R \in \mathcal{R}^N$ be given as above. Let $z \equiv \varphi(R, \omega)$. Since R_i is risk neutral, then by *individual rationality,* $\pi \cdot z_i \geq \pi \cdot \omega_i$. So, by the assumption on ω, z_i is in the interior. Clearly, $\pi \cdot \sum_{j \neq i} z_j \leq \pi \cdot \left(\sum_{j \neq i} \omega_j\right)$. By *efficiency*, there exists $p \in \mathbb{R}^S_+$ that supports R_h at z_h for all $h \in N$. Then since z_i is an interior bundle and agent i is risk neutral, p is parallel to π. Therefore, we only have to show that for all $j \neq i$, $\pi \cdot z_j = \pi \cdot \omega_j$.

Suppose by contradiction that there exists $j \neq i$ such that $\pi \cdot z_j < \pi \cdot \omega_j$.[14] Then there exists R'_j such that $\omega_j \ P'_j \ z_j$ and R'_j is strictly risk averse. By Lemma 6, agent j is fully insured at both (R, ω) and $((R'_j, R_{-j}), \omega)$, that is, both $\varphi_j(R, \omega) = z_j$ and $\varphi_j((R'_j, R_{-j}), \omega)$ are on the monotonic path FI_0. Hence, by *strategy-proofness*, they should be identical; $\varphi_j((R'_j, R_{-j}), \omega) = z_j$. Therefore, $\omega_j \ P'_j \ \varphi_j((R'_j, R_{-j}), \omega)$, contradicting *individual rationality.* ∎

Using Lemmas 6 and 7, we prove Theorem 2.

[14] Since $\pi \cdot z_i \geq \pi \cdot \omega_i$, then if we deny that for all $j \neq i$, $\pi \cdot z_j = \pi \cdot \omega_j$, then there must exist $j \neq i$ such that $\pi \cdot z_j < \pi \cdot \omega_j$.

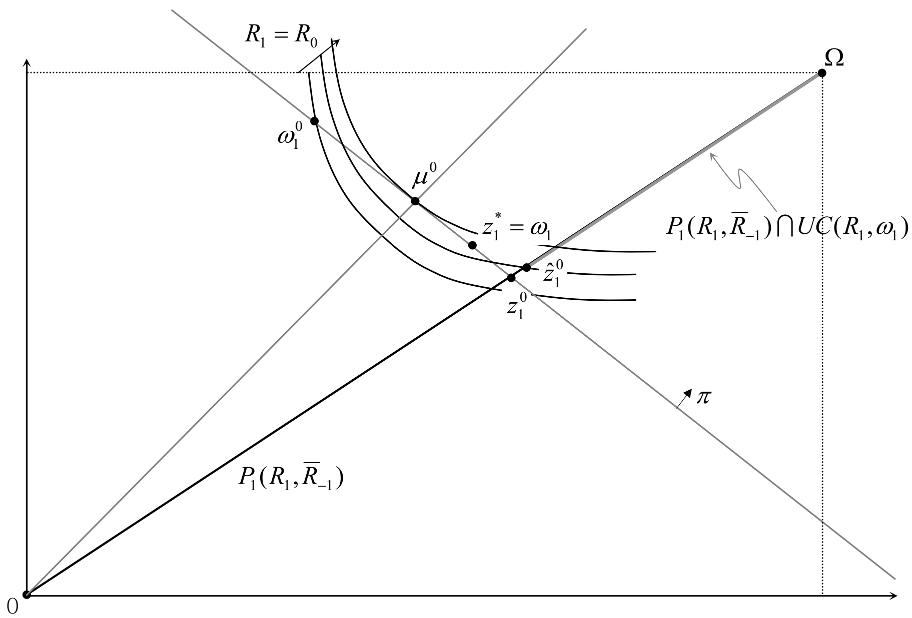

Figure 3: Let R_0 be homothetic and strictly risk averse. Let $R_1 \equiv R_0$ and $\bar{R}_{-1} \equiv (R_0, \cdots, R_0)$. Note that $P_1\left(R_1, \bar{R}_{-1}\right) = \overline{0, \Omega}$ and $P_1\left(R_1, \bar{R}_{-1}\right) \cap UC\left(R_1, \omega_1\right) = \overline{\hat{z}_1^0, \Omega}$.

Proof of Theorem 2. Suppose by contradiction that there exists an *efficient, individually rational,* and *strategy-proof* rule φ. We derive a contradiction considering the endowment profile and preferences constructed in the following claim.

Claim 1. *There exist $\omega \in \mathcal{W}$, $\bar{R} \in \mathcal{R}^N$, and $R_1 \in \mathcal{R}$ such that (i) every bundle $z_1 \in Z_0$ with $\pi \cdot z_1 \geq \pi \cdot \omega_1$ is in the interior; (ii) $\bar{R}_1$ is risk neutral; (iii) for all $i \neq 1$, $\bar{R}_i$ is strictly risk averse; (iv) for all efficient allocation for $\left(R_1, \bar{R}_{-1}\right)$, if $z_1 \; R_1 \; \omega_1$, $\pi \cdot z_1 > \pi \cdot \omega_1$.*

Proof. Throughout the proof, see Figure 3 for illustration. Let R_0 be a strictly risk averse and homothetic preference.[15] Let $\omega^0 \in \mathcal{W}$ be a profile of individual endowments satisfying part (i). Let z_1^0 be the intersection of $\{z_1 \in Z_0 : \pi \cdot z_1 = \pi \cdot \omega_1^0\}$ and $\overline{0, \Omega}$. Let $\mu^0 \equiv \pi \cdot z_1^0$. Since $z_1^0 \in \overline{0, \Omega}$, by aggregate uncertainty, $z_1^0 \neq (\mu^0, \cdots, \mu^0)$. Let $\lambda^* \in (0,1)$ and $z_1^* \equiv \lambda^* z_1^0 + (1 - \lambda^*)\left(\mu^0, \cdots, \mu^0\right)$. Then by strict risk aversion, $z_1^* \; P_0 \; z_1^0$. Therefore, there exists $\hat{z}_1^0 \in \overline{0, \Omega}$ such that $\hat{z}_1^0 \; I_0 \; z_1^*$ and $\hat{z}_1^0 > z_1^0$. Clearly, for all $z_1 \in \overline{\hat{z}_1^0, \Omega}$, $\pi \cdot z_1 > \pi \cdot z_1^0 = \pi \cdot z_1^*$.

Let $\omega \in \mathcal{W}$ be such that $\omega_1 = z_1^*$. Let $\bar{R}$ be such that $\bar{R}_1$ is risk neutral and for all $i \neq 1$, $\bar{R}_i \equiv R_0$. Let $R_1 \equiv R_0$. We only have to show part (iv). Since R_0 is

[15] Let $\alpha \in (0,1)$ and u^α be such that $u^\alpha(m) \equiv m^\alpha$ for all $m \geq 0$. Let R_0 be the expected utility preference with utility index u^α.

homothetic, $P_1(R_0, \cdots, R_0) \left(= P_1\left(R_1, \bar{R}_{-1}\right)\right) = \overline{0, \Omega}$. Hence if z_1 is *efficient* for $\left(R_1, \bar{R}_{-1}\right)$, then $z_1 \in \overline{0, \Omega}$. So, if, in addition, $z_1 \; R_1 \; \omega_1 \left(= z_1^*\right) \; I_1 \; \hat{z}_1^0$, then $z_1 \geq \hat{z}_1^0$. Hence $\pi \cdot z_1 > \pi \cdot \omega_1 = \pi \cdot z_1^*$. □

Let ω, $\bar{R}$, and R_1 be given as in Claim 1. Let $\bar{z} \equiv \varphi\left(\bar{R}, \omega\right)$. Then by Lemma 7, for all $h \in N$, $\pi \cdot \bar{z}_h = \pi \cdot \omega_h$. By *efficiency* and *individual rationality,* $\varphi\left(\left(R_1, \bar{R}_{-1}\right), \omega\right)$ is *efficient* for $(R_1, \bar{R}_{-1})$ and $\varphi_1\left(\left(R_1, \bar{R}_{-1}\right), \omega\right) \; R_1 \; \omega_1$. Hence by part (iv), $\pi \cdot \varphi_1\left(\left(R_1, \bar{R}_{-1}\right), \omega\right) > \pi \cdot \omega_1 = \pi \cdot \bar{z}_1$. Therefore, since $\bar{R}_1$ is risk neutral, $\varphi_1\left(\left(R_1, \bar{R}_{-1}\right), \omega\right) \; \bar{P}_1 \; \bar{z}_1 = \varphi_1\left(\left(\bar{R}_1, \bar{R}_{-1}\right), \omega\right)$, contradicting *strategy-proofness.* ∎

In what follows, we considers the two agents case. In order to prove Theorem 3, we first study how each *strategy-proof* and *efficient* rule behaves over the following subdomain of preference profiles. For all $\gamma_0 \geq 0$, let $u_{CARA}^{\gamma_0} \colon \mathbb{R}_+ \to \mathbb{R}$ be defined as follows: for all $m \in \mathbb{R}_+$,

$$u_{CARA}^{\gamma_0}(m) \equiv \begin{cases} -e^{-\gamma_0 m}, \text{ if } \gamma_0 > 0; \\ m, \qquad \text{if } \gamma_0 = 0. \end{cases}$$

