Non-Manipulable Division Rules in Claim Problems and Generalizations

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Abstract

When resources are divided among agents, the characteristics of the agents are taken into consideration. A simple example is the bankruptcy problem, where the liquidation value of a bankrupt firm is divided among the creditors based on their claims. We characterize division rules under which no group of agents can increase the total amount they receive by transferring their characteristics within the group. By allowing agents' characteristics to be multi-dimensional and choosing the meaning of variables appropriately, our model can subsume a number of existing and new allocation problems, such as cost sharing, social choice with transferable utilities, income redistribution, bankruptcy with multiple types of assets, probability updating, and probability aggregation. A number of existing and new results in specific problems are obtained as corollaries.

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1 Introduction

Resource allocation problems often take the following form. There is an amount of a homogeneous and infinitely divisible good (e.g., money) to be divided among a set of agents, and each agent's relevant characteristics are summarized by a vector. For example, when the liquidation value of a bankrupt firm is divided among its creditors, the relevant characteristics of each creditor are the amount of his claim, possibly categorized by the type of assets. Similarly, when the cost of a service is divided among its users, each user's usage level will be taken into account. For allocation problems of this kind, we study allocation rules that assign an allocation to each possible problem in a way that is immune to strategic transfers of characteristics among agents. That is, we search for allocation rules such that no group of agents can increase the total amount they receive by reallocating their characteristic vectors within the group in advance. This non-manipulability condition, introduced by Moulin [16], is called *reallocation-proofness*.

As an illustration, consider the standard bankruptcy problem (with a single type of assets) studied by O'Neill [20] and Aumann and Maschler [2].[1] Suppose that a bankrupt firm has 3 creditors and the amounts that the firm owes to these creditors are $(c_1, c_2, c_3) = (1, 3, 3)$. Suppose also that the firm's liquidation value is 6, which is not enough to pay off all the creditors in full. For this specific problem, an allocation rule assigns an award vector (x_1, x_2, x_3) with $x_1 + x_2 + x_3 = 6$.

A well-known allocation rule in this context is the *constrained equal award rule*, which chooses the award vector that is closest to equal division subject to the constraint that no creditor gets more than his claim: $x_i \leq c_i$ for each i.[2] For the above problem, the rule chooses $(x_1, x_2, x_3) = (1, 2.5, 2.5)$. While this rule has great appeal and plays a prominent role in the literature, it is manipulable via transfers of claims among creditors. Indeed, if creditor 2 transfers one unit of his claim to creditor 1, the claim vector changes to $(c'_1, c'_2, c'_3) = (2, 2, 3)$ and the constrained equal award rule chooses $(x'_1, x'_2, x'_3) = (2, 2, 2)$. Thus the total award to creditors 1 and 2 increases to 4 from 3.5. With an appropriate side payment from 1 to 2, both creditors gain from the manipulation.

An example of a reallocation-proof rule is the *proportional rule*, which divides the firm's value proportionally to claims. For the above example, this rule chooses $(x_1, x_2, x_3) = (6/7, 18/7, 18/7)$. The total awards to creditors 1 and 2, $x_1 + x_2 = 6(c_1 + c_2)/(c_1 + c_2 + c_3)$, depend on c_1 and c_2 only through the sum $c_1 + c_2$, which shows that the rule is reallocation-proof.

We consider a general class of allocation problems including bankruptcy problems as a special case. Characteristics c_i are allowed to be multi-dimensional, which enables us to deal with, for example, social choice with transferable utilities (where c_i denotes i's valuation function) as well as bankruptcy problems with multiple types of assets. Further, the amount to divide may depend on characteristic vectors, as in the problems of cost sharing (where c_i is i's usage level) and income redistribution (where c_i is i's income level).

The model can also formulate rather different sets of problems if we vary the meaning of vari-

[1] The class of bankruptcy problems with a single type of assets was introduced by O'Neill [20], and since then a large literature has been developed. See Moulin [19] and Thomson [24, 26] for surveys.

[2] Formally, the rule assigns $x_i = \min\{\lambda, c_i\}$ where λ is uniquely determined by $\sum_{i=1}^{3} \min\{\lambda, c_i\} = E$, where E is the liquidation value ($E = 6$ in the example).

ables in the model. For example, by replacing "agents" with "states of the world" and "awards" with "probabilities," we can consider problems of probability updating (Rubinstein and Zhou [22] and Majumdar [14][3]) and probability aggregation (McConway [15] and Rubinstein and Fishburn [21]), where probabilities are allocated among the states. In the probability aggregation problem, for instance, a set of forecasters have their own forecasts, or probabilistic beliefs over the states. What an allocation rule does is to pool these forecasts as inputs and specifies a single forecast. Thus c_i is the vector of probabilities assigned to state i by the forecasters. A well-known aggregation scheme is to take a weighted average of the probability distributions, which is called a *linear opinion pool* (McConway [15]) and is reallocation-proof. In this context, reallocation-proofness means informational efficiency: when the forecasters are interested in an event but not the individual states constituting it, they can treat the event as a single composite state without any loss.

Our main result characterizes reallocation-proof rules. We show that any reallocation-proof rule can be written as the sum of two parts: a "priority part," which may treat agents differently based on their identities but ignores differences among their characteristics, and an "additive part," which treats agents equally and depends on characteristics additively. If the rule satisfies a mild boundedness condition, the additive part is proportional to characteristics. This class of rules includes the proportional rule, equal division, and weighted versions of "equal-distance" type rules, and is closed under convex combinations.

Several existing results in specialized contexts are obtained as corollaries. In particular, our results generate the characterizations of the proportional rule in O'Neill [20], Chun [8], de Frutos [9], Ching and Kakker [7], Chambers and Thomson [6], and Moulin [19]; some families of rules in Chun [8] and Moulin [16, 17]; and linear opinion pools in McConway [15]. We also show that, for the characterization of the proportional rule, reallocation-proofness can be weakened to a version that considers only coalitions of size two.

Our results together with the generality of the model generate new results as well. For example, the multi-dimensional setting enables us to consider priorities among types of assets in multi-dimensional bankruptcy problems: e.g., claims based on bonds should be reimbursed prior to claims based on stocks. We characterize proportional rules that respect exogenously given priorities. We also give a new characterization in income redistribution problems. We show that the only way for an income redistribution scheme to satisfy reallocation-proofness and avoid a transfer paradox is to use income tax with a flat tax rate and personalized lump-sum transfers.

The remainder of the paper is organized as follows. The next section introduces the model. Section 3 defines axioms. Section 4 defines proportional rules and generalizations. Section 5 presents the main results. Section 6 gives applications of the results in specialized problems where the set of agents is fixed. Section 7 considers problems where the set of agents is also variable; in particular, we give a characterization of a closely related axiom called *merging-splitting-proofness*. Section 8 discusses a few ways to extend the model and the robustness of our results.

[3]Belief updating is also studied by Gilboa and Schmeidler [10] in a preference-based framework with non-additive probabilities and multiple priors, and by Stalnaker [23] in a theory of conditionals.

2 Model

There is a finite set $N = \{1, 2, \dots, |N|\}$ of *entities*.[4] Each entity $i \in N$ is characterized by a finite dimensional vector $c_i \equiv (c_{ik})_{k \in K} \in \mathbb{R}^K_+$ where $K = \{1, 2, \dots, |K|\}$ is a finite set of issues.[5] We refer to c_i as i's *characteristic vector*. A profile of characteristic vectors is denoted by $c \equiv (c_i)_{i \in N} \in \mathbb{R}^{N \times K}_+$ and the sum of these vectors is denoted by

$$\bar{c} \equiv (\bar{c}_k)_{k \in K} \equiv (\sum_{i \in N} c_{ik})_{k \in K} \in \mathbb{R}^K_+.$$

A *problem* is a pair $(c, E) \in \mathbb{R}^{N \times K}_+ \times \mathbb{R}_{++}$, where $c \in \mathbb{R}^{N \times K}_+$ is a profile of characteristic vectors and $E \in \mathbb{R}_{++}$ is the amount of a homogeneous and infinitely divisible good to be divided.[6] To avoid introducing uninteresting complication to the exposition, we only consider problems such that $\bar{c}_k > 0$ for each $k \in K$.[7]

A *domain* is a non-empty set of problems and is denoted by $\mathscr{D}$. A division rule, or briefly, a *rule* over a domain $\mathscr{D}$ is a function f associating with each problem $(c, E) \in \mathscr{D}$ a vector of awards $f(c, E) \in \mathbb{R}^N$. A domain $\mathscr{D}$ is *rich* if it is closed under reallocations of characteristic vectors: for each problem $(c, E) \in \mathscr{D}$ and each profile $c' \in \mathbb{R}^{N \times K}_+$ such that $\bar{c'} = \bar{c}$, we have $(c', E) \in \mathscr{D}$. We restrict our attention to rich domains.

Here are examples of rich domains.

Bankruptcy. As mentioned in the introduction, a bankruptcy problem deals with the division of the liquidation value E of a bankrupt firm among the set of its creditors N. Here, K is the set of assets and c_{ik} the claim that creditor i holds in the form of asset k. Thus, the domain of bankruptcy problems is given by $\mathscr{D} = \{(c, E) \in \mathbb{R}^{N \times K}_+ \times \mathbb{R}_{++} : \sum_{k \in K} \bar{c}_k \geq E, \text{ and } \bar{c}_k > 0 \text{ for all } k \in K\}$.

Bankruptcy problems with $|K| = 1$ can also be interpreted as problems of collecting income tax, where c_i is taxpayer i's income level and E is the amount of tax revenues to be collected (Young [27]).

In what follows, bankruptcy problems refer to the case of $|K| = 1$, unless stated otherwise.

Surplus Sharing. The problem is to divide the profit from a project among contributors (Young [27]). Here, $|K| = 1$, c_i is the amount of the opportunity cost for contributor i, and $E \geq \sum_{i \in N} c_i$ is the profit that the project generates. Thus $\mathscr{D} = \{(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++} : 0 < \sum_{i \in N} c_i \leq E\}$.

Claim Problems. This domain is simply the union of the domains of (single-dimensional) bank-

[4]This rather neutral term is used because the meaning of N varies with the context.

[5]We use the following notation for vector inequalities: given $x, y \in \mathbb{R}^M$, $x \geqq y$ means that $x_m \geq y_m$ for each m; $x \geq y$ means that $x \geqq y$ and $x \neq y$; and $x > y$ means that $x_m > y_m$ for each m.

[6]For simplicity, we only consider non-negative characteristic vectors and positive amounts to divide. Our results hold in more general settings, as we discuss in Section 8.

[7]Even if we allow for problems such that $\bar{c}_k = 0$ for some k, the main results and proofs go through with no technical difficulty. We just need to replace summations $\sum_{k \in K}$ with $\sum_{\{k \in K : \bar{c}_k > 0\}}$. On the other hand, if we allow for problems where $\bar{c}_k = 0$ for all k, then reallocation-proofness has no bite and allows for any allocation for those problems. However, for those problem, two basic axioms, efficiency and no award for null (to be defined later), are incompatible.

ruptcy and surplus-sharing problems (Moulin [17] and Chun [8]).[8] That is, no inequality between E and $\sum_{i\in N} c_i$ is imposed. Thus $\mathscr{D} = \{(c,E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++} : \sum_{i\in N} c_i > 0\}$.

Social Choice with Transferable Utilities. Let N be the set of agents and K be the set of possible public projects, one of which has to be chosen. Each agent $i \in N$ has a quasi-linear utility function $u_i(k,m_i) = c_{ik} + m_i$ ($c_{ik} \geq 0$) where $k \in K$ denotes the chosen project and $m_i \in \mathbb{R}$ denotes the side-payment to agent i. A feasible allocation is a list $(k,m) \in K \times \mathbb{R}^N$ such that $\sum_{i\in N} m_i = 0$. Note that $\sum_{i\in N} u_i(k,m_i) = \overline{c}_k$. Under Pareto efficiency, a project $k \in \arg\max_{k'\in K} \overline{c}_{k'}$ is chosen. Given such a project k, any utility allocation $x \in \mathbb{R}^N$ with $\sum_{i\in N} x_i = \overline{c}_k$ is attainable through monetary transfers. Hence, in utility terms, the problem is to divide $E \equiv \max_{k\in K} \overline{c}_k$ among the agents. Thus $\mathscr{D} = \{(c,E) \in \mathbb{R}^{N\times K}_+ \times \mathbb{R}_{++} : E = \max_{k\in K} \overline{c}_k$, and $\overline{c}_k > 0$ for each $k \in K\}$. This class of problems is studied by Moulin [16]. It differs from the previous examples in that E depends on c.

Cost Sharing. Let N be the set of agents and $|K| = 1$. Each agent $i \in N$ has a demand $c_i \geq 0$ for a service. For each profile of demands $c \in \mathbb{R}^N_+$, the aggregate cost to be shared among the agents is given by $C(\overline{c})$, where $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ is the cost function. Then $\mathscr{D} = \{(c,E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++} : E = C(\overline{c})\}$.

Income Redistribution. Let N be the set of agents and $|K| = 1$. Each agent $i \in N$ has income $c_i \geq 0$. The problem is to redistribute the incomes among the agents. Thus $\mathscr{D} = \{(c,E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++} : E = \overline{c}\}$.

Probability Updating. Let N^* be the set of all states of the world. A person initially has a probability distribution over N^*. We then consider a situation in which the person is informed that event $N \subseteq N^*$ has occurred. The problem is how to update the person's probability distribution. For each state $i \in N$, $c_i \in \mathbb{R}_+$ denotes the probability that the person initially assigns to state i (thus $|K| = 1$). Since $N \subseteq N^*$, we have $\sum_{i\in N} c_i \leq 1$. Since the total probability to be allocated is 1, we always have $E = 1$. Thus $\mathscr{D} = \{(c,1) \in \mathbb{R}^N_+ \times \{1\} : 0 < \sum_{i\in N} c_i \leq 1\}$.

Probability Aggregation. Here N is the set of possible states of the world, one of which is realized in the future (e.g., N is the set of possible weather conditions tomorrow). There is a set K of forecasters, and each forecaster $k \in K$ has a probability distribution over N, denoted by $(c_{ik})_{i\in N} \in \Delta^{|N|-1}$. The problem is how to aggregate the set of probability distributions into a single distribution. Since the total probability to be allocated over the states is 1, we have $E = 1$. Thus $\mathscr{D} = \{(c,1) \in \mathbb{R}^{N\times K}_+ \times \{1\} : \overline{c}_k = 1$ for each $k \in K\}$.

3 Axioms

This section defines a number of properties that might be satisfied by division rules.

We start with our main axiom, which states that no group of entities can change the total amount of their awards by reallocating their characteristic vectors within the group.

[8] Moulin [17] interprets this problem as surplus sharing after all opportunity costs are returned to contributors.

Reallocation-Proofness. For each $(c,E) \in \mathscr{D}$, each $S \subseteq N$, and each $c' \in \mathbb{R}_+^{N\times K}$, if $\sum_{i\in S} c'_i = \sum_{i\in S} c_i$, then

$$\sum_{i\in S} f_i(c'_S, c_{N\setminus S}, E) = \sum_{i\in S} f_i(c,E).$$

In the contexts of claim problems and their variants, if the left-hand side of the equation exceeds the right-hand side, then group S with claim profile $(c_i)_{i\in S}$ can increase their total awards by reallocating the members' claims into $(c'_i)_{i\in S}$. If the reverse inequality holds, group S with claim profile $(c'_i)_{i\in S}$ can gain from the reverse arrangement. This axiom was introduced by Moulin [16] in the context of social choice with transferable utilities.[9]

In the context of probability aggregation, *reallocation-proofness* has a meaning of informational efficiency. Given a set of states $S \subseteq N$, consider two profiles of beliefs $(c_k)_{k\in K}$ and $(c'_k)_{k\in K}$ such that, for each forecaster $k \in K$, c_k and c'_k differ only in probabilities assigned to the states in S. So the probability of the event S itself is the same under c_k and c'_k. Then, any *reallocation-proof* aggregation rule assigns the same probability to the event S as a whole under $(c_k)_{k\in K}$ and $(c'_k)_{k\in K}$. Thus, one can treat the event S as a single composite state without any loss and does not have to collect information about the forecasters' beliefs over individual states in S.

Similarly, in the context of probability updating, *reallocation-proofness* states that the updated probability of a given event S depends on the initial belief over the states in S only through the total probability that the initial belief puts on S as a whole.[10]

We also consider a pairwise version of *reallocation-proofness*, which deals only with reallocation of characteristics between two entities:

Pairwise Reallocation-Proofness. For each $(c,E) \in \mathscr{D}$, each $i, j \in N$ with $i \neq j$, and each $c'_i, c'_j \in \mathbb{R}_+^K$, if $c'_i + c'_j = c_i + c_j$, then

$$f_i(c'_i, c'_j, c_{N\setminus\{i,j\}}, E) + f_j(c'_i, c'_j, c_{N\setminus\{i,j\}}, E) = f_i(c,E) + f_j(c,E).$$

The pairwise version is particularly relevant for problems in which N is the set of agents (e.g., claim problems), since strategic reallocations of characteristics would be easier to implement for smaller groups of agents.

The remainder of this section defines a number of basic axioms. The following axiom requires that awards add up to the amount to divide:

Efficiency. For each $(c,E) \in \mathscr{D}$, $\sum_{i\in N} f_i(c,E) = E$.

For each problem $(c,E) \in \mathscr{D}$, let

$$\mathscr{D}(\overline{c},E) \equiv \{(c',E) \in \mathbb{R}_+^{N\times K} \times \mathbb{R}_{++} : \overline{c'} = \overline{c}\}.$$

The following axiom basically excludes rules whose image of the compact set $\mathscr{D}(\overline{c},E)$ is unbounded above and below, but it is stated in a weak form:

[9] Moulin calls the axiom "no advantageous reallocation."

[10] An axiom based on a similar idea can also be found in inductive probability theory (Carnap [5], Axiom C9).

One-Sided Boundedness. For each $(c,E) \in \mathscr{D}$, there exists $i \in N$ such that $f_i(\cdot, E)$ is bounded from either above or below over a non-empty open subset of $\mathscr{D}(\bar{c}, E)$.

Since $\mathscr{D}(\bar{c}, E)$ is compact, *one-sided boundedness* is satisfied by any rule that is continuous in claims. The following weak form of continuity is stronger than *one-sided boundedness*.

Continuity. For each $(c,E) \in \mathscr{D}$, there exists $i \in N$ such that $f_i(\cdot, E)$ is continuous at least at one point in $\mathscr{D}(\bar{c}, E)$.

The following two axioms are also stronger than *one-sided boundedness*. The first one requires that awards be non-negative:

Non-Negativity. For each $(c,E) \in \mathscr{D}$ and each $i \in N$, $f_i(c,E) \geq 0$.

Another axiom that implies *one-sided boundedness* is *no transfer paradox* (Moulin [16]), which states that no entity can increase its award by transferring part of its characteristic vector to another entity:

No Transfer Paradox. For each $(c,E) \in \mathscr{D}$, each $i, j \in N$ with $i \neq j$, and each $t \in \mathbb{R}^K$ such that $0 \leqq t \leqq c_i$,

$$f_i(c_i - t, c_j + t, c_{N \setminus \{i,j\}}, E) \leq f_i(c,E).$$

The next axiom states that no amount is awarded to entities whose characteristic vectors are zero:

No Award for Null. For each $(c,E) \in \mathscr{D}$ and each $i \in N$, if $c_i = 0$, then $f_i(c,E) = 0$.

For example, in the context of probability updating, *no award for null* means that, if a state initially receives no probability, so does it after updating.

The next axiom states that if *all* entities have the same characteristic vector, then they all receive the same amount:

Uniform Treatment of Uniforms. For each $(c,E) \in \mathscr{D}$, if $c_1 = c_2 = \cdots = c_{|N|}$, then $f_1(c,E) = f_2(c,E) = \cdots = f_{|N|}(c,E)$.

The next axiom, which is stronger than *uniform treatment of uniforms*, says that for any pair of entities, if they have the same characteristic vector, they receive the same amount:

Equal Treatment of Equals. For each $(c,E) \in \mathscr{D}$ and each $i, j \in N$, if $c_i = c_j$, then $f_i(c,E) = f_j(c,E)$.

The next axiom, which is stronger than *equal treatment of equals*, states that the names of entities do not matter:

Anonymity. For each permutation $\tau \colon N \to N$, each $(c,E) \in \mathscr{D}$, and each $i \in N$, $f_i(c^\tau, E) = f_{\tau(i)}(c,E)$ where c^τ is defined by $c_i^\tau \equiv c_{\tau(i)}$.

The next axiom, which is also stronger than *equal treatment of equals*, states that if *i*'s characteristic vector weakly dominates j's in every dimension, then i receives at least as much as j:

Order Preservation in Gains. For each $(c,E) \in \mathscr{D}$ and each $i, j \in N$, if $c_i \geqq c_j$, then $f_i(c,E) \geq f_j(c,E)$.

4 Generalized proportional rules

For the case when characteristic vectors are single-dimensional (i.e., $|K| = 1$), one of the simplest and best-known rules is the proportional rule:

Definition 1 (Proportional Rule, $|K| = 1$). For each $(c,E) \in \mathscr{D}$ and each $i \in N$,

$$f_i(c,E) = \frac{c_i}{\bar{c}} E.$$

The right-hand side is well-defined since we rule out problems for which $\bar{c} = 0$. In the context of probability updating, the proportional rule is *Bayes rule*. In the context of cost sharing, it is the *average-cost rule*.