For each $\gamma_0 \geq 0$, utility index $u_{CARA}^{\gamma_0}$ exhibits constant "Arrow-Pratt coefficient of absolute risk aversion" equal to γ_0, that is, for all $m \in \mathbb{R}_+$, $-\frac{d^2 u_{CARA}^{\gamma_0}(m)/dm^2}{du_{CARA}^{\gamma_0}(m)/dm} = \gamma_0$.[16] Let $\boldsymbol{\mathcal{R}_{CARA}}$ be the family of all preferences represented by $u_{CARA}^{\gamma_0}$ for some $\gamma_0 \geq 0$. For each $(\gamma_1, \gamma_2) \in \mathbb{R}_+^2$, let $R(\gamma_1, \gamma_2)$ be the preference profile in $\mathcal{R}_{CARA}^N$ consisting of two preferences, $R_1^{\gamma_1}$ and $R_2^{\gamma_2}$, with utility indices, $u_{CARA}^{\gamma_1}$ and $u_{CARA}^{\gamma_2}$, respectively. Let $\boldsymbol{\mathcal{E}_{CARA}} \equiv \mathcal{R}_{CARA}^N \times \mathcal{W}$. Then a rule φ is *strategy-proof* over $\mathcal{E}_{CARA}$ if and only if for all $i \in N$, all $\omega \in \mathcal{W}$, all $\left(\gamma_i, \gamma_{-i}\right) \in \mathbb{R}_+^2$, and all $\gamma_i' \in \mathbb{R}_+$,

$$\varphi_i\left(R\left(\gamma_i, \gamma_{-i}\right), \omega\right) \; R_i^{\gamma_i} \; \varphi_i\left(R\left(\gamma_i', \gamma_{-i}\right), \omega\right).$$

Let $\boldsymbol{C}^{\ulcorner}$ be the monotonic path from 0 to Ω defined as follows: for all $x \in \mathbb{R}_+^S$ with $0 \leq x \leq \Omega$, $x \in C^{\ulcorner}$ if and only if there exists $s \in \{1, \cdots, S\}$ such that (i) $s = S$ and $x_1 = \cdots = x_S$ or (i) $s < S$, $x_1 = \cdots = x_s \geq \Omega_{s+1}$, and $x_{s'} = \Omega_{s'}$ for all $s' \geq s+1$. Similarly, let $\boldsymbol{C}^{\lrcorner}$ be the monotonic path from 0 to Ω defined as follows: for all $x \in \mathbb{R}_+^S$ with $0 \leq x \leq \Omega$, $x \in C^{\lrcorner}$ if and only if there exists $s \in \{1, \cdots, S\}$ such that (i) $s = S$ and $\Omega_1 - x_1 = \cdots = \Omega_S - x_S$ or (ii) $s < S$, $\Omega_1 - x_1 = \cdots = \Omega_s - x_s \geq \Omega_{s+1}$, and $x_{s'} = 0$ for all $s' \geq s+1$. For example, in the two states case with $\Omega_1 > \Omega_2$, $C^{\ulcorner} = \{x \in \mathbb{R}_+^2 : x_1 = x_2 \text{ or } x_2 = \Omega_2\}$ is the upper piecewise linear path of the Edgeworth box in Figure 4 and $C^{\lrcorner} = \{x \in \mathbb{R}_+^2 : \Omega_1 - x_1 = \Omega_2 - x_2$ or

[16] CARA stands for constant absolute risk aversion.

$x_2 = 0\}$ is the lower piecewise linear path.

The following three lemmas are used to prove Theorem 3.

Lemma 8. *Let $N \equiv \{1,2\}$. For all $(\gamma_1, \gamma_2) \in \mathbb{R}^2_+$, (i) $P_1(R(\gamma_1,\gamma_2)) = C^{\ulcorner}$ if and only if $\gamma_1 > 0$ and $\gamma_2 = 0$; (ii) $P_1(R(\gamma_1,\gamma_2)) = C^{\lrcorner}$ if and only if $\gamma_1 = 0$ and $\gamma_2 > 0$.*

Proof. We prove part (i) (the proof of part (ii) is the same).

Let $\gamma_1 > 0$ and $\gamma_2 = 0$. In order to prove $P_1(R(\gamma_1,\gamma_2)) \subseteq C^{\ulcorner}$, suppose $x \notin C^{\ulcorner}$. Then there exists $s^* \in \{2, \cdots, S-1\}$ such that $x_1 = \cdots = x_{s^*-1} \neq x_{s^*}$. When $x_{s^*-1} < x_{s^*}$, consider the bundle x' that is obtained by changing incomes at states s^*-1 and s^* in x to $\frac{\pi_{s^*-1}}{\pi_{s^*-1}+\pi_{s^*}} x_{s^*-1} + \frac{\pi_{s^*}}{\pi_{s^*-1}+\pi_{s^*}} x_{s^*}$, preserving all other components of x.[17] Note that each agent has the same expected income in the new allocation $(x', \Omega - x')$ as in $(x, \Omega - x)$. Then by risk aversion, agent 1 is better off and by risk neutrality, agent 2 is indifferent. Hence $(x, \Omega - x)$ is not *efficient*. When $x_{s^*-1} > x_{s^*}$, since $x \notin C^{\ulcorner}$, then there exists $s \geq s^*$ such that $x_s < \Omega_s$ and for all $r \in \{s^*, \cdots, s-1\}$, $x_r = \Omega_r$. Consider the bundle x' that is obtained by changing incomes at states s^*-1 and s in x to $\frac{\pi_{s^*-1}}{\pi_{s^*-1}+\pi_s} x_{s^*-1} + \frac{\pi_s}{\pi_{s^*-1}+\pi_s} x_s$, preserving all other components of x. Let $x^\delta \equiv (1-\delta)x + \delta x'$, where $\delta \in (0,1)$. Note that since $x_{s^*-1} > x_s$, $x \neq x'$ and $x \neq x^\delta$ for all $\delta \in (0,1)$ and that since $x_s < \Omega_s$, for sufficiently small $\delta \in (0,1)$, $\left(x^\delta, \Omega - x^\delta\right)$ is feasible. Note that by strict risk aversion, agent 1 prefers x' to x and so prefers x^δ to x. Since the expected income of x^δ is the same as that of x, risk neutral agent 2 is indifferent between $\Omega - x^\delta$ and $\Omega - x$. Hence $(x, \Omega - x)$ is not *efficient*.

In order to prove $C^{\ulcorner} \subseteq P_1(R(\gamma_1,\gamma_2))$, let $x \in C^{\ulcorner}$. Let $s^* \in S$ and $\alpha \in \mathbb{R}_+$ be such that $\alpha \geq \Omega_{s^*}$ and $x \equiv (\alpha, \cdots, \alpha, \Omega_{s^*}, \cdots, \Omega_S)$. Let $u(\cdot) \equiv u_1^{\gamma_1}(\cdot)$ be the utility index of agent 1. Let $\hat{\pi} \equiv (u'(\alpha)\pi_1, \cdots, u'(\alpha)\pi_{s^*-1}, u'(\Omega_{s^*})\pi_{s^*}, \cdots, u'(\Omega_S)\pi_S)$. Then $\hat{\pi}$ supports agent 1's preference at x. In order to show *efficiency* of $(x, \Omega - x)$, we only have to show that $\hat{\pi}$ also supports agent 2's preference at $\Omega - x$, that is, by risk neutrality, for all $x' \in \mathbb{R}^S_+$ with $\pi \cdot x' \leq \pi \cdot x$, $\hat{\pi} \cdot x' \leq \hat{\pi} \cdot x$. Let $x' \in \mathbb{R}^S_+$ be such that $\pi \cdot x' \leq \pi \cdot x$. Since $\alpha \geq \Omega_{s^*} \geq \cdots \geq \Omega_S$ and u is strictly concave, $\frac{u'(\Omega_{s^*})}{u'(\alpha)} \geq 1, \cdots, \frac{u'(\Omega_S)}{u'(\alpha)} \geq 1$. Hence

$$\begin{aligned}
&\frac{1}{u'(\alpha)}\left(\hat{\pi} \cdot x' - \hat{\pi} \cdot x\right) \\
&= \pi_1(x'_1 - \alpha) + \cdots + \pi_{s^*-1}\left(x'_{s^*-1} - \alpha\right) + \frac{u'(\Omega_{s^*})}{u'(\alpha)}\pi_{s^*}(x'_{s^*} - \Omega_{s^*}) + \cdots + \frac{u'(\Omega_S)}{u'(\alpha)}\pi_S(x'_S - \Omega_S) \\
&\leq \pi_1(x'_1 - \alpha) + \cdots + \pi_{s^*-1}\left(x'_{s^*-1} - \alpha\right) + \pi_{s^*}(x'_{s^*} - \Omega_{s^*}) + \cdots + \pi_S(x'_S - \Omega_S) \\
&= \pi \cdot x' - \pi \cdot x \leq 0.
\end{aligned}$$

Therefore $\hat{\pi} \cdot x' \leq \hat{\pi} \cdot x$.

[17] Such a change is feasible, since $x_{s^*-1} < x_{s^*} \leq \Omega_{s^*} \leq \Omega_{s^*-1}$.