We now extend the definition of the proportional rule to the case in which characteristic vectors are multi-dimensional. Let us define a *weight function* as a function $W: \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \Delta^{|K|-1}$, which assigns a vector of weights $W(\bar{c},E)$ over K as a function of $(\bar{c},E)$. With this definition, we define proportional rules in the multi-dimensional case as follows:

Definition 2 (Proportional Rule). There exists a weight function W such that, for each $(c,E) \in \mathscr{D}$ and each $i \in N$,

$$f_i(c,E) = \sum_{k \in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c},E) E. \tag{1}$$

Let P^W denote the proportional rule associated with W.

This rule P^W first applies the proportional rule to each single-dimensional sub-problem (c^k,E) where $c^k \equiv (c_{ik})_{i \in N}$ and then takes the weighted average of the solutions to the sub-problems using the vector of weights $W(\bar{c},E)$. The weights depend on the problem being considered but depend only on $(\bar{c},E)$. Proportional rules are *efficient* since $\sum_{k \in K} W_k(\bar{c},E) = 1$. They also satisfy all the other axioms defined in Section 3. Evidently, if $|K| = 1$, Definition 2 reduces to Definition 1.

In the context of probability aggregation, $E = 1$ and $\bar{c}_k = 1$ for each $k \in K$. Thus a weight function reduces to a single weight vector $w \equiv W((1,\dots,1),1)$. A proportional rule then simply takes a weighted average of individual probability distributions using a fixed weight vector. This rule is called a *linear opinion pool* (McConway [15]).

We now introduce what we call *generalized proportional rules*. These rules are characterized by two functions $A: \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $W: \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$, and the award to i is given by the sum of the following two terms. The first term is $A_i(\bar{c},E)$, which is independent of i's characteristic vector but may treat i differently from others based on i's identity. The second term is proportional to i's characteristic vector and treats entities symmetrically. This term, on the other hand, may treat issues asymmetrically, and the degree of importance attached to each issue $k \in K$

is given by $W_k(\bar{c},E)$. Formally,

Definition 3 (Generalized Proportional Rule). There exist two functions $A\colon \mathbb{R}^K_{++}\times\mathbb{R}_{++}\to\mathbb{R}^N$ and $W\colon \mathbb{R}^K_{++}\times\mathbb{R}_{++}\to\mathbb{R}^K$ such that, for each $(c,E)\in\mathscr{D}$ and each $i\in N$,

$$f_i(c,E)=A_i(\bar{c},E)+\sum_{k\in K}\frac{c_{ik}}{\bar{c}_k}W_k(\bar{c},E)E. \tag{2}$$

Note that W is not required to be a weight function, i.e., neither $W_k(\bar{c},E)\geq 0$ nor $\sum_{k\in K}W_k(\bar{c},E)=1$ is required. Proportional rules are a special case where $A_i=0$ and W is a weight function.

Since, given $(\bar{c},E)$, the second term on the right-hand side of (2) is linear in c_{ik}, generalized proportional rules satisfy *reallocation-proofness* and *one-sided boundedness*. On the other hand, these rules do not necessarily satisfy other axioms in Section 3. We will specify necessary and sufficient conditions for (A,W) to satisfy each of those axioms.

An example of a generalized proportional rule that is not a proportional rule is the *equal division rule*, which simply divides E equally among entities: $f_i(c,E)=E/|N|$. This rule is a generalized proportional rule with $A_i(\bar{c},E)=E/|N|$ and $W_k(\bar{c},E)=0$.

Another example, in the case of $|K|=1$, is a *weighted rights egalitarian rule* (Bergantiños and Vidal-Puga [4]), which first awards c_i to each i and then distributes the difference $E-\sum_{i\in N}c_i$ among the entities according to a weight vector $(\lambda_1,\ldots,\lambda_n)\in\mathrm{int}(\Delta^{|N|-1})$: $f_i(c,E)=c_i+\lambda_i(E-\bar{c})$. This rule is a generalized proportional rule with $A_i(\bar{c},E)=\lambda_i(E-\bar{c})$ and $W(\bar{c},E)=\bar{c}/E$. If the weights are equal ($\lambda_i=1/|N|$ for each i), the rule is what is called the *rights egalitarian rule* in Herrero, Maschler, and Villar [11].

5 Main results

Our first main result is a characterization of *reallocation-proof* rules.

Theorem 1. *Assume $|N|\geq 3$. A rule f on a rich domain $\mathscr{D}$ is reallocation-proof if and only if there exist two functions $A\colon \mathbb{R}^K_{++}\times\mathbb{R}_{++}\to\mathbb{R}^N$ and $\hat{W}\colon \mathbb{R}_{+}\times\mathbb{R}^K_{++}\times\mathbb{R}_{++}\to\mathbb{R}^K$ such that, for each $(c,E)\in\mathscr{D}$ and each $i\in N$,*

$$f_i(c,E)=A_i(\bar{c},E)+\sum_{k\in K}\hat{W}_k(c_{ik},\bar{c},E),$$

and for each $k\in K$, $\hat{W}_k(\,\cdot\,,\bar{c},E)$ is additive.

Proof. Since the "if" part is straightforward, we prove the "only if" part. Let f be a *reallocation-proof* rule defined on a rich domain $\mathscr{D}$. We fix $(d,E)\in\mathbb{R}^K_{++}\times\mathbb{R}_{++}$ and consider problems (c,E) such that $\bar{c}=d$. Let $\mathscr{C}\equiv\{c\in\mathbb{R}^{K\times N}_{+}:\bar{c}=d\}$.

Step 1. We first show that *reallocation-proofness* implies the following property: for each $c,c'\in\mathscr{C}$ and each $S\subseteq N$, if $\sum_{i\in S}c_i=\sum_{i\in S}c'_i$, then $\sum_{i\in S}f_i(c,E)=\sum_{i\in S}f_i(c',E)$. Indeed, by applying *reallocation-proofness* to S, N, and $N \setminus S$, we obtain

$$\sum_{i \in S} f_i(c,E) = \sum_{i \in S} f_i(c'_S, c_{N\setminus S}, E) = \sum_{i \in N} f_i(c'_S, c_{N\setminus S}, E) - \sum_{i \in N\setminus S} f_i(c'_S, c_{N\setminus S}, E)$$
$$= \sum_{i \in N} f_i(c', E) - \sum_{i \in N\setminus S} f_i(c', E) = \sum_{i \in S} f_i(c', E).$$

Step 2. For each $i \in N$, let $A_i(d,E) \equiv f_i(c,E)$ where $c \in \mathscr{C}$ and $c_i = 0$. That is, $A_i(d,E)$ is the amount that i receives whenever its characteristic vector is zero, given (d,E). By Step 1 with $S = \{i\}$, $A_i(d,E)$ is uniquely determined. By applying the same observation to coalitions $S \subsetneq N$, we can also define a function $w\colon (2^N \setminus \{\emptyset, N\}) \times \prod_{k=1}^K [0,d_k] \to \mathbb{R}$ by

$$w(S,x) \equiv \sum_{i \in S} f_i(c,E) - \sum_{i \in S} A_i(d,E) \tag{3}$$

where $c \in \mathscr{C}$ is such that $\sum_{i \in S} c_i = x$. For brevity, the dependence of $w(S,x)$ on (d,E) is omitted.

Step 3. We show that for each $x \in \prod_{k=1}^K [0,d_k]$ and each $S, S' \subsetneq N$, $w(S,x) = w(S',x)$. We first consider the case when $S' \subseteq S$. Let $c \in \mathscr{C}$ be such that $\sum_{i \in S} c_i = x$ and $c_i = 0$ for each $i \in S \setminus S'$. Since $f_i(c,E) - A_i(d,E) = 0$ for each $i \in S \setminus S'$,

$$w(S,x) = \sum_{i \in S'} f_i(c,E) - \sum_{i \in S'} A_i(d,E) = w(S',x),$$

as desired. Now, consider the case in which no inclusion holds between S and S'. Let $i \in S$ and $j \in S'$. The result just obtained implies $w(S,x) = w(\{i\},x) = w(\{i,j\},x) = w(\{j\},x) = w(S',x)$.

This step enables us to write $w(S,x)$ as $w^*(x)$.

Step 4. We show that for each $x,y \in \prod_{k=1}^K [0,d_k]$ such that $x+y \in \prod_{k=1}^K [0,d_k]$, $w^*(x) + w^*(y) = w^*(x+y)$. Let $i,j \in N$ ($i \neq j$) and $c \in \mathscr{C}$ be such that $c_i = x$ and $c_j = y$; although $x+y \leq d$, such a $c \in \mathscr{C}$ exists since there are 3 or more entities. Then

$$\begin{aligned} w^*(x) + w^*(y) &= w(\{i\},x) + w(\{j\},y) \\ &= f_i(c,E) - A_i(d,E) + f_j(c,E) - A_j(d,E) \\ &= w(\{i,j\},x+y) = w^*(x+y). \end{aligned}$$

Step 5. For each $i \in N$, each $k \in K$, and each $c_{ik} \in [0,d_k]$, let

$$\hat{W}_k(c_{ik}, d, E) \equiv w^*(0,\ldots,0,c_{ik},0,\ldots,0)$$

where c_{ik} appears in the kth entry. Then for each $c \in \mathscr{C}$ and each $i \in N$,

$$\begin{aligned} f_i(c,E) &= A_i(\bar{c},E) + w(\{i\},c_i) \\ &= A_i(\bar{c},E) + \sum_{k \in K} \hat{W}_k(c_{ik},\bar{c},E). \end{aligned}$$

The additivity of $\hat{W}_k(\cdot,\bar{c},E)$ follows from Step 4.[11] □

This and the subsequent results require at least three entities. However, the same results hold even in the two-entity case in an extended framework where the set of entities is variable, as we discuss in Section 7.

The following result shows that *reallocation-proofness* together with *one-sided boundedness* characterizes generalized proportional rules:

Theorem 2. *Assume $|N| \geq 3$. A rule on a rich domain satisfies reallocation-proofness and one-sided boundedness if and only if it is a generalized proportional rule.*

Proof. The "if" part has been discussed. The "only if" part holds since $W_k(\cdot,\bar{c},E)$ is additive over $[0,\bar{c_k}]$ and bounded either above or below over a non-empty open subset, which implies that $W_k(\cdot,\bar{c},E)$ is linear [1, Corollary 2.5]. Therefore, $\hat{W}_k(c_{ik},\bar{c},E) = (c_{ik}/\bar{c_k})\hat{W}_k(\bar{c_k},\bar{c},E)$. Letting $W_k(\bar{c},E) \equiv \hat{W}_k(\bar{c_k},\bar{c},E)/E$ completes the proof. □

The two axioms in Theorem 2 are independent. Indeed, a number of rules in the literature satisfy *one-sided boundedness* but not *reallocation-proofness*. A rule that satisfies *reallocation-proofness* but not *one-sided boundedness* can be constructed using an additive nonlinear function [1, Theorem 2.2.10].

We can obtain necessary and sufficient conditions on $(A,\hat{W})$ under which the *reallocation-proof* rules characterized in Theorem 1 satisfy additional basic axioms. We omit the proof since it is straightforward.

Theorem 3. *Assume $|N| \geq 3$. Let f be a reallocation-proof rule on a rich domain $\mathscr{D}$, and $(A,\hat{W})$ be the list of associated functions. Then*

1. *Rule f satisfies no award for null if and only if, for each $(c,E) \in \mathscr{D}$ and each $i \in N$,*

$$A_i(\bar{c},E) = 0. \tag{4}$$

2. *Rule f satisfies uniform treatment of uniforms if and only if, for each $(c,E) \in \mathscr{D}$,*

$$A_1(\bar{c},E) = A_2(\bar{c},E) = \cdots = A_{|N|}(\bar{c},E), \tag{5}$$

which holds if and only if f satisfies anonymity. Hence, for reallocation-proof rules, anonymity, equal treatment of equals, and uniform treatment of uniforms are all equivalent. By (4) *and* (5)*, if f satisfies no award for null, then f satisfies anonymity.*

3. *Rule f satisfies no transfer paradox if and only if, for each $(c,E) \in \mathscr{D}$, each $k \in K$, and each $i \in N$,*

$$\hat{W}_k(c_{ik},\bar{c},E) \geq 0, \tag{6}$$

[11] We defined $\hat{W}_k(c_{ik},\bar{c},E)$ only for $c_{ik} \leq \bar{c_k}$ and Step 4 shows only that $\hat{W}_k(\cdot,\bar{c},E)$ is additive over $[0,\bar{c_k}]$. But we can easily extend the definition and the additivity to $\mathbb{R}_+$.

which is the case if and only if, for each $k \in K$, $\hat{W}_k(\cdot, \bar{c}, E)$ *is non-decreasing in* c_{ik}.

4. *Rule f satisfies order preservation in gains if and only if f satisfies uniform treatment of uniforms and no transfer paradox (i.e., f satisfies* (5) *and* (6)*).*

5. *Rule f satisfies one-sided boundedness (hence f is a generalized proportional rule) if and only if, for each $k \in K$ and each $(c, E) \in \mathscr{D}$, $\hat{W}_k(\cdot, \bar{c}, E)$ is monotonic, i.e., either non-decreasing or non-increasing.*

6. *Rule f satisfies continuity if and only if it satisfies one-sided boundedness.*

7. *Rule f satisfies non-negativity if and only if f satisfies one-sided boundedness and, for each $(c, E) \in \mathscr{D}$,*

$$\min_{j \in N} A_j(\bar{c}, E) + \sum_{k \in K} \min\{0, \hat{W}_k(\bar{c}_k, \bar{c}, E)\} \geq 0. \tag{7}$$

   A necessary condition for (7) *is*

$$A_i(\bar{c}, E) \geq 0 \quad \text{for each } i \in N. \tag{8}$$

8. *Rule f satisfies efficiency if and only if, for each $(c, E) \in \mathscr{D}$,*

$$\sum_{k \in K} \hat{W}_k(\bar{c}_k, \bar{c}, E) = E - \sum_{i \in N} A_i(\bar{c}, E). \tag{9}$$

   Therefore, when $|K| = 1$, f satisfies efficiency and one-sided boundedness if and only if f takes the following form:

$$f_i(c, E) = A_i(\bar{c}, E) + \frac{c_i}{\bar{c}}\big[E - \sum_{i \in N} A_i(\bar{c}, E)\big]. \tag{10}$$

   Thus f first allocates $A_i(\bar{c}, E)$ to each i and then divides the remainder among the entities proportionally to their characteristics. This rule satisfies non-negativity if and only if, for each $(c, E) \in \mathscr{D}$ and each $i \in N$,

$$A_i(\bar{c}, E) \geq \max\{0, \sum_{j \in N} A_j(\bar{c}, E) - E\}.$$

The following result is a characterization of proportional rules. We omit the proof since it follows easily from Theorem 3.

Corollary 1. *Assume $|N| \geq 3$. A rule on a rich domain satisfies reallocation-proofness, efficiency, no award for null, and non-negativity (or no transfer paradox) if and only if it is a proportional rule.*

The following result is a characterization of another subfamily of generalized proportional rules. We again omit the proof since it also follows easily from Theorem 3.

Corollary 2. *Assume* $|N| \geq 3$. *A rule* f *on a rich domain* $\mathscr{D}$ *satisfies reallocation-proofness, efficiency, uniform treatment of uniforms, and one-sided boundedness if and only if there exists a function* $W: \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ *such that for each* $(c,E) \in \mathscr{D}$ *and each* $i \in N$,

$$f_i(c,E) = \frac{E}{|N|}\left[1 - \sum_{k\in K} W_k(\bar{c},E)\right] + \sum_{k\in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c},E)E. \tag{11}$$

This rule satisfies non-negativity and no transfer paradox if and only if, for each $(c,E) \in \mathscr{D}$, $W(\bar{c},E) \geqq 0$ *and* $\sum_{k\in K} W_k(\bar{c},E) \leq 1$.

Moulin [17, Lemma 2] considers (single-dimensional) claim problems and obtains the functional form (11) using *reallocation-proofness*, *efficiency*, *uniform treatment of uniforms*, *non-negativity*, "homogeneity" (f being linear in (c,E)), "claim monotonicity," and "resource monotonicity." Chun [8, Theorem 1] also considers claim problems and obtains (11) using *reallocation-proofness*, *efficiency*, *anonymity*, and *continuity*.

We now show that, for the characterization of proportional rules in Corollary 1, *reallocation-proofness* can be weakened to its pairwise version.

Theorem 4. *Assume* $|N| \geq 3$. *A rule on a rich domain satisfies pairwise reallocation-proofness, efficiency, no award for null, and non-negativity (or no transfer paradox) if and only if it is a proportional rule.*

Proof. Let f be a rule on a rich domain $\mathscr{D}$ with $|N| \geq 3$ satisfying all the axioms. For each $S \subseteq N$, let $\mathscr{D}_S \equiv \{(c,E) \in \mathscr{D} : c_i = 0 \text{ for all } i \notin S\}$. By *no award for null*, we can treat problems in $\mathscr{D}_S$ as those in which only entities in S are present.

On $\mathscr{D}_S$ such that $|S| = 3$, *pairwise reallocation-proofness* and *efficiency* imply *reallocation-proofness*. Corollary 1 then implies that, on $\mathscr{D}_S$, f coincides with a proportional rule. Let W^S denote the associated weight function. For each $S,T \subseteq N$ such that $|S| = |T| = 3$ and $|S\cap T| \geq 2$, since $\mathscr{D}_S \cap \mathscr{D}_T \neq \emptyset$, we have $W^S = W^T$. Thus, weight functions for all triples are identical and we can write them simply by W. Hence, on $\cup_{|S|\leq 3}\mathscr{D}_S$, f coincides with the proportional rule associated with W.

To prove that f is the proportional rule on the entire domain, we use an induction argument. Given $k \geq 3$, suppose that, on $\cup_{|S|\leq k}\mathscr{D}_S$, f coincides with the proportional rule associated with a weight function W, and let $S \subseteq N$ contain $k+1$ entities. To prove that f also coincides with the proportional rule on $\mathscr{D}_S$, let $(c,E) \in \mathscr{D}_S$. Consider a pair $\{i,j\} \subseteq S$, and let $c' \in \mathbb{R}^{S\times K}_+$ be such that $(c'_i, c'_j) = (c_i + c_j, 0)$ and $c'_h = c_h$ for each $h \notin \{i,j\}$. Then by *pairwise reallocation-proofness* and *no award for null*, $f_i(c,E) + f_j(c,E) = f_i(c',E) + f_j(c',E) = f_i(c',E)$. Since $(c',E) \in \mathscr{D}_{S\setminus\{j\}}$, the induction hypothesis implies $f_i(c',E) = P^W_i(c',E) = P^W_i(c,E) + P^W_j(c,E)$. Therefore $f_i(c,E) + f_j(c,E) = P^W_i(c,E) + P^W_j(c,E)$. Since this holds for every pair $\{i,j\} \subseteq S$, we obtain $f(c,E) = P^W(c,E)$. □

A similar result is obtained by replacing *non-negativity* (or *no transfer paradox*) in Theorem 4 with *one-sided boundedness*. Indeed, one can easily show that, for any rule f that satisfies the modified list of axioms, there exists a function $W : \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that, for each $(c,E) \in \mathscr{D}$ and each $i \in N$, $\sum_{k\in K} W_k(\bar{c},E) = 1$ and (1) holds. This family of rules is strictly larger than that of proportional rules since W is allowed to take negative values. However, if $|K| = 1$, then $W_k(\bar{c},E) = 1$ and hence any rule that satisfies the modified list of axioms also satisfies *non-negativity* and *no transfer paradox*. This implies that if $|K| = 1$, *non-negativity* (and *no transfer paradox*) in Theorem 4 can be replaced with *one-sided boundedness*. Thus we obtain

Corollary 3. *Assume $|N| \geq 3$ and $|K| = 1$. A rule on a rich domain satisfies pairwise reallocation-proofness, efficiency, no award for null, and one-sided boundedness if and only if it is the proportional rule.*

A few papers consider the case when $|K| = 1$ and prove results similar to Corollary 3. Chun [8, Theorem 2] considers claim problems in the framework where the set of agents is variable, and characterizes the proportional rule using *reallocation-proofness*, *efficiency*, *anonymity*, *continuity*, and "null consistency" (defined later in Section 7.1). We will strengthen the result in Section 7.1: see the equivalence between (ii) and (iii) in Corollary 11. Ching and Kakkar [7, Corollary 3] consider bankruptcy problems and characterize the proportional rule using *reallocation-proofness*, *efficiency*, *no award for null*, and *non-negativity*, thereby showing that *anonymity*, *continuity*, and "null consistency" in Chun's result can be replaced with *no award for null* and *non-negativity*.[12] As we observe below, *no award for null* is weaker than "null consistency" in the presence of *efficiency*. Our Corollary 3 strengthens Ching and Kakkar's result by showing that it holds for any rich domain, *non-negativity* can be weakened to *one-sided boundedness*, and *reallocation-proofness* can be weakened to its pairwise version.

6 Application I: Fixed set of entities

6.1 Claim problems and variants

This subsection presents applications of our results in the contexts of bankruptcy, surplus sharing, and claim problems.

We consider the following three additional axioms. The first one says that if the amount to divide is split into two parts and the award vector is computed separately for each part, then the sum of the award vectors should coincide with the award vector obtained from a single calculation applied to the total amount to divide:

Resource Additivity. For each $(c,E) \in \mathscr{D}$ and each $(c,E') \in \mathscr{D}$ such that $(c,E+E') \in \mathscr{D}$, $f(c,E) + f(c,E') = f(c,E+E')$.