Remaining is to prove the opposite direction, which is evident so omitted. ∎

Lemma 9. *Let* $N \equiv \{1,2\}$. *Let* φ *be efficient and strategy-proof. Then for all* $\gamma, \gamma' \in \mathbb{R}^2_+$ *and all* $\omega \in \mathcal{W}$,

$$P\left(R\left(\gamma_1,\gamma_2\right)\right) = P\left(R\left(\gamma_1',\gamma_2'\right)\right) = C^{\ulcorner} \quad \Rightarrow \quad \varphi\left(R\left(\gamma_1,\gamma_2\right),\omega\right) = \varphi\left(R\left(\gamma_1',\gamma_2'\right),\omega\right);$$
$$P(R\left(\gamma_1,\gamma_2\right)) = P\left(R\left(\gamma_1',\gamma_2'\right)\right) = C^{\lrcorner} \quad \Rightarrow \quad \varphi\left(R\left(\gamma_1,\gamma_2\right),\omega\right) = \varphi\left(R\left(\gamma_1',\gamma_2'\right),\omega\right).$$

Proof. Let $\omega \in \mathcal{W}$. Let $\left(\gamma_1,\gamma_2\right),\left(\gamma_1',\gamma_2'\right) \in \mathbb{R}^2_+$ be such that $P\left(R\left(\gamma_1,\gamma_2\right)\right) = P\left(R\left(\gamma_1',\gamma_2'\right)\right) = C^{\ulcorner}$. Let $z \equiv \varphi\left(R\left(\gamma_1,\gamma_2\right),\omega\right)$ and $z' \equiv \varphi\left(R\left(\gamma_1',\gamma_2'\right),\omega\right)$. Then by Lemma 8, $\gamma_2 = \gamma_2' = 0$.

Suppose to the contrary that $z_1 \neq z_1'$. Then since $C^{\ulcorner}$ is a monotone increasing path from 0 to Ω, either $z_1 \leq z_1'$ or $z_1 \geq z_1'$. Hence, $\varphi_1\left(R\left(\gamma_1,0\right),\omega\right) \leq \varphi_1\left(R\left(\gamma_1',0\right),\omega\right)$ or $\varphi_1\left(R\left(\gamma_1,0\right),\omega\right) \geq \varphi_1\left(R\left(\gamma_1',0\right),\omega\right)$. In either case, we have a contradiction to *strategy-proofness*. ∎

Lemma 10. *Let* $N \equiv \{1,2\}$. *Let* φ *be efficient and strategy-proof. Then for all* $\omega \in \mathcal{W}$, *all* $i \in N$, *and all* $\left(\gamma_1,\gamma_2\right),\left(\gamma_1',\gamma_2'\right) \in \mathbb{R}^2_+$, *if* $P_1\left(R\left(\gamma_1,\gamma_2\right)\right) = C^{\ulcorner}$ *and* $P_1\left(R\left(\gamma_1',\gamma_2'\right)\right) = C^{\lrcorner}$, *then* $\pi \cdot \varphi_i\left(R\left(\gamma_1,\gamma_2\right),\omega\right) = \pi \cdot \varphi_i\left(R\left(\gamma_1',\gamma_2'\right),\omega\right)$.

Proof. Let $\omega \in \mathcal{W}$. Let $\left(\gamma_1,\gamma_2\right),\left(\gamma_1',\gamma_2'\right) \in \mathbb{R}^2_+$ be such that $P\left(R\left(\gamma_1,\gamma_2\right)\right) = C^{\ulcorner}$ and $P\left(R\left(\gamma_1',\gamma_2'\right)\right) = C^{\lrcorner}$. Then $\gamma_1, \gamma_2' > 0$ and $\gamma_2 = \gamma_1' = 0$. Let $z \equiv \varphi\left(R\left(\gamma_1,\gamma_2\right),\omega\right)$, $z' \equiv \varphi\left(R\left(\gamma_1',\gamma_2'\right),\omega\right)$, and $\bar{z} \equiv \varphi\left(R\left(0,0\right),\omega\right)$. We only have to show that $\pi \cdot \bar{z}_1 = \pi \cdot z_1$ and $\pi \cdot \bar{z}_2 = \pi \cdot z_2'$. By *strategy-proofness*, $\bar{z}_1 = \varphi_1\left(R(0,0),\omega\right) \; R_1^0 \; \varphi_1\left(R(\gamma_1,0),\omega\right) = z_1$. Therefore, $\pi \cdot \bar{z}_1 \geq \pi \cdot z_1$. Suppose $\pi \cdot \bar{z}_1 > \pi \cdot z_1$. Then there exists $\gamma_1'' > 0$ such that $\bar{z}_1 \; P_1^{\gamma_1''} \; z_1$. Clearly, $P(R(\gamma_1'',0)) = C^{\ulcorner}$. Hence, by Lemma 9, $\varphi(R(\gamma_1'',0),\omega) = z$. Hence, $\varphi_1\left(R(0,0),\omega\right) \; P_1^{\gamma_1''} \; \varphi_1(R(\gamma_1'',0),\omega)$, contradicting *strategy-proofness*. Therefore, $\pi \cdot \bar{z}_1 = \pi \cdot z_1$. Similarly, we can show $\pi \cdot \bar{z}_2 = \pi \cdot z_2'$. ∎

Now we are ready to prove Theorem 3.

Proof of Theorem 3. Let $\varphi \colon \mathcal{E} \to Z$ be *efficient* and *strategy-proof*. Throughout the proof, we fix $\omega \in \mathcal{W}$ and so we omit this notation. We assume without loss of generality that $\Omega_1 \geq \Omega_2 \geq \cdots \geq \Omega_S$. Let $z^{\ulcorner} \in C^{\ulcorner}$ and $z^{\lrcorner} \in C^{\lrcorner}$ be such that for all $\left(\gamma_1,\gamma_2\right) \in \mathbb{R}^2_+$,

$$P\left(R\left(\gamma_1,\gamma_2\right)\right) = C^{\ulcorner} \quad \Rightarrow \quad \varphi\left(R\left(\gamma_1,\gamma_2\right)\right) = z^{\ulcorner};$$
$$P\left(R\left(\gamma_1,\gamma_2\right)\right) = C^{\lrcorner} \quad \Rightarrow \quad \varphi\left(R\left(\gamma_1,\gamma_2\right)\right) = z^{\lrcorner}.$$

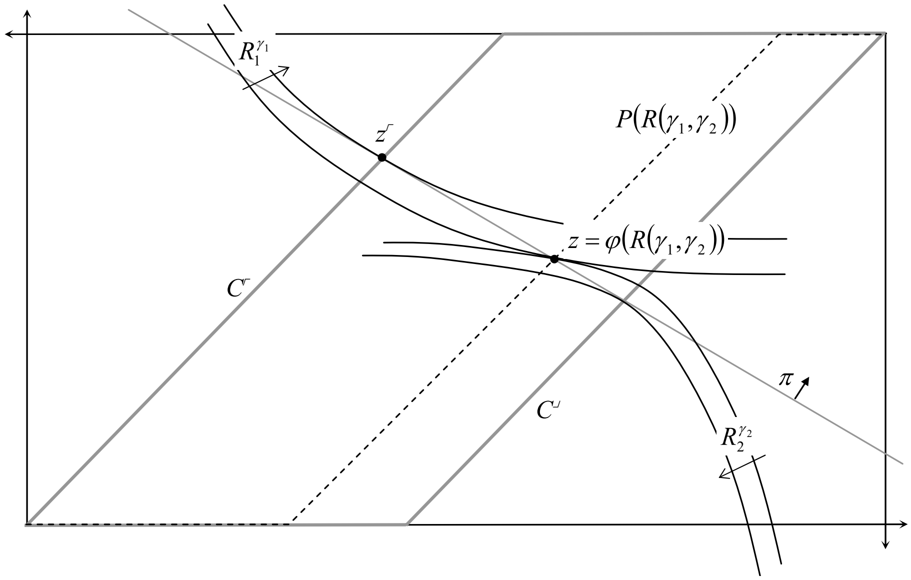

Figure 4: Pareto sets for CES preference economies in the two states case. When $\gamma_1, \gamma_2 > 0$, $P_1(R(\gamma_1, 0)) = C^{\ulcorner}$, $P_1(R(0, \gamma_2)) = C^{\lrcorner}$, and $P_1(R(\gamma_1, \gamma_2))$ is the dotted path depicted above between $C^{\ulcorner}$ and $C^{\lrcorner}$. We show in Step 1 of Proof of Theorem 3 that when φ is *efficient* and *strategy-proof*, $\varphi(R(\gamma_1, \gamma_2))$ is the intersection z of $P(R(\gamma_1, \gamma_2))$ and the hyperplane that passes through $z^{\ulcorner}$ and that is normal to π. Note that $z_1^{\ulcorner}\ P_1^{\gamma_1}\ z_1$.

By Lemma 9, such allocations $z^{\ulcorner}$ and $z^{\lrcorner}$ are well-defined. Our proof is composed of the following five steps.

Step 1. *For all* $(\gamma_1, \gamma_2) \in \mathbb{R}^2_{++}$, $\pi \cdot \varphi_1(R(\gamma_1, \gamma_2)) = \pi \cdot z_1^{\ulcorner}$ (see Figure 4).

Let $z \equiv \varphi(R(\gamma_1, \gamma_2))$. Since $\varphi(R(\gamma_1, 0)) = z^{\ulcorner}$, then when agent 2 has preference R_2^0, by *strategy-proofness*, $\pi \cdot z_2^{\ulcorner} \geq \pi \cdot z_2$. Hence, $\pi \cdot z_1^{\ulcorner} \leq \pi \cdot z_1$. On the other hand, since $\varphi(R(0, \gamma_2)) = z^{\lrcorner}$, then when agent 1 has preference R_1^0, by *strategy-proofness,* $\pi \cdot z_1^{\lrcorner} \geq \pi \cdot z_1$. Therefore, since by Lemma 10, $\pi \cdot z_1^{\lrcorner} = \pi \cdot z_1^{\ulcorner}$, then $\pi \cdot z_1^{\ulcorner} \geq \pi \cdot z_1$.