The next axiom says that division should be independent of the unit with which the data of the problems are measured. That is, the rule should be linear in (c,E) jointly:

Homogeneity. For each $(c,E) \in \mathscr{D}$ and each $\lambda > 0$, $f(\lambda c, \lambda E) = \lambda f(c,E)$.

[12] The comparison between Chun and Ching–Kakkar is not precise since Chun considers a variable-population model and covers the two-agent case.

The next axiom says that no agent loses when his claim increases:

Claim Monotonicity. For each $(c,E) \in \mathscr{D}$, each $i \in N$, and each $c_i' \geq c_i$, if $(c_i', c_{-i}, E) \in \mathscr{D}$, then $f_i(c_i', c_{-i}, E) \geq f_i(c,E)$.

We begin by characterizing a subfamily of generalized proportional rules that satisfy *resource additivity*.

Theorem 5. *Assume that $\mathscr{D}$ is any of the three domains— bankruptcy, surplus sharing, or claim problems—with at least 3 agents. A rule f on $\mathscr{D}$ satisfies reallocation-proofness, efficiency, non-negativity, and resource additivity if and only if there exists a function $A\colon \mathbb{R}_{++} \to \mathbb{R}_+^N$ such that, for each $(c,E) \in \mathscr{D}$ and each $i \in N$,*

$$f_i(c,E) = E\Big[A_i(\bar{c}) + \frac{c_i}{\bar{c}}\big[1 - \sum_{j\in N} A_j(\bar{c})\big]\Big].$$

This rule satisfies no transfer paradox if and only if $\sum_{j\in N} A_j(\bar{c}) \leq 1$.

Proof. Let $\mathscr{D}$ be the class of bankruptcy problems with $|N| \geq 3$ (proofs for the other classes are similar). Let f be a rule on $\mathscr{D}$ satisfying the axioms. By Theorem 3 (equations (8) and (10)), there exists a function $A\colon \mathbb{R}_{++}^2 \to \mathbb{R}_+^N$ such that, for each $(c,E) \in \mathscr{D}$ and each $i \in N$,

$$f_i(c,E) = A_i(\bar{c},E) + \frac{c_i}{\bar{c}}\big[E - \sum_{j\in N} A_j(\bar{c},E)\big].$$

Let $c \in \mathbb{R}_+^N$ and $i \in N$. We shall show that $A_i(\bar{c},\cdot)$ is linear on $[0,\bar{c}]$. To prove this, we can assume $c_i = 0$. Then *resource additivity* implies that, for each $E, E' \in [0,\bar{c}]$, we have $A_i(\bar{c},E) + A_i(\bar{c},E') = A_i(\bar{c},E+E')$ as long as $0 \leq E + E' \leq \bar{c}$; i.e., $A_i(\bar{c},\cdot)$ is additive on $[0,\bar{c}]$. Since f satisfies *non-negativity*, a standard argument of Cauchy's equation yields (as in the proof of Theorem 2) that $A_i(\bar{c},\cdot)$ is linear on $[0,\bar{c}]$. Thus, for each $E \in [0,\bar{c}]$, we can write $A_i(\bar{c},E)$ as $A_i(\bar{c})E$. □

It is easy to show that if *homogeneity* is added, $A_i(\cdot)$ in Theorem 5 is constant for each $i \in N$, and for these rules, *no transfer paradox* is equivalent to *claim monotonicity*. Thus we obtain

Corollary 4. *Assume that $\mathscr{D}$ is any of the three domains—bankruptcy, surplus sharing, or claim problems—with at least 3 agents. A rule f on $\mathscr{D}$ satisfies reallocation-proofness, efficiency, uniform treatment of uniforms, non-negativity, no transfer paradox (or claim monotonicity), homogeneity, and resource additivity if and only if there exists $\alpha \in [0,1]$ such that, for each $(c,E) \in \mathscr{D}$,*

$$f_i(c,E) = \alpha\frac{1}{|N|}E + (1-\alpha)\frac{c_i}{\bar{c}}E, \tag{12}$$

i.e., f is a convex combination of the proportional rule and equal division.

Moulin [17, Theorem 3] characterizes the same family of rules for claim problems. Corollary 4 strengthens his result by removing "resource monotonicity" from his characterization and making it applicable to other domains.

The family of rules characterized in Corollary 4 is indexed by $\alpha \in [0,1]$. The axiom to be considered next, called *composition down*, further contracts this family. To motivate this axiom, consider a problem (c,E) and suppose that, after an award vector x is agreed upon, it is revealed that the amount to divide is actually less than expected, i.e., $E' < E$. There are at least two ways to adjust the award vector. One is to re-calculate the award vector for the problem with the right amount to divide, (c,E'). Another is to consider the previous agreement x as the relevant claim vector and calculate the award vector for the problem (x,E'). The axiom states that in either way, we reach the same award vector.[13]

Composition Down. For each $(c,E) \in \mathscr{D}$ and each $E' < E$ with $(c,E') \in \mathscr{D}$, $f(c,E') = f(f(c,E),E')$.

Corollary 5. *Assume that $\mathscr{D}$ is the class of either bankruptcy problems or claim problems, with at least 3 agents. A rule on $\mathscr{D}$ satisfies all the axioms in Corollary 4 and composition down if and only if it is either the proportional rule or equal division.*

Proof. The "if" part follows since the proportional rule and equal division satisfy *composition down*. To prove the converse, let f be a rule satisfying the axioms. By Corollary 4, f is a convex combination of the proportional rule and equal division with a weight $\alpha \in [0,1]$ on equal division. Let $(c,E) \in \mathscr{D}$ and $E' \in (0,E)$. Notice that $(c,E') \in \mathscr{D}$ and $(f(c,E),E') \in \mathscr{D}$. By *composition down*, $f(c,E') = f(f(c,E),E')$, which implies

$$\frac{\alpha}{|N|} + (1-\alpha)\frac{c_i}{\bar{c}} = \frac{\alpha}{|N|} + (1-\alpha)\frac{E[\frac{\alpha}{|N|} + (1-\alpha)\frac{c_i}{\bar{c}}]}{E}.$$

Hence $(1-\alpha)\alpha[\frac{1}{|N|} - \frac{c_i}{\bar{c}}] = 0$. Since c was chosen arbitrarily, $\alpha = 0$ or $\alpha = 1$. □

Moulin [17, Theorem 2] also characterizes the pair of the proportional rule and equal division, in the context of claim problems, using "path independence" instead of *composition down*. "Path independence" is also a condition of dynamic consistency in calculating awards, but it is not well-defined in the class of bankruptcy problems.

6.2 Bankruptcy with multiple types of assets

We now give an application of Theorem 4 in the context of bankruptcy problems with multiple types of assets. There are often exogenously determined priorities among different types of assets. For example, the standard legal code states that claims based on bonds should be reimbursed prior to claims based on stocks.[14] Without loss of generality, suppose that assets of type k have priority over assets of type k' for all $k' > k$. Given the priorities, we consider a requirement that if there exists a creditor whose claim based on the first k types of assets is not fully reimbursed, then there should not exist a creditor who gets strictly more than his claim for these k types of assets.

[13] This axiom, introduced by Moulin [19], is well-defined under *efficiency* and *non-negativity* in the classes of bankruptcy and claim problems (but not surplus sharing).

[14] Priorities of the United States Code, Title 11 (Bankruptcy), are stated in Sections 507 (Priorities) and 726 (Distribution of property of the estate).

Formally, a rule f conforms to *asset priorities* if for each $(c,E) \in \mathscr{D}$ and each $k \in K$, if there exists a creditor $i \in N$ such that $f_i(c,E) < \sum_{h=1}^{k} c_{ih}$, then $f_j(c,E) \leq \sum_{h=1}^{k} c_{jh}$ for each $j \in N$.[15] The next result characterizes the proportional rule that conforms to *asset priorities*, by identifying the exact form of its weight function.

Theorem 6. *For the class of bankruptcy problems with multiple assets with at least 3 agents, a rule satisfies pairwise reallocation-proofness, efficiency, non-negativity, and asset priorities if and only if it is the proportional rule with the weight function W defined as follows: for each possible* $(\bar{c},E)$*, if* $k^* \in K$ *is the minimum index such that* $\sum_{k=1}^{k^*} \bar{c}_k \geq E$*, then for each* $k \in K$*,*

$$W_k(\bar{c},E) = \begin{cases} \frac{\bar{c}_k}{E} & \text{if } k < k^*, \\ 1 - \sum_{k<k^*} \frac{\bar{c}_k}{E} & \text{if } k = k^*, \\ 0 & \text{if } k > k^*. \end{cases} \tag{13}$$

Proof. Let f be a rule satisfying the axioms. By *asset priorities*, no creditor gets more than his total claim ($\sum_{k\in K} c_{ik}$) and therefore, by *non-negativity*, the rule satisfies *no award for null*. Thus by Theorem 4, f is a proportional rule with some weight function W. To show that W satisfies (13), consider any possible $(\bar{c},E)$ and let k^* be defined as above. Let $(c,E) \in \mathscr{D}(\bar{c},E)$. If there exists a creditor i such that $f_i(c,E) > \sum_{h=1}^{k^*} c_{ih}$, then by *efficiency*, there exists another creditor j such that $f_j(c,E) < \sum_{h=1}^{k^*} c_{jh}$, but then f violates *asset priorities*. This shows that for each $i \in N$,

$$f_i(c,E) \leq \sum_{h=1}^{k^*} c_{ih}. \tag{14}$$

Now, given any $k \in K$ and any $i \in N$, consider a problem $(c,E) \in \mathscr{D}(\bar{c},E)$ such that $c_{ik} = \bar{c}_k$ and $c_{ih} = 0$ for each $h \neq k$. Then $f_i(c,E) = W_k(\bar{c},E)E$. Since this value should not exceed i's total claim, we have $W_k(\bar{c},E) \leq \bar{c}_k/E$. If $k > k^*$, (14) implies $f_i(c,E) \leq 0$ and hence $W_k(\bar{c},E) = 0$. If $k < k^*$, we claim $W_k(\bar{c},E) = \bar{c}_k/E$. Indeed, if $W_k(\bar{c},E) < \bar{c}_k/E$, then $f_i(c,E) < \bar{c}_k = \sum_{h=1}^{k} c_{ih}$. Thus, by *asset priorities*, $\sum_{j\in N} f_j(c,E) < \sum_{h=1}^{k} \bar{c}_h < E$, contradicting *efficiency*. □

6.3 Income redistribution

For income redistribution problems, by using Theorem 3, we can characterize the family of income-tax schedules with a flat tax rate and personalized lump-sum transfers:

Theorem 7. *For the class of income redistribution problems with at least 3 agents, a rule* f *satisfies reallocation-proofness, efficiency, non-negativity, and no transfer paradox if and only if there exist two functions* $T\colon \mathbb{R}_{++} \to [0,1]$ *and* $R\colon \mathbb{R}_{++} \to \mathbb{R}^N_+$ *such that, for each* $(c,E) \in \mathscr{D}$ *and*

[15] In the context of bankruptcy problems with $|K| = 1$, Moulin [18] shows that a certain set of independence axioms characterizes a family of rules that conform to exogenously given priorities among *agents*. Our priorities, on the other hand, pertain to assets, not agents.

each $i \in N$,

$$f_i(c,E) = (1-T(\bar{c}))c_i + R_i(\bar{c}) \quad \textit{and} \quad \sum_{j \in N} R_j(\bar{c}) = T(\bar{c})\bar{c}.$$

In these rules, T determines the flat tax rate $T(\bar{c})$ as a function of the size of the economy, $\bar{c}$, while R determines the reallocation scheme $(R_1(\bar{c}), R_2(\bar{c}), \ldots, R_{|N|}(\bar{c}))$ as a function of individuals' identities subject to the budget balance: $\sum_{j \in N} R_j(\bar{c}) = T(\bar{c})\bar{c}$. It is easy to see that these rules also satisfy *homogeneity* if and only if T is constant and each R_i is linear.

6.4 Social choice with transferable utilities

In social choice problems with transferable utilities, the vector c_i denotes agent i's valuations for alternatives. Thus it is immaterial how the vector is normalized. This motivates the following axiom. Let $\mathbf{1} \in \mathbb{R}^K$ denote the vector consisting of 1 only.

Translation Invariance. For each $(c,E) \in \mathscr{D}$, each $i \in N$, and each $\lambda \in \mathbb{R}_+$,

$$f_i((c_i + \lambda\mathbf{1}, c_{-i}), E+\lambda) = f_i(c,E) + \lambda \text{ and } f_{-i}((c_i + \lambda\mathbf{1}, c_{-i}), E+\lambda) = f_{-i}(c,E).$$

For each $c \in \mathbb{R}_+^{N \times K}$, let $\bar{c}_{\max} \equiv \max_{k \in K} \bar{c}_k$. Since $E = \bar{c}_{\max}$, we suppress E throughout this subsection. Moulin [16] introduced the following family of rules:

Definition 4 (Equal Sharing Above a Convex Decision, ESCD). There exists a function $\rho : \mathbb{R}_{++}^K \to \Delta^{|K|-1}$ such that, for each $\bar{c} \in \mathbb{R}_+^K$ and each $\lambda \geq 0$,

$$\rho(\bar{c} + \lambda\mathbf{1}) = \rho(\bar{c}), \tag{15}$$

and, for each $c \in \mathbb{R}_+^{N \times K}$ and each $i \in N$,

$$f_i(c) = \frac{1}{|N|}\left[\bar{c}_{\max} - \sum_{k \in K} \bar{c}_k \rho_k(\bar{c})\right] + \sum_{k \in K} c_{ik}\rho_k(\bar{c}). \tag{16}$$

Let ES^ρ denote the ESCD rule associated with ρ.

It is easy to see that ES^ρ is *efficient* and *translation invariant*. Note that ES^ρ is the generalized proportional rule associated with $A \colon \mathbb{R}_{++}^K \to \mathbb{R}^N$ and $W : \mathbb{R}_{++}^K \to \mathbb{R}^K$ defined by $W_k(\bar{c}) \equiv \bar{c}_k \rho_k(\bar{c})/\bar{c}_{\max}$ and $A_i(\bar{c}) \equiv \frac{\bar{c}_{\max}}{|N|}[1 - \sum_{k \in K} W_k(\bar{c})]$.

Moulin [16, Theorem 1] characterizes the family of ESCD rules by *reallocation-proofness*, *efficiency*, *no transfer paradox*, *translation invariance*, and *anonymity*. The next result, which relies on Corollary 2, shows that his characterization remains valid if *anonymity* is weakened to *uniform treatment of uniforms*. The proof is in Appendix.

Corollary 6. *For the class of social choice problems with transferable utilities with at least 3 agents, a rule satisfies reallocation-proofness, efficiency, no transfer paradox, translation invariance, and uniform treatment of uniforms if and only if it is an ESCD rule.*

Moulin [16] also introduced the following subfamily of ESCD rules.

Definition 5. A *utilitarian rule* is an ESCD rule whose weight function $\rho: \mathbb{R}^K_{++} \to \Delta^{|K|-1}$ is such that, for each $c \in \mathbb{R}^{N\times K}_+$, (15) is satisfied and

$$\rho_k(\bar{c}) = 0 \quad \text{for each } k \in K \text{ with } \bar{c}_k < \bar{c}_{\max}. \tag{17}$$

Let U^ρ denote this rule. By (17), the first term of (16) is zero. Thus

$$U_i^\rho(c) = \sum_{k\in K} c_{ik}\rho_k(\bar{c}) = \sum_{k\in K} \frac{c_{ik}}{\bar{c}_k}\rho_k(\bar{c})\bar{c}_{\max}.$$

Utilitarian rules are proportional rules that assign zero weights on inefficient alternatives. Under these rules, each agent receives a weighted average of his valuations for efficient alternatives. Thus, when agents have expected utility preferences, utilitarian rules can be considered as rules that simply select an efficient alternative randomly without side-payments.

Among ESCD rules, only utilitarian rules satisfy *no award for null*. This suggests a characterization of utilitarian rules in the manner of Theorem 4. Indeed, Moulin [16, Theorem 3] characterizes utilitarian rules using *no award for null* together with *reallocation-proofness*, *efficiency*, *non-negativity*, and *anonymity*. However, the characterization holds without *anonymity* since *anonymity* is implied by *reallocation-proofness* and *no award for null* by Theorem 3 (Item 2). Furthermore, *reallocation-proofness* can be weakened to the pairwise version, and *non-negativity* can be replaced with *no transfer paradox*, as the following result shows. The proof is in Appendix.

Corollary 7. *For the class of social choice problems with transferable utilities with at least 3 agents, a rule satisfies pairwise reallocation-proofness, efficiency, no award for null, non-negativity (or no transfer paradox), and translation invariance if and only if it is a utilitarian rule.*

Although Corollaries 6–7 are shown on $\mathbb{R}^{N\times K}_+$, *translation invariance* enables us to extend these results to $\mathbb{R}^{N\times K}$, which is in fact the domain considered in Moulin [16].

6.5 Probability updating and aggregation

For probability updating problems, Theorem 4 and Corollary 1 give a characterization of Bayes rule.

Corollary 8. *For the class of probability updating problems with at least 3 states (i.e., $|N| \geq 3$), a rule satisfies pairwise reallocation-proofness, efficiency, no award for null, and non-negativity if and only if it is Bayes rule.*

For probability aggregation, McConway [15] considers the following axiom. A rule f satisfies the *strong setwise function property* if there is a function $h\colon [0,1]^K \to [0,1]$ such that, for each $(c,E) \in \mathscr{D}$ and each $S \subseteq N$, $\sum_{i\in S} f_i(c,E) = h(\sum_{i\in S} c_i)$. Since function h is independent of S, this axiom is stronger than *reallocation-proofness*. Since $S = \emptyset$ is allowed in the definition, the axiom also implies *no award for null*. Hence, we obtain the following result of McConway as a corollary.

Corollary 9 (McConway [15], Theorem 3.3). *For the class of probability aggregation problems with at least 3 states, a rule satisfies the strong setwise function property, efficiency, and non-negativity if and only if it is a linear opinion pool.*

7 Application II: Variable set of entities

We extend the model in the previous sections to allow the set of entities to vary. Let $I \subseteq \{1,2,\ldots\}$ be the set of *potential* entities, which may be finite or infinite. Let $\mathscr{N}$ be the set of all non-empty finite subsets of I. For each $N \in \mathscr{N}$, let $\mathscr{A}^N$ be the class of all division problems associated with N. We retain our simplifying assumption that for each $k \in K$, $\overline{c}_k > 0$. For each $N \in \mathscr{N}$, let $\mathscr{D}^N \subseteq \mathscr{A}^N$ and $\mathscr{D} \equiv \cup_{N \in \mathscr{N}} \mathscr{D}^N$. A *rule* is now a function f that associates with each $N \in \mathscr{N}$ and each problem $(c,E) \in \mathscr{D}^N$ an award vector $f(c,E) \in \mathbb{R}^N$. We say that $\mathscr{D}$ is *rich** if for each $N, N' \in \mathscr{N}$, each $(c,E) \in \mathscr{D}^N$, and each $c' \in \mathbb{R}^{N'}_+$, if $\sum_{i \in N'} c'_i = \sum_{i \in N} c_i$, then $(c',E) \in \mathscr{D}^{N'}$. Note that if $\mathscr{D}$ is rich*, $\mathscr{D}^N$ is rich for all $N \in \mathscr{N}$. The axioms and notions defined in the previous sections can be easily redefined in this extended setup by simply adding "for each $N \in \mathscr{N}$" in the definitions.

7.1 Merging-splitting-proofness

This subsection considers an axiom, *merging-splitting-proofness*, which is closely related to *reallocation-proofness*. In the context of claim problems, a rule is *merging-splitting-proof* if no group of agents can increase their total awards by merging their claims and, conversely, no single agent can increase his award by creating dummy agents and splitting his claim among those dummy agents and himself. This axiom was introduced by O'Neill [20] in the context of bankruptcy problems.

Merging-Splitting-Proofness. For each $N \in \mathscr{N}$, each $(c,E) \in \mathscr{D}^N$, each non-empty $S \subseteq N$, each $i \in S$, and each $c'_i \in \mathbb{R}^K_+$, if $c'_i = \sum_{j \in S} c_j$, then

$$f_i(c'_i, c_{N \setminus S}, E) = \sum_{j \in S} f_j(c,E).$$

Note that the problem on the left-hand side is well-defined since $\mathscr{D}$ is rich*. We also consider a pairwise version of the axiom:[16]

Pairwise Merging-Splitting-Proofness. For each $N \in \mathscr{N}$, each $(c,E) \in \mathscr{D}^N$, each pair $\{i,j\} \subseteq N$ with $i \neq j$, and each $c'_i \in \mathbb{R}^K_+$, if $c'_i = c_i + c_j$, then

$$f_i(c'_i, c_{N \setminus \{i,j\}}, E) = f_i(c,E) + f_j(c,E). \tag{18}$$

The following axiom, introduced by O'Neill [20], states that, if $c_i = 0$ for an entity i, the awards to the other entities are independent of whether entity i is present:

[16] Banker [3] considers a stronger version of *pairwise merging-splitting-proofness* requiring that the merger of a pair should not affect the award for anyone else.