Step 2. *Suppose that* $\gamma_1, \gamma_2 > 0$. *Then the projection,* $P_1(R(\gamma_1, \gamma_2))$, *of the Pareto set for CES preference profile* $R(\gamma_1, \gamma_2)$ *to agent 1's consumption space is the monotonic path connecting* 0 *and* Ω, *defined as follows (see Figure 4 for the illustration in the two states case): for all* $x \in P_1(R(\gamma_1, \gamma_2))$.
(i) if for some $r \in S$, $x_r = 0$, *then there exists the minimal index* $s \in S$ *such that (i-1)* $x = (x_1, \cdots, x_{s-1}, 0, \cdots, 0)$ and for all $r = 1, \cdots, s-1$, $0 < x_r < \Omega_r$ *and (i-2) for all* $r = 1, \cdots, s-2$, $x_r = x_{s-1} + \frac{\gamma_2}{\gamma_1+\gamma_2}(\Omega_r - \Omega_{s-1})$ *and* $x_{s-1} \leq$

$\frac{\gamma_2}{\gamma_1+\gamma_2}(\Omega_{s-1}-\Omega_s)$*;*
(ii) if for some $r \in S$*,* $x_r = \Omega_r$*, then there exists the minimal index* $s \in S$ *such that (ii-1)* $x = (x_1, \cdots, x_{s-1}, \Omega_s, \cdots, \Omega_S)$ *and for all* $r = 1, \cdots, s-1$*,* $0 < x_r < \Omega_r$ *and (ii-2) for all* $r = 1, \cdots, s-2$*,* $x_r = x_{s-1} + \frac{\gamma_2}{\gamma_1+\gamma_2}(\Omega_r - \Omega_{s-1})$ *and* $x_{s-1} \geq \Omega_s + \frac{\gamma_2}{\gamma_1+\gamma_2}(\Omega_{s-1}-\Omega_s)$*;*
(iii) if for all $r \in S$*,* $0 < x_r < \Omega_r$*, then for all* $r = 1, \cdots, S-1$*,* $x_r = x_S + \frac{\gamma_2}{\gamma_1+\gamma_2}(\Omega_r - \Omega_S)$*.*

First note that if $\gamma_1, \gamma_2 > 0$ and an allocation $(x, \Omega - x)$ is *efficient* for the CES preference profile $R(\gamma_1, \gamma_2)$, that is, $x \in P_1(R(\gamma_1, \gamma_2))$, then[18]

$$x_1 \geq x_2 \geq \cdots \geq x_S; \tag{1}$$

$$\Omega_1 - x_1 \geq \Omega_2 - x_2 \geq \cdots \geq \Omega_S - x_S. \tag{2}$$

Let $x \in P_1(R(\gamma_1, \gamma_2))$. Consider case (i). By (1), $x_s = x_{s+1} = \cdots = x_S = 0$. By the minimality of s, for all $r \leq s-1$, $x_r > 0$. Since $\Omega_s - x_s = \Omega_s$, then by (2), for all $r \leq s-1$, $x_r < \Omega_r$. So we obtain (i-1). By *efficiency* and (i-1), for all $r = 1, \cdots, s-2$, agent 1's marginal rate of substitution of state-r consumption with state-$(s-1)$ consumption at x is the same as agent 2's marginal rate of substitution at $\Omega - x$. Hence $x_r = x_{s-1} + \frac{\gamma_2}{\gamma_1+\gamma_2}(\Omega_r - \Omega_{s-1})$. Since $x_s = 0$, agent 1's marginal rate of substitution of state-$(s-1)$ consumption with state-s consumption at x is greater than or equal to agent 2's marginal rate of substitution at $\Omega - x$. So $x_{s-1} \leq \frac{\gamma_2}{\gamma_1+\gamma_2}(\Omega_{s-1}-\Omega_s)$. Proofs of the other two cases are similar. It can be shown using (i)-(iii) that $P_1(R(\gamma_1, \gamma_2))$ is a monotonic path from 0 to Ω.

Step 3. *Suppose that* $\gamma_1, \gamma_2 > 0$*. Then the projection of the Pareto set for the CES economy* $R(\gamma_1, \gamma_2)$ *to agent 1's consumption space,* $P_1(R(\gamma_1, \gamma_2))$*, converges to* $C^{\ulcorner}$ *as agent 1's degree of risk aversion* γ_1 *increases infinitely and agent 2's degree of risk aversion* γ_2 *is fixed. Similarly, the projection* $P_1(R(\gamma_1, \gamma_2))$ *converges to* $C^{\lrcorner}$ *as agent 2's degree of risk aversion* γ_2 *increases infinitely and agent 1's degree of risk aversion* γ_2 *is fixed.*

[18]Suppose to the contrary that for some $r, s \in S$ with $r < s$, $x_r < x_s$. Then since $\Omega_r \geq \Omega_s$, $\Omega_r - x_r > \Omega_s - x_s$. Therefore since both agents are strictly risk averse, then by increasing state-r consumption and decreasing state-s consumption from x, not changing the expected value of x, and making the opposite change in $\Omega - x$, we can make both better off than $(x, \Omega - x)$, contradicting *efficiency*.

Let $x \in P_1(R(\gamma_1, \gamma_2))$. If x satisfies part (i) of Step 2, then by (i-2),

$$\begin{aligned} ||x - 0|| &\leq \frac{\gamma_2}{\gamma_1 + \gamma_2} \sqrt{\sum_{r=1}^{s-2} (\Omega_{s-1} - \Omega_s + \Omega_r - \Omega_{s-1})^2} \\ &\leq \frac{\gamma_2}{\gamma_1 + \gamma_2} \sqrt{\sum_{r=1}^{s-2} (\Omega_r - \Omega_s)^2}. \end{aligned} \tag{3}$$

If x satisfies part (ii) of Step 2, then by (ii-1) and (ii-2),

$$||x - (x_{s-1}, \cdots, x_{s-1}, \Omega_s, \Omega_{s+1}, \cdots, \Omega_S)|| = \frac{\gamma_2}{\gamma_1 + \gamma_2} \sqrt{\sum_{r=1}^{s-2} (\Omega_r - \Omega_{s-1})^2}. \tag{4}$$

Finally if x satisfies part (iii) of Step 2, then

$$||x - (x_S, x_S, \cdots, x_S)|| = \frac{\gamma_2}{\gamma_1 + \gamma_2} \sqrt{\sum_{r=1}^{S-1} (\Omega_r - \Omega_S)^2}. \tag{5}$$

Since three bundles 0, $(x_{s-1}, \cdots, x_{s-1}, \Omega_s, \Omega_{s+1}, \cdots, \Omega_S)$, and $(x_S, x_S, \cdots, x_S)$ in (3)-(5) are all in $C^\ulcorner$, then for all $x \in P_1(R(\gamma_1, \gamma_2))$,

$$\inf_{y \in C^\ulcorner} ||x - y|| \leq \frac{\gamma_2}{\gamma_1 + \gamma_2} \sqrt{\sum_{r=1}^{S-1} (\Omega_r - \Omega_S)^2}. \tag{6}$$

Note that the right hand side of (6) is independent of $x \in P_1(\gamma_1, \gamma_2)$ and converges to zero as γ_1 increases infinitely and γ_2 is fixed. Therefore the Hausdorff distance between the two sets $P_1(R(\gamma_1, \gamma_2))$ and $C^\ulcorner$ converges to zero.[19]

Step 4. $z_1^\ulcorner \in \{0, \Omega\}$.

Suppose, by contradiction, $z_1^\ulcorner \notin \{0, \Omega\}$. Let $H(\pi, z_1^\ulcorner) \equiv \{x \in \mathbb{R}_+^2 : \pi \cdot x = \pi \cdot z^\ulcorner\}$. Let $\gamma_1, \gamma_2 > 0$. Since $P_1(R(\gamma_1, \gamma_2))$ converges to $C^\lrcorner$ as γ_2 increases infinitely and γ_1 is fixed, then by aggregate uncertainty, we can make $z_1^\ulcorner$ not to be in $P_1(R(\gamma_1, \gamma_2))$ by choosing sufficiently large $\gamma_2 > 0$ (see Figure 4). Let z_1 be the intersection between $P_1(R(\gamma_1, \gamma_2))$ and $H(\pi, z_1^\ulcorner)$ (since $P_1(R(\gamma_1, \gamma_2))$ is a monotonic path and $\pi > 0$, the intersection is a singleton). Then by Step 1,

[19] $\lim_{\gamma_1 \to \infty} \sup_{x \in P_1(R(\gamma_1, \gamma_2))} \inf_{y \in C^\ulcorner} ||x - y|| \leq \lim_{\gamma_1 \to \infty} \frac{\gamma_2}{\gamma_1 + \gamma_2} \sqrt{\sum_{r=1}^{S-1} (\Omega_r - \Omega_S)^2} = 0$. Similarly, we can show $\lim_{\gamma_1 \to \infty} \sup_{x \in C^\ulcorner} \inf_{y \in P_1(R(\gamma_1, \gamma_2))} ||x - y|| = 0$.