Null Consistency. For each $N \in \mathscr{N}$, each $(c,E) \in \mathscr{D}^N$, and each $i \in N$, if $c_i = 0$, then for each $j \in N \setminus \{i\}$, $f_j(c_{N\setminus\{i\}},E) = f_j(c,E)$.

This axiom differs from *no award for null*. If $c_i = 0$, *no award for null* says $f_i(c,E) = 0$ but allows the other entities $j \in N \setminus \{i\}$ to receive different amounts at (c,E) and $(c_{N\setminus\{i\}},E)$.

We first use *null consistency* to extend the characterization of generalized proportional rules in Theorem 2 to the current variable-population framework. The definition of generalized proportional rules, which is given below, is the same as before but it should be noted that the pair of functions (A,W) is independent of the set N of entities.

Corollary 10. *Assume $|I| \geq 3$ and let f be a rule on a rich* domain $\mathscr{D}$. A rule f satisfies reallocation-proofness, one-sided boundedness, and null consistency if and only if it is a generalized proportional rule, i.e., there exist two functions $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^I$ and $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that, for each $N \in \mathscr{N}$, each $(c,E) \in \mathscr{D}^N$, and each $i \in N$,*

$$f_i(c,E) = A_i(\bar{c},E) + \sum_{k\in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c},E)E.$$

Proof. Let f be a rule on $\mathscr{D}$ satisfying the axioms. Theorem 2 and *null consistency* imply that, for each $N \in \mathscr{N}$, f coincides with a generalized proportional rule on $\mathscr{D}^N$. Let (A^N,W^N) denote the associated pair. By *null consistency*, (A^N,W^N) is independent of N. □

The next result characterizes *merging-splitting-proof* rules. The result also gives a relation between *merging-splitting-proofness* and *reallocation-proofness*: *merging-splitting-proofness* is equivalent to the combination of *reallocation-proofness, no award for null*, and *null consistency*.

Theorem 8. *Assume $|I| \geq 3$ and let f be a rule on a rich* domain $\mathscr{D}$. Then the following three statements are equivalent: (i) f satisfies merging-splitting-proofness; (ii) f satisfies reallocation-proofness, no award for null, and null consistency; (iii) there exists a function $\hat{W}\colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that, for each $N \in \mathscr{N}$, each $(c,E) \in \mathscr{D}^N$, and each $i \in N$,*

$$f_i(c,E) = \sum_{k\in K} \hat{W}_k(c_{ik},\bar{c},E),$$

and, for each $k \in K$, $\hat{W}_k(\cdot,\bar{c},E)$ is additive.

Intuitively, *merging-splitting-proofness* implies *reallocation-proofness* since a reallocation of claims within a group can be done in two steps: merge the claims first and then split them among the members. By *merging-splitting-proofness*, the total awards stay the same in each step, and hence *reallocation-proofness* is satisfied. The fact that *reallocation-proofness, no award for null*, and *null consistency* imply *merging-splitting-proofness* is obtained as follows. By *reallocation-proofness* and *no award for null*, the rule is given by the sum of $\hat{W}_k$. By *null consistency*, $\hat{W}_k$ is independent of the set of agents. This independence and the additivity of $\hat{W}_k$ imply that merging or splitting claims does not affect the total awards.

Proof. Let f be a rule on a rich* domain $\mathscr{D}$ with $|I| \geq 3$. Clearly, (iii) implies (i). The fact that (ii) implies (iii) follows from Theorem 1 as in the proof of Corollary 10. To show that (i) implies (ii), let f be *merging-splitting-proof*.

We first show that f is *reallocation-proof*. Let $N \in \mathscr{N}$, $S \subseteq N$, $i \in S$, $(c,E) \in \mathscr{D}^N$, and $c'_i \in \mathbb{R}_+$ be such that $c'_i = \sum_{j \in S} c_j$. By *merging-splitting-proofness*, $f_i(c'_i, c_{N \setminus S}, E) = \sum_{j \in S} f_j(c,E)$. This equality implies that $\sum_{j \in S} f_j(c,E)$ is invariant under any reallocation of characteristic vectors within S.

We now show that f satisfies *no award for null* and *null consistency*. Let $N \in \mathscr{N}$ and $(c,E) \in \mathscr{D}^N$ be such that $c_h = 0$ for some $h \in N$.

We first consider the case when $|N| \geq 3$. Let $x \equiv f(c,E)$ and $y \equiv f(c_{N \setminus \{h\}}, E)$. Let $j \in N \setminus \{h\}$ and let $\alpha = f_j(\hat{c}_j, E)$ be the award to entity j in the single-entity problem where $\hat{c}_j = \sum_{i \in N} c_i$. By applying *merging-splitting-proofness* to each of (c,E) and $(c_{N \setminus \{h\}}, E)$, we obtain $\sum_{i \in N} x_i = \alpha$ and $\sum_{i \in N \setminus \{h\}} y_i = \alpha$. On the other hand, for each $i \in N \setminus \{h\}$, *merging-splitting-proofness* for the pair $\{i,h\}$ implies $x_i + x_h = y_i$. Hence $\sum_{i \in N} x_i + (|N|-2)x_h = \sum_{i \in N \setminus \{h\}} y_i$. Since $\sum_{i \in N} x_i = \sum_{i \in N \setminus \{h\}} y_i$ and $|N| \geq 3$, we obtain $x_h = 0$, which proves *no award for null*. This in turn implies $x_i = y_i$ for each $i \in N \setminus \{h\}$, which proves *null consistency*.

We now consider N such that $|N| = 2$, say $N = \{1,2\}$. Let $(c_1, c_2, E) \in \mathscr{D}^N$ be such that $c_2 = 0$, and let $y \equiv f(c_1, c_2, E)$. Consider the three-entity problem (c_1, c_2, c_3, E) where $c_3 = 0$, and let $x \equiv f(c_1, c_2, c_3, E)$. Since the result in the previous paragraph applies to the three-entity problem, *null-consistency* implies $(y_1, y_2) = (x_1, x_2)$ and *no award for null* implies $x_2 = 0$. Thus $y_2 = 0$, which proves *no award for null*. Finally, *merging-splitting-proofness* implies $f_1(c_1, E) = y_1 + y_2 = y_1$, which proves *null-consistency*. $\square$

The following result characterizes proportional rules as in Theorem 4. The definition of proportional rules is the same as in the previous sections but the vector of weights $W(\bar{c}, E)$ is independent of the set of entities N.

Theorem 9. *Assume $|I| \geq 3$ and let f be a rule on a rich* domain $\mathscr{D}$. Then the following three statements are equivalent: (i) f satisfies pairwise merging-splitting-proofness, efficiency, and non-negativity (or no transfer paradox); (ii) f satisfies pairwise reallocation-proofness, efficiency, non-negativity (or no transfer paradox), and null consistency; (iii) f is a proportional rule, i.e., there exists a weight function $W: \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \Delta^{|K|-1}$ such that, for each $N \in \mathscr{N}$, each $(c,E) \in \mathscr{D}^N$, and each $i \in N$,* (1) *holds.*

Proof. Clearly, (iii) implies (i) and (ii).

(ii) $\Rightarrow$ (iii). Let f satisfy the axioms in (ii). Note that *efficiency* and *null consistency* imply *no award for null*. Theorem 4 and *null consistency* then imply that, on $\mathscr{D}^N$ for a given $N \in \mathscr{N}$, f is a proportional rule for some weight function W^N. By *null consistency*, W^N is identical for all N.

(i) $\Rightarrow$ (ii). Let f satisfy the axioms in (i). To prove that f is *pairwise reallocation-proof*, we can use the argument in the proof of Theorem 8 ((i) $\Rightarrow$ (ii)) for S such that $|S| = 2$. To show that f satisfies *null consistency*, let $N \in \mathscr{N}$ and $(c,E) \in \mathscr{D}^N$ be such that $c_h = 0$ for some $h \in N$. Let $x \equiv f(c,E)$ and $y \equiv f(c_{N \setminus \{h\}}, E)$. In the proof of Theorem 8 ((i) $\Rightarrow$ (ii)), we used *merging-splitting-proofness* with respect to coalitions with more than two entities only to obtain

$\sum_{i\in N} x_i = \sum_{i\in N\setminus\{h\}} y_i$. This equality now holds by *efficiency*. We can use the remaining argument in the proof of Theorem 8 ((i) $\Rightarrow$ (ii)) to show that f satisfies *null consistency*. $\square$

If $|K| = 1$, the same argument that led us to Corollary 3 also implies that *non-negativity* (or *no transfer paradox*) in Theorem 9 can be weakened to *one-sided boundedness*.

Corollary 11. *Assume* $|I| \geq 3$ *and* $|K| = 1$ *and let* f *be a rule on a rich* domain. Then the following three statements are equivalent: (i)* f *satisfies pairwise merging-splitting-proofness, efficiency, and one-sided boundedness; (ii)* f *satisfies pairwise reallocation-proofness, efficiency, one-sided boundedness, and null consistency; (iii)* f *is the proportional rule.*

Several papers consider the case where $|K| = 1$ and prove results similar to Corollary 11. O'Neill [20, Theorem C.1] considers bankruptcy problems and characterizes the proportional rule using *merging-splitting-proofness*, *efficiency*, *anonymity*, *continuity*, and *null consistency*. Chun [8, Theorem 3] considers claim problems and shows that *null consistency* in O'Neill's result is redundant. de Frutos [9, Theorem 1] considers bankruptcy problems and characterizes the proportional rule using *merging-splitting-proofness*, *efficiency*, and *non-negativity*, thereby showing that *anonymity* and *continuity* in Chun's result can be replaced with *non-negativity*. Our result [(i) $\Leftrightarrow$ (iii)] strengthens de Frutos's by weakening *non-negativity* to *one-sided boundedness* and showing that the pairwise version of *merging-splitting-proofness* suffices. As we mentioned after Corollary 3, Chun [8, Theorem 2] also characterizes the proportional rule using *reallocation-proofness*, *efficiency*, *anonymity*, *continuity*, and *null consistency*. Our result [(ii) $\Leftrightarrow$ (iii)] strengthens Chun's Theorem 2 by removing *anonymity*, weakening *continuity* to *one-sided boundedness*, and showing that the pairwise version of *reallocation-proofness* suffices. Ju [12] considers bankruptcy problems and shows that for rules that satisfy *efficiency*, *non-negativity*, and "claim boundedness" (requiring $f_i(c,E) \leq c_i$), *pairwise merging-splitting-proofness* is equivalent to the combination of *pairwise reallocation-proofness* and *null consistency*. Our result [(i) $\Leftrightarrow$ (ii)] strengthens Ju's by removing "claim boundedness" and weakening *non-negativity* to *one-sided boundedness*. All the existing results mentioned above are proved under the assumption that there exist an infinite number of potential agents ($|I| = \infty$).