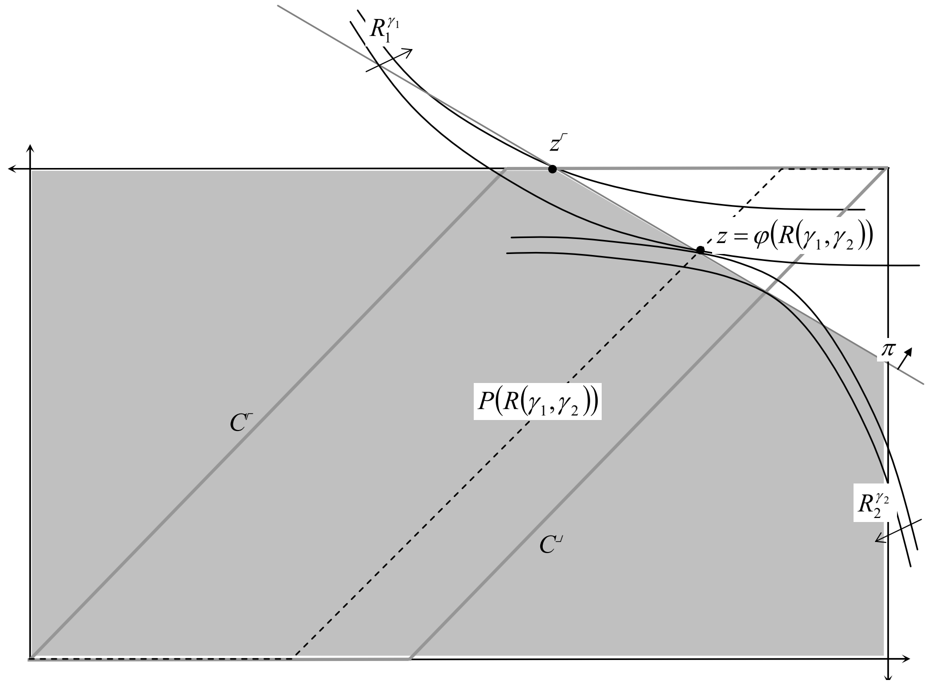

Figure 5: Although $z_1^\ulcorner$ is not a full insurance bundle, $z_1^\ulcorner$ attains the highest welfare level for $R_1^{\gamma_1}$ among feasible allocations that guarantee agent 2 the same expected value (or income) as in $z^\ulcorner$ (the set of such allocations is depicted as the gray area). Hence $z_1^\ulcorner \; P_1^{\gamma_1} \; z_1$.

$z_1 = \varphi_1 (R(\gamma_1, \gamma_2))$. When $z_1^\ulcorner$ is a full insurance bundle, since both $z_1^\ulcorner$ and z_1 have the same expected value, then by strict risk aversion, $z_1^\ulcorner \; P_1^{\gamma_1} \; z_1$. This relation holds, although $z_1^\ulcorner$ is not a full insurance bundle. To explain this, note that $z^\ulcorner$ is an *efficient* allocation for the economy $R(\gamma_1, 0)$ where agent 1 has the same strictly risk averse CES preference $R_1^{\gamma_1}$ and agent 2 has the risk neutral preference R_2^0. Hence $z_1^\ulcorner$ attains the highest welfare level for $R_1^{\gamma_1}$ among feasible allocations that guarantee agent 2 the same expected value as in $z^\ulcorner$. Since allocation $(z_1, \Omega - z_1)$ satisfies this constraint and $R_1^{\gamma_1}$ is *strictly* risk averse, then $z_1^\ulcorner \; P_1^{\gamma_1} \; z_1$ (see Figure 5).

Since $P_1 (R(\gamma_1, \gamma_2))$ converges to $C^\ulcorner$ as γ_1 increases to infinity and γ_2 is fixed, then the intersection between $P_1 (R(\gamma_1, \gamma_2))$ and $H\left(\pi, z_1^\ulcorner\right)$ converges to $z_1^\ulcorner$. Therefore, since $z_1^\ulcorner \; P_1^{\gamma_1} \; z_1$, then for sufficiently large value of γ_1, denoted by γ_1', the intersection between $P_1 (R(\gamma_1', \gamma_2))$ and $H\left(\pi, z_1^\ulcorner\right)$, denoted by z_1', is preferred to z_1 by agent 1 with preference $R_1^{\gamma_1}$ (see Figure 6). Since, by

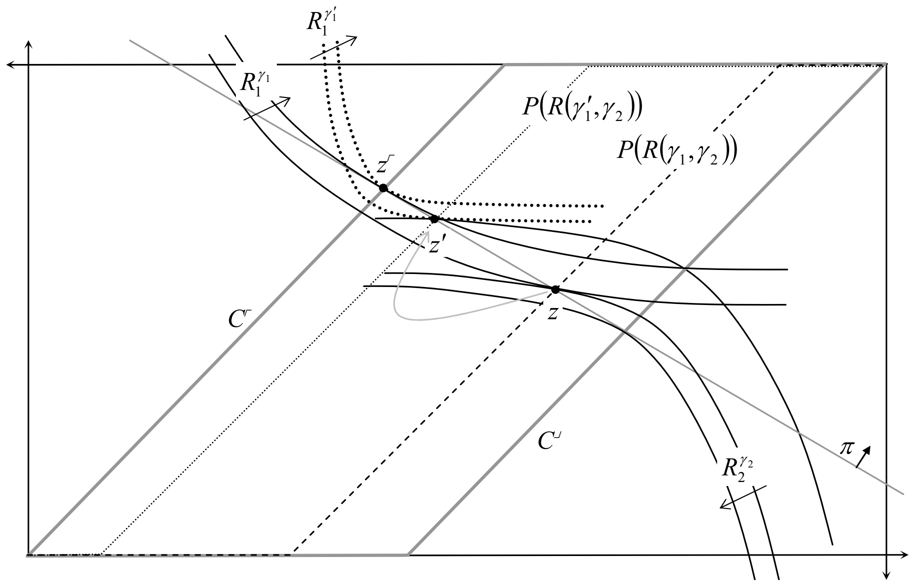

Figure 6: Proof of Step 4 in Proof of Theorem 3. When agent 1 increases his degree of risk aversion sufficiently to γ_1', the outcome changes to z', that is, $\varphi(R(\gamma_1', \gamma_2)) = z'$, which is better than the truthful outcome $z = \varphi(R(\gamma_1, \gamma_2))$.

Step 1, $z_1' = \varphi(R(\gamma_1', \gamma_2))$, φ violates *strategy-proofness*, contradicting the initial assumption.

Step 5. *There exists $i \in \{1, 2\}$ such that for all $R \in \mathcal{R}^N$, $\varphi_i(R) = \Omega$.*

By Step 4, $z_1^{\ulcorner} = 0$ or $z_1^{\ulcorner} = \Omega$. In the first case, consider an arbitrary profile $R \in \mathcal{R}^N$. If (γ_1, γ_2) is such that $P(R(\gamma_1, \gamma_2)) = C^{\ulcorner}$, then $\varphi(R(\gamma_1, \gamma_2)) = (0, \Omega)$. Hence by *strategy-proofness*, when agent 1 has preference $R_1^{\gamma_1}$, he cannot avoid the worst bundle 0 reporting R_1, given the report $R_2^{\gamma_2}$ by agent 2. So $\varphi(R_1, R_2^{\gamma_2}) = (0, \Omega)$. Next by *strategy-proofness*, when agent 2 has preference R_2, he still gets the best bundle Ω, given the report R_1 by agent 1. So $\varphi(R) = (0, \Omega)$. Therefore φ is dictatorial. We apply the same argument for the other case with $z_1^{\ulcorner} = \Omega$. ∎

Remark 2. Proof of Theorem 3 shows that every *efficient* and *strategy-proof* rule defined over $\mathcal{E}_{CARA}$ is dictatorial. Therefore, since preferences are strictly monotonic, the same impossibility result applies in every restricted domain including $\mathcal{E}_{CARA}$.

4.3 Proof of Theorem 4

Throughout this section, we assume that $N \equiv \{1,2\}$ and $\Omega \equiv (\varsigma, \cdots, \varsigma)$, where $\varsigma > 0$ and that each agent $i \in N$ has a fixed non-degenerate belief $\pi^i \in \Delta^{S-1}$. The following subfamilies of preferences are useful. For each $i \in N$, let $\mathcal{R}_{B,\pi^i}$ be the family of *strictly risk averse* EU preferences associated with π^i, of which utility indices are *differentiable* with derivatives nowhere equal to 0 or ∞ over $[0,\varsigma]$. Let $\mathcal{D}_B \equiv \mathcal{R}_{B,\pi^1} \times \mathcal{R}_{B,\pi^2} \times \mathcal{W}$ be the family of all economies with preference profiles in $\mathcal{R}_{B,\pi^1} \times \mathcal{R}_{B,\pi^2}$. Note that for all $R_i \in \mathcal{R}_{B,\pi^i}$ represented by utility index u_i and for all $s,t \in S$, $\{MRS_{s,t}(x;R_i) : x \in Z_0\} = [\frac{\pi_s^i Du_i(\varsigma)}{\pi_t^i Du_i(0)}, \frac{\pi_s^i Du_i(0)}{\pi_t^i Du_i(\varsigma)}]$, where $Du_i(\cdot)$ is the first order derivative of u_i and $MRS_{s,t}(x;R_i) \equiv \frac{\pi_s^i Du_i(x_s)}{\pi_t^i Du_i(x_t)}$ is the "marginal rate of substitution" of state s income with state t income at x. So $MRS_{s,t}(\cdot;R_i)$ is maximized at x with $(x_s,x_t) = (0,\varsigma)$ and minimized at x with $(x_s,x_t) = (\varsigma,0)$. Thus, for all $z \in Z$, $MRS_{s,t}(z_1;R_1) \geq MRS_{s,t}(z_2;R_2)$ if and only if $\frac{\pi_s^1 Du_1(\varsigma)}{\pi_t^1 Du_1(0)} \geq \frac{\pi_s^2 Du_2(0)}{\pi_t^2 Du_2(\varsigma)}$.

In most of our proofs below, we will assume, without loss of generality,[20] that the ratio of two agents' probabilities of each state is decreasing in state index, that is, $\pi_1^1/\pi_1^2 > \cdots > \pi_S^1/\pi_S^2$. Under this assumption, there are economies of which Pareto sets can be described by the following set. Let $\boldsymbol{C^{lr}} \equiv \{x \in Z_0 : x \equiv (\varsigma, \cdots, \varsigma, \alpha, 0, \cdots, 0)$ where $\alpha \in [0,\varsigma]$ is the s-th component of x for some $s \in S\}$ be the "lower right corner" of the Edgeworth box.

Lemma 11. *Assume that* $N \equiv \{1,2\}$ *and* $\Omega \equiv (\varsigma, \cdots, \varsigma)$*, where* $\varsigma > 0$*, and that* $\pi_1^1/\pi_1^2 > \cdots > \pi_S^1/\pi_S^2$*. For all* $R \in \mathcal{D}_B$*,* $P_1(R) = C^{lr}$ *if and only if for all* $s \in S$ *and all* $t > s$*,* $\frac{\pi_s^1 Du_1(\varsigma)}{\pi_t^1 Du_1(0)} \geq \frac{\pi_s^2 Du_2(0)}{\pi_t^2 Du_2(\varsigma)}$.

Proof. For simplicity, we prove the lemma in the two states case. However, our argument can be extended straightforwardly. Let $R \in \mathcal{R}_{B,\pi^1} \times \mathcal{R}_{B,\pi^2}$. Suppose $P_1(R) = C^{lr}$. Then if $\frac{\pi_1^1 Du_1(\varsigma)}{\pi_2^1 Du_1(0)} < \frac{\pi_1^2 Du_2(0)}{\pi_2^2 Du_2(\varsigma)}$, then $((\varsigma,0),(0,\varsigma))$ is not *efficient*, contradicting $P_1(R) = C^{lr}$. Hence $\frac{\pi_1^1 Du_1(\varsigma)}{\pi_2^1 Du_1(0)} \geq \frac{\pi_1^2 Du_2(0)}{\pi_2^2 Du_2(\varsigma)}$. In order to prove the converse, suppose $\frac{\pi_1^1 Du_1(\varsigma)}{\pi_2^1 Du_1(0)} \geq \frac{\pi_1^2 Du_2(0)}{\pi_2^2 Du_2(\varsigma)}$. Then by strict risk aversion, for all $z \in Z$, if $z_1 \notin C^{lr}$, $\frac{\pi_1^1 Du_1(z_{11})}{\pi_2^1 Du_1(z_{12})} > \frac{\pi_1^2 Du_2(z_{21})}{\pi_2^2 Du_2(z_{22})}$. Therefore we can make both agents better off moving toward C^{lr} and so z is not *efficient*. Hence $P_1(R) \subseteq C^{lr}$. The opposite inclusion is evident, since for all $z \in Z$ with $z_1 \in C^{lr}$, $MRS_{1,2}(z_1;R_1) \geq MRS_{1,2}(z_2;R_2)$. ∎

[20] If $\pi_s^1/\pi_s^2 = \pi_{s+1}^1/\pi_{s+1}^2$, then at every *efficient* allocation for economies with strict risk aversion, each agent gets the same amount in state s as in state $s+1$. Otherwise, a Pareto improvement exists because of aggregate certainty and strict risk aversion. Therefore, as long as we are interested in *efficiency*, we can regard both states as a single composite state.

Lemma 12. *Assume that $N \equiv \{1,2\}$ and $\Omega \equiv (\varsigma, \cdots, \varsigma)$, where $\varsigma > 0$, and that $\pi_1^1/\pi_1^2 > \cdots > \pi_S^1/\pi_S^2$. If φ is efficient and strategy-proof, then for all $(R,\omega), (R',\omega') \in \mathcal{D}_B$ with $P_1(R) = P_1(R') = C^{lr}$ and $\omega = \omega'$, $\varphi(R,\omega) = \varphi(R',\omega')$.*

Proof. For simplicity, we prove the lemma in the two states case. However, our argument can be extended straightforwardly. Let φ and $(R,\omega), (R',\omega') \in \mathcal{D}_B$ be given as above. By Lemma 11, $\frac{\pi_1^1 Du_1(\varsigma)}{\pi_2^1 Du_1(0)} \geq \frac{\pi_1^2 Du_2(0)}{\pi_2^2 Du_2(\varsigma)}$ and $\frac{\pi_1^1 Du_1'(\varsigma)}{\pi_2^1 Du_1'(0)} \geq \frac{\pi_1^2 Du_2'(0)}{\pi_2^2 Du_2'(\varsigma)}$. Let R_1'' be one of R_1 and R_1', which has greater MRS at $(\varsigma, 0)$ (or equivalently, R_1'' solves $\max\{\frac{Du_1(\varsigma)}{Du_1(0)}, \frac{Du_1'(\varsigma)}{Du_1'(0)}\}$). Let R_2'' be one of R_2 and R_2', which has smaller MRS at $(0,\varsigma)$ (or equivalently, R_2'' solves $\min\{\frac{Du_2(0)}{Du_2(\varsigma)}, \frac{Du_2'(0)}{Du_2'(\varsigma)}\}$). Then by Lemma 11, $P_1(R_1'', R_2) = P_1(R_1'', R_2') = C^{lr}$ and so $P_1(R_1'', R_2'') = C^{lr}$. Now by *efficiency*, both $\varphi_1((R_1,R_2),\omega)$ and $\varphi_1((R_1'',R_2),\omega)$ are in C^{lr}. Since C^{lr} is a monotonic path and preferences are strictly monotonic, by *strategy-proofness*, $\varphi_1((R_1,R_2),\omega) = \varphi_1((R_1'',R_2),\omega)$. Hence $\varphi((R_1,R_2),\omega) = \varphi((R_1'',R_2),\omega)$. Again by *efficiency*, $\varphi_1((R_1'',R_2''),\omega) \in C^{lr}$. Applying *strategy-proofness* for agent 2, $\varphi_2((R_1'',R_2''),\omega) = \varphi_2((R_1'',R_2),\omega)$. Therefore, $\varphi((R_1,R_2),\omega) = \varphi((R_1'',R_2''),\omega)$. We can show $\varphi((R_1',R_2'),\omega) = \varphi((R_1'',R_2''),\omega)$, using the same argument as above. ∎

Lemma 13. *If $\pi_1^1/\pi_1^2 > \cdots > \pi_S^1/\pi_S^2$ and $x \in C^{lr}\backslash\{0,\Omega\}$, then $\pi^1 \cdot x > \pi^2 \cdot x$.*

Proof. Suppose $\pi_1^1/\pi_1^2 > \cdots > \pi_S^1/\pi_S^2$. We show that for all $s = 0, \cdots, S-1$ and all $\alpha \in [0,\varsigma]$, if (i) $s \neq S-1$ or $\alpha < \varsigma$ and (ii) $s \neq 0$ or $\alpha > 0$,

$$\pi_1^1 \varsigma + \cdots + \pi_s^1 \varsigma + \pi_{s+1}^1 \alpha > \pi_1^2 \varsigma + \cdots + \pi_s^2 \varsigma + \pi_{s+1}^2 \alpha.$$

Note that $\pi_1^1/\pi_1^2 > 1 > \pi_S^1/\pi_S^2$. We first show that for all $s = 1, \cdots, S-1$,

$$\pi_1^1 + \cdots + \pi_s^1 > \pi_1^2 + \cdots + \pi_s^2. \qquad (*)$$

When $s = 1$ or $S-1$, the inequality $(*)$ follows from $\pi_1^1/\pi_1^2 > 1 > \pi_S^1/\pi_S^2$. Now let $s \equiv S-2$. Note that for all $a,b,c,d > 0$, if $a/b > c/d$, then $a/b > (a+c)/(b+d) > c/d$. Using this fact, we obtain $\frac{\pi_1^1 + \cdots + \pi_{S-2}^1}{\pi_1^2 + \cdots + \pi_{S-2}^2} > \frac{\pi_{S-1}^1}{\pi_{S-1}^2}$ and also

$$\frac{\pi_1^1 + \cdots + \pi_{S-2}^1}{\pi_1^2 + \cdots + \pi_{S-2}^2} > \frac{\pi_1^1 + \cdots + \pi_{S-2}^1 + \pi_{S-1}^1}{\pi_1^2 + \cdots + \pi_{S-2}^2 + \pi_{S-1}^2} > \frac{\pi_{S-1}^1}{\pi_{S-1}^2}.$$

Since $\frac{\pi_1^1 + \cdots + \pi_{S-2}^1 + \pi_{S-1}^1}{\pi_1^2 + \cdots + \pi_{S-2}^2 + \pi_{S-1}^2} > 1$, then $\frac{\pi_1^1 + \cdots + \pi_{S-2}^1}{\pi_1^2 + \cdots + \pi_{S-2}^2} > 1$, establishing the inequality $(*)$ for $s = S-2$. Proceeding this way, we can establish the inequality $(*)$ for all

$s = 1, \cdots, S-1$.

Now we complete the proof using the above inequality $(*)$. Let $\alpha \in [0, \varsigma]$ and $s = 2, \cdots, S-2$. Then

$$\begin{aligned}
&\pi_1^1 \varsigma + \cdots + \pi_s^1 \varsigma + \pi_{s+1}^1 \alpha \\
&= (\varsigma - \alpha)\left(\pi_1^1 + \cdots + \pi_s^1\right) + \alpha\left(\pi_1^1 + \cdots + \pi_s^1 + \pi_{s+1}^1\right) \\
&> (\varsigma - \alpha)\left(\pi_1^2 + \cdots + \pi_s^2\right) + \alpha\left(\pi_1^2 + \cdots + \pi_s^2 + \pi_{s+1}^2\right) \\
&= \pi_1^2 \varsigma + \cdots + \pi_s^2 \varsigma + \pi_{s+1}^2 \alpha.
\end{aligned}$$

Remaining is the proof for $s = 1$ and $s = S-1$, which is the same. ∎

Proof of Theorem 4. Let $\Omega \equiv (\varsigma, \cdots, \varsigma)$, where $\varsigma > 0$. By Theorem 1, if $\pi^1 = \pi^2$, there exist *efficient*, *individually rational*, and *strategy-proof* rules. We prove the converse by showing that when $\pi^1 \neq \pi^2$, there exists no rule satisfying the three requirements. For simplicity, we prove the non-existence in the two states case. However, our argument can be extended straightforwardly.

Suppose by contradiction that there is a rule φ satisfying the three requirements. In what follows, we will fix $\omega_1 = \omega_2 = \Omega/2$ and, therefore, we will skip notation ω. In what follows, we suppose, without loss of generality, that $\pi_1^1/\pi_1^2 > \cdots > \pi_S^1/\pi_S^2$. By Lemma 12, there is $z^{lr} \in Z$ such that for all $R \in \mathcal{D}_B$, if $P_1(R) = C^{lr}$, then $\varphi(R) = z^{lr}$. Note that by *individual rationality*, $z_1^{lr} \notin \{0, \Omega\}$. For each $i \in N$, let R_i^{neut} be the risk neutral preference with belief π_i. We derive a contradiction through the following three steps.

Step 1. *For all $i \in N$ and all $R_i \in \mathcal{R}_B$ with belief π_i, if $P_1\left(R_i, R_{-i}^{neut}\right) = C^{lr}$, then $\varphi\left(R_i, R_{-i}^{neut}\right) = z^{lr}$.*

Without loss of generality, let $i = 2$. Let R_2 be a preference in $\mathcal{R}_B$ with belief π^2 such that $P_1\left(R_1^{neut}, R_2\right) = C^{lr}$. Thus, $MRS_{1,2}$ of R_2 at $(0, \varsigma)$ is less than or equal to π_1^1/π_2^1. Note that using Lemma 11, we can find R_1 such that $(R_1, R_2) \in \mathcal{D}_B$ and $P_1(R_1, R_2) = C^{lr}$.[21] Then by *strategy-proofness*, $\varphi_1\left(R_1^{neut}, R_2\right) = \varphi_1(R_1, R_2) = z_1^{lr}$. Therefore $\varphi\left(R_1^{neut}, R_2\right) = z^{lr}$.

Step 2. *For all $i \in N$, there is a preference $R_i^* \in \mathcal{R}_B$ with belief π_i such that $P_1\left(R_i^*, R_{-i}^{neut}\right) = C^{lr}$ and the certainty equivalent of z_i^{lr}, denoted by $CE\left(z_i^{lr}; R_i^*\right)$, is less than $\left(\mu\left(z_i^{lr}; \pi^1\right) + \mu\left(z_i^{lr}; \pi^2\right)\right)/2$, where $\mu\left(z_i^{lr}; \pi_j\right) = \pi_j \cdot z_i^{lr}$ for all $j = i, -i$.*

Without loss of generality, let $i = 2$. Since $z_1^{lr} \in C^{lr} \backslash \{0, \Omega\}$, then by Lemma 13, $\mu\left(z_1^{lr}; \pi^1\right) > \mu\left(z_1^{lr}; \pi^2\right)$ and so $\mu\left(z_2^{lr}; \pi^1\right) < \mu\left(z_2^{lr}; \pi^2\right)$. Let $\hat{u}_2$ be the piecewise

[21] This is possible, since we assume $\frac{\pi_1^1}{\pi_1^2} > \cdots > \frac{\pi_S^1}{\pi_S^2}$.

linear index function defined as follows: for all $m \geq 0$,

$$\hat{u}_2(m) \equiv \begin{cases} \alpha m, \text{ if } m \leq \mu\left(z_2^{lr};\pi^1\right); \\ \beta m, \text{ if } m > \mu\left(z_2^{lr};\pi^1\right), \end{cases}$$

where $\alpha > \beta > 0$ and $\frac{\pi_1^2\alpha}{\pi_2^2\beta} = \frac{\pi_1^1}{\pi_2^1}$. Let $\hat{R}_2$ be represented by $\hat{u}_2$ and π^2. Then $\hat{R}_2$ is risk averse and $CE(z_2^{lr}, \hat{R}_2) = \mu\left(z_2^{lr};\pi^1\right)$. For each $\delta \in (0,1)$, smoothen $\hat{u}_2$ over $\left(\mu\left(z_2^{lr};\pi^1\right) - \delta, \mu\left(z_2^{lr};\pi^1\right) + \delta\right)$ and preserve $\hat{u}_2$ outside this area. Then we obtain a concave and differentiable index function $\hat{u}_2^\delta$. Note that $\hat{u}_2^\delta$ converges to $\hat{u}_2$ as δ goes to 0 and that the slope of $\hat{u}_2^\delta$ is between α and β. Let $\bar{u}_2$ be a differentiable and strictly concave index function of which slope is bounded above by α and below by β. For each $\delta \in (0,1)$, let $u_2^\delta \equiv (1-\delta)\hat{u}_2^\delta + \delta\bar{u}_2$. Then as δ goes to 0, u_2^δ converges to $\hat{u}_2$ and moreover for all δ and all $x \in Z_0$, $\frac{\pi_1^2 Du_2^\delta(x_1)}{\pi_2^2 Du_2^\delta(x_2)} \leq \frac{\pi_1^2\alpha}{\pi_2^2\beta} = \frac{\pi_1^1}{\pi_2^1}$. Therefore, if we let R_2^δ be the preference represented by u_2^δ and π^2, then $P_1\left(R_1^{neut}, R_2^\delta\right) = C^{lr}$ and $CE\left(z_2^{lr}, R_2^\delta\right)$ converges to $CE(z_2^{lr}; \hat{R}_2) = \mu\left(z_2^{lr};\pi^1\right)$ as δ goes to 0. Therefore, for sufficiently small $\delta^* > 0$, $P_1\left(R_1^{neut}, R_2^{\delta^*}\right) = C^{lr}$ and $CE\left(z_2^{lr}, R_2^{\delta^*}\right) < \left(\mu\left(z_2^{lr};\pi^1\right) + \mu\left(z_2^{lr};\pi^2\right)\right)/2$. So if we let $R_2^* \equiv R_2^{\delta^*}$, the desired property is met.

Step 3. We now complete the proof. Let R_1^* and R_2^* be the two preferences described in Step 2. Since $\varphi\left(R_1^{neut}, R_2^*\right) = z^{lr}$, then by *individual rationality* of agent 2 with R_2^*, $\varsigma/2 \leq CE\left(z_2^{lr}; R_2^*\right)$ (note that both agents have the same individual endowment $(\varsigma/2, \varsigma/2)$). Therefore, $\varsigma/2 < \left(\mu\left(z_2^{lr};\pi^1\right) + \mu\left(z_2^{lr};\pi^2\right)\right)/2$. Applying the same argument for (R_1^*, R_2^{neut}), $\varsigma/2 < \left(\mu\left(z_1^{lr};\pi^1\right) + \mu\left(z_1^{lr};\pi^2\right)\right)/2$, that is, $\varsigma/2 > \left(\mu\left(z_2^{lr};\pi^1\right) + \mu\left(z_2^{lr};\pi^2\right)\right)/2$, which contradicts the previous inequality. ∎

4.4 Proof of Theorem 5

Consider the multiple goods domain $\mathcal{E}_{l\text{-goods}}$ with $l \geq 2$. In what follows, we focus on the family of economies with fixed aggregate endowment $\mathbf{\Omega} \equiv (\mathbf{\Omega}_s)_{s\in S} \in \mathbb{R}^{l\times S}$ and fixed individual endowments $(\boldsymbol{\omega}_i)_{i\in N} \in \mathbb{R}^{l\times n}$. Since both endowments and the common prior π are fixed, we can skip endowments and denote preferences by their utility indices, when describing an economy. To distinguish notation in the multiple goods model from the single good model, we use $\boldsymbol{\varphi}$ to denote rules over $\mathcal{E}_{l\text{-goods}}$, $\mathbf{z}$ to denote allocations, and $\mathbf{z}_i$ to denote i's bundle at $\mathbf{z}$. We first establish a useful lemma.

Lemma 14. *Suppose that there exist $l \geq 2$ goods and $\mathbf{\Omega}_1 = \cdots = \mathbf{\Omega}_S$. For all agent $i \in N$, let f_i be a continuous, monotonic, and concave real valued function* *over $\mathbb{R}$ and let $u_i \in \mathcal{U}$ be a concave utility index function over $\mathbb{R}^l_+$. If there is an agent $i \in N$ such that f_i or u_i is strictly concave, then for all efficient allocations $\mathbf{z}$ for the economy with preferences $(f_j \circ u_j)_{j \in N}$, we have $u_i(\mathbf{z}_{i1}) = \cdots = u_i(\mathbf{z}_{iS})$.*

Proof. Let $i \in N$ be the agent for whom f_i or u_i is strictly concave. Suppose to the contrary that for some $r, s \in S$, $u_i(\mathbf{z}_{ir}) \neq u_i(\mathbf{z}_{is})$. Let $\mathbf{z}'$ be such that for all $j \in N$, $\mathbf{z}'_j \equiv \sum_{s \in S} \pi_s \mathbf{z}_{js}$. Then by aggregate certainty, $\mathbf{z}'$ is also feasible. For all $j \in N$, by concavity and monotonicity of f_j and u_j,

$$f_j\left(u_j\left(\sum_{s \in S} \pi_s \mathbf{z}_{is}\right)\right) \geq f_j\left(\sum_{s \in S} \pi_s u_j(\mathbf{z}_{js})\right) \geq \sum_{s \in S} \pi_s f_j(u_j(\mathbf{z}_{js})).$$

If f_i is strictly concave, then the second inequality is strict for $j = i$, since $u_i(\mathbf{z}_{ir}) \neq u_i(\mathbf{z}_{is})$. If u_i is strictly concave, then the first inequality is strict for $j = i$. Therefore, $\mathbf{z}'$ is a Pareto improvement upon $\mathbf{z}$, contradicting *efficiency.* ∎

Now we are ready to prove Theorem 5.

Proof of Theorem 5. Suppose that $\boldsymbol{\varphi}$ is *efficient* and *strategy-proof.* We will show that $\boldsymbol{\varphi}$ is dictatorial, applying a result by Ju (2001) and also Theorem 3. Note first that for all $u \in \mathcal{U}^N$, $\boldsymbol{\varphi}(u)$ is "ex post efficient", that is, for all $s \in S$, $\boldsymbol{\varphi}_s(u)$ is *efficient* for the ex post exchange economy with preference profile u and endowment $\boldsymbol{\Omega}_s$. Our proof is divided into two cases.

Case 1. *Aggregate certainty case, that is,* $\boldsymbol{\Omega}_1 = \cdots = \boldsymbol{\Omega}_S$.

Let $\mathcal{U}_0$ be a family of concave utility functions u in $\mathcal{U}$ of the following quasi-linear form: for all $\mathbf{x} \in \mathbb{R}^l_+$, $u(\mathbf{x}) \equiv a\frac{\mathbf{x}_1}{\boldsymbol{\Omega}_{s1}} + \sum_{k=2}^l \left(\frac{\mathbf{x}_k}{\boldsymbol{\Omega}_{sk}} + 1\right)^\rho$, where $a > 0$ and $\rho \in (0,1)$. Note that the projection of *Pareto set* of the ex post economy $(u, \boldsymbol{\Omega}_s)$ to agent 1's consumption space is a monotonic path from $\mathbf{0}$ to $\boldsymbol{\Omega}_s$. Fix a strictly concave function f over $\mathbb{R}$. Let $\mathcal{U}_*$ be the family of utility functions $v \equiv f \circ u$, where $u \in \mathcal{U}_0$. We show that $\boldsymbol{\varphi}$ is "ex post strategy-proof over $\mathcal{U}^N_*$", that is, for all $u \in \mathcal{U}^N_*$ and all $u'_i \in \mathcal{U}_*$, $u_i(\boldsymbol{\varphi}_{si}(u_i, u_{-i})) \geq u_i(\boldsymbol{\varphi}_{si}(u'_i, u_{-i}))$. To show this, suppose by contradiction that $u_i(\boldsymbol{\varphi}_{si}(u_i, u_{-i})) < u_i(\boldsymbol{\varphi}_{si}(u'_i, u_{-i}))$. Then by *strategy-proofness*, there exists another state $r \neq s$ such that $u_i(\boldsymbol{\varphi}_{ri}(u_i, u_{-i})) > u_i(\boldsymbol{\varphi}_{ri}(u'_i, u_{-i}))$. By aggregate certainty, both $\boldsymbol{\varphi}_{si}(u'_i, u_{-i})$ and $\boldsymbol{\varphi}_{ri}(u'_i, u_{-i})$ are *efficient* for the ex post economy $((u'_i, u_{-i}), \boldsymbol{\Omega}_s)$. Since the Pareto set for the ex post economy is a monotonic path, then $u'_i(\boldsymbol{\varphi}_{si}(u'_i, u_{-i})) \neq u'_i(\boldsymbol{\varphi}_{ri}(u'_i, u_{-i}))$, contradicting the previous lemma.

It is shown by Ju (2001) (see Proof of Proposition 5 in Ju (2001)) that if a rule over the domain $\mathcal{U}^N_*$ of ex post economies is *efficient* and *strategy-proof*, then it is dictatorial. Therefore, $\boldsymbol{\varphi}$ is "ex post dictatorial over $\mathcal{U}_*^N$", that is, for all $s \in S$, $\boldsymbol{\varphi}_s$ is *dictatorial* over $\mathcal{U}_*^N$. By the previous lemma, we conclude that $\boldsymbol{\varphi}$ is *dictatorial* over $\mathcal{U}_*^N$. Finally, using the same argument as in the final paragraph of the proof of Theorem 3, we can show that $\boldsymbol{\varphi}$ is *dictatorial* over the entire domain $\mathcal{U}^N$.

Case 2. *Aggregate uncertainty case.*

By aggregate uncertainty, there are at least two states, $r, s \in S$, with different endowments, that is, $\boldsymbol{\Omega}_r \neq \boldsymbol{\Omega}_s$. Then there exists a vector $\bar{u} \in \mathbb{R}^l_{++}$ such that $\bar{u} \cdot \boldsymbol{\Omega}_r \neq \bar{u} \cdot \boldsymbol{\Omega}_s$. Consider the linear function $\bar{u} \colon \mathbb{R}^l_+ \to \mathbb{R}$ defined as follows: for all $\mathbf{x} \in \mathbb{R}^l_+$, $\bar{u}(\mathbf{x}) \equiv \bar{u} \cdot \mathbf{x}$. Now for each monotonic and concave function $f \colon \mathbb{R}_+ \to \mathbb{R}$, let $u^f \equiv f \circ \bar{u}$ be the concave transformation of $\bar{u}$ by f. Let $\mathcal{U}_{\bar{u}}$ be the collection of all such concave transformations of $\bar{u}$ and $\mathcal{E}_{\bar{u},l\text{-goods}} \equiv \mathcal{U}_{\bar{u}} \times \mathcal{W}$. Then we can embed the restricted domain $\mathcal{E}_{\bar{u},l\text{-goods}}$ in the single good model as follows. With each l-goods problem $\left(\left(u^{f_i}\right)_{i \in N}, (\boldsymbol{\omega}_i)_{i \in N}\right) \in \mathcal{E}_{\bar{u},l\text{-goods}}$, we associate the single good problem $(f_i, \omega_i)_{i \in N}$, where f_i is i's utility index and $\omega_i \equiv \bar{u} \cdot \boldsymbol{\omega}_i$. For each single good problem $(f_i, \omega_i)_{i \in N}$ and each $j \in N$, let $\varphi_j\left((f_i)_{i \in N}\right) \equiv \bar{u} \cdot \boldsymbol{\varphi}_j(\left(u^{f_i}\right)_{i \in N})$. Then since $\boldsymbol{\varphi}$ is *efficient* and *strategy-proof*, so is φ over the single good domain. Now applying Theorem 3, φ is dictatorial. Hence $\boldsymbol{\varphi}$ is also dictatorial over $\mathcal{E}_{\bar{u},l\text{-goods}}$. Finally, using the same argument as in the final paragraph of the proof of Theorem 3, we can show that $\boldsymbol{\varphi}$ is *dictatorial* over the entire domain. ∎

5 Concluding remarks

We showed that the well-known conflict between *efficiency* and *strategy-proofness* in exchange economies prevails also in risk sharing problems when aggregate uncertainty holds (Theorems 2 and 3). The conflict disappears when aggregate certainty holds. In this case, rules selecting from the Walrasian equilibrium allocations based on a fixed price are shown to be the only rules satisfying the two requirements as well as *individual rationality* (Theorem 1).

In our model, the profile of individual endowments is variable. Such a feature does not play an essential role in our results except Theorem 2. Proof of Theorem 2 crucially relies on the admissibility of individual endowments profiles in which the endowment of an agent is sufficiently large. We leave the proof of the same impossibility result for fixed individual endowments economies for the future research.

In the aggregate uncertainty case, dropping *individual rationality*, we showed that when there are only two agents, dictatorial rules are the only *efficient* and *strategy-proof* rules (Theorem 3). It remains as an open question whether a similar result holds for the case of arbitrary number of agents.

It is also left for the future research to investigate *strategy-proofness*, apart from *efficiency*, however, in conjunction with several auxiliary requirements such as *individual rationality,* "non-bossiness" (Satterthwaite and Sonnenschein, 1981), "symmetry", "continuity", etc. In exchange economies, Barberà and Jackson (1995) establish several characterization results in this regard. They identify certain interesting families of *strategy-proof* rules violating *efficiency.* These rules continue to be *strategy-proof* in risk sharing problems. However, it remains as an open question whether the same uniqueness results can be obtained.

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