7.2 *Equal treatment of equal groups*

This subsection considers another axiom that is also closely related to *reallocation-proofness*. The axiom, called *equal treatment of equal groups*, extends the idea of *equal treatment of equals* to groups, requiring that two groups with the same aggregate claims should receive the same amount. The axiom was introduced by Chambers and Thomson [6] and Ching and Kakkar [7] in the context of bankruptcy problems.

Equal Treatment of Equal Groups. For each $N \in \mathscr{N}$, each $N', N'' \subseteq N$, and each $(c,E) \in \mathscr{D}^N$, if $\sum_{i\in N'} c_i = \sum_{i\in N''} c_i$, then $\sum_{i\in N'} f_i(c,E) = \sum_{i\in N''} f_i(c,E)$.

In this subsection, we focus on the classes of bankruptcy, surplus sharing, and claim problems. We say that a rule is *regular* if it satisfies *efficiency, non-negativity, no award for null,* and the following condition of *claim boundedness*:[17]

$$\begin{cases} f(c,E) \leqq c & \text{for bankruptcy problems,} \\ f(c,E) \geqq c & \text{for surplus sharing problems,} \\ \text{no condition} & \text{for claim problems.} \end{cases}$$

We also consider what is called *consistency* or the *reduced-game property*. Suppose that, after awards are determined by a rule, a subset of agents "leave the scene" with their awards. Then *consistency* says that reapplying the rule to the problem with the remaining agents and the remaining amount to divide does not change the award vector for those agents.

Consistency. For each $N \in \mathscr{N}$, each $(c,E) \in \mathscr{D}^N$, and each $N' \subseteq N$, $f_{N'}(c,E) = f(c_{N'}, E - \sum_{i \in N \setminus N'} f_i(c,E))$.[18]

The next result shows a relation among *consistency*, *equal treatment of equal groups*, and *reallocation-proofness*. The relation in turn yields an alternative characterization of the proportional rule.

Theorem 10. *Assume that $\mathscr{D}$ is any of the three domains—bankruptcy, surplus sharing, or claim problems—with at least 6 potential agents. If a regular rule f on $\mathscr{D}$ satisfies equal treatment of equal groups and consistency, then for each $N \in \mathscr{N}$ with $3 \leq |N| \leq |I|/2$, the restriction of f on $\mathscr{D}^N$ is reallocation-proof; in fact, f is necessarily the proportional rule over the entire domain. Therefore, the proportional rule is the only regular rule that satisfies equal treatment of equal groups and consistency.*

Proof. Let f be a *regular* rule satisfying the axioms on any of the three domains with at least 6 potential agents. We first prove that f is *reallocation-proof* for each $N \in \mathscr{N}$ such that $3 \leq |N| \leq |I|/2$. Without loss of generality, assume $N = \{1,2,\ldots,n\}$, where $n \equiv |N|$. Let $(c,E) \in \mathscr{D}^N$, $N' \subseteq N$, and $\hat{c} \in \mathbb{R}^N_+$ be such that $\sum_{i \in N'} \hat{c}_i = \sum_{i \in N'} c_i$ and $\hat{c}_i = c_i$ for all $i \in N \setminus N'$. Without loss of generality, assume $N' = \{1,2,\ldots,n'\}$, where $n' \equiv |N'|$. To show $\sum_{i \in N'} f_i(\hat{c},E) = \sum_{i \in N'} f_i(c,E)$, let $M \equiv \{n+1, n+2, \ldots, 2n\}$, $M' \equiv \{n+1, n+2, \ldots, n+n'\}$, and $c' \in \mathbb{R}^M_+$ be such that $\sum_{i \in M} c'_i = \sum_{i \in N} c_i$ and $\sum_{i \in M'} c'_i = \sum_{i \in N'} c_i$. Consider the problem for $N \cup M$, $(c, c', 2E)$. By *equal treatment of equal groups* and *efficiency*, $\sum_{i \in N} f_i(c,c',2E) = \sum_{i \in M} f_i(c,c',2E) = E$ and $\sum_{i \in N'} f_i(c,c',2E) = \sum_{i \in M'} f_i(c,c',2E)$. By *consistency*, $f_N(c,c',2E) = f(c,E)$ and $f_M(c,c',2E) = f(c',E)$. The last three equalities imply $\sum_{i \in N'} f_i(c,E) = \sum_{i \in M'} f_i(c',E)$. By replacing c with $\hat{c}$, the same argument yields $\sum_{i \in N'} f_i(\hat{c},E) = \sum_{i \in M'} f_i(c',E)$. Hence $\sum_{i \in N'} f_i(c,E) = \sum_{i \in N'} f_i(\hat{c},E)$. This shows that f is *reallocation-proof* on $\mathscr{D}^N$.

It remains to show that f is the proportional rule. By Corollary 1, f coincides with the proportional rule on $\mathscr{D}^N$ for all $N \in \mathscr{N}$ such that $3 \leq |N| \leq |I|/2$. By *consistency*, f is also the

[17] Note that for bankruptcy problems, *claim boundedness* and *non-negativity* imply *no award for null*.

[18] If $E - \sum_{i \in N \setminus N'} f_i(c,E) = 0$, the last term is not well-defined because of our simplifying assumption that $E > 0$ for all problems. Thus we complete the definition by saying that, if $E - \sum_{i \in N \setminus N'} f_i(c,E) = 0$, then $f_{N'}(c,E) = 0$. Given this, the last term is always well-defined for regular rules because of *efficiency* and *claim boundedness*. Thomson [25] surveys the large literature of the consistency principle.

proportional rule on two-agent problems. To show that f is the proportional rule on the entire domain, take any $N \in \mathscr{N}$ and let $(c,E) \in \mathscr{D}^N$ and $x \equiv f(c,E)$. Since the rule is the proportional rule on two-agent problems, *consistency* for a given pair $\{i,j\} \subseteq N$ implies $x_i c_j = c_i x_j$. Aggregating this equation for all $j \in N$ yields $x_i \sum_{j \in N} c_j = c_i \sum_{j \in N} x_j = c_i E$ by *efficiency*. This shows that f is the proportional rule on the entire domain. □

Chambers and Thomson [6, Theorem 5] show that, in the class of bankruptcy problems with $|I| \geq 3$, the proportional rule is the only *regular* rule satisfying *equal treatment of equal groups*, *consistency*, and *continuity*. It has been known that if $|I| = \infty$, *continuity* is redundant in their characterization. Indeed, as they show, *equal treatment of equal groups* and *consistency* imply *merging-splitting-proofness* if $|I| = \infty$ (Chambers and Thomson [6, Theorem 7]). This and the result of de Frutos [9] (or our Corollary 11) imply that, if $|I| = \infty$, the proportional rule is the only *regular* rule that satisfies *equal treatment of equal groups* and *consistency*. It has been an open question whether, when $|I|$ is finite, their result holds without *continuity*. Theorem 10 shows that *continuity* is in fact redundant if $|I| \geq 6$.[19]

Theorem 10 also holds even if we weaken *equal treatment of equal groups* by requiring the equal treatment condition only to a pair of *disjoint* groups of the same *size*: N' and N'' such that $N' \cap N'' = \emptyset$ and $|N'| = |N''|$. It is easy to see that the proof of Theorem 10 also works with this axiom. We remark that, on a rich* domain with $|I| = \infty$, the weaker version of *equal treatment of equal groups*, *no award for null*, and *null consistency* together imply the original version of *equal treatment of equal groups*. We leave the easy proof to the reader.

8 Extensions

We conclude the paper by discussing a few ways to extend our model. First, the model can be extended to allow (c,E) to take negative values. For example, there may be debtors as well as creditors, and the surplus to divide may be sometimes negative. Theorem 1 extends to the case where the domain is $\mathscr{D} = \mathbb{R}^{N \times K} \times \mathbb{R}$. For general domains $\mathscr{D} \subseteq \mathbb{R}^{N \times K} \times \mathbb{R}$, Appendix gives a generalized condition of richness. Second, the proof of Theorem 1 also extends to the model where all values are restricted to integers. However, if awards have to be integers too, proportional rules are not admissible and an impossibility result obtains. Third, one might want to extend the model to allow for multi-dimensional resources (multiple goods). Doing so, however, requires one to introduce entities' preferences over different types of resources and generalize *reallocation-proofness*. Finally, we assumed that any subset or pair of agents can form a coalition. Ju [13] extends our results to the case where there are exogenously given restrictions on what coalitions can form.

[19] Actually, we can show that *continuity* is redundant if $|I| \geq 4$. The proof, however, does not use *reallocation-proofness* and hence is not given here; it is available from the authors upon request.

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Appendix

Proof of Corollary 6. Let f be a rule satisfying the axioms. Then by Corollary 2, f is given by (11) for some non-negative valued function $W : \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K_+$. Define ρ by $\rho_k(\bar{c}) \equiv W_k(\bar{c})\bar{c}_{\max}/\bar{c}_k$. With this definition and $E = \bar{c}_{\max}$, (11) reduces to (16). It remains to show that ρ satisfies (15) and $\sum_{k\in K} \rho_k(\bar{c}) = 1$.

We first prove that ρ satisfies (15). Let $d \in \mathbb{R}^K_{++}$ and $\lambda > 0$. Pick $h \in K$ and $j, \ell \in N$ arbitrarily, and let $c \in \mathbb{R}^{K\times N}_+$ be such that $\bar{c} = d$, $c_{jh} > 0$, $c_{jk} = 0$ for each $k \in K\backslash\{h\}$, and $c_{\ell k} = 0$ for each $k \in K$. Since $|N| \geq 3$, there exists another agent $m \in N \setminus \{j, \ell\}$. Let $c' \in \mathbb{R}^{K\times N}_+$ be the profile defined by $c' \equiv (c_m + \lambda \mathbf{1}, c_{-m})$. By *translation invariance*, $f_j(c') = f_j(c)$ and $f_\ell(c') = f_\ell(c)$. By definition of c and (16), $f_j(c) = f_\ell(c) + c_{jh}\rho_h(\bar{c})$ and $f_j(c') = f_\ell(c') + c_{jh}\rho_h(\bar{c} + \lambda\mathbf{1})$. Since $c_{jh} > 0$, we obtain $\rho_h(\bar{c} + \lambda\mathbf{1}) = \rho_h(\bar{c})$.

We now prove that $\sum_{k\in K} \rho_k(d) = 1$ for all d. By the previous paragraph, it suffices to prove the result for d such that $d_k > 1$ for all $k \in K$. Pick two agents $j, \ell \in N$ arbitrarily, and let $c \in \mathbb{R}^{K\times N}_+$ be such that $\bar{c} = c_\ell = d$. Let $c' \in \mathbb{R}^{K\times N}_+$ be defined by $c' \equiv (c_j + \mathbf{1}, c_\ell - \mathbf{1}, c_{N\setminus\{j,\ell\}})$. Then $\bar{c'} = \bar{c}$, and *translation invariance* implies $f_j(c') = f_j(c) + 1$. Since $f_j(c)$ and $f_j(c')$ differ only in the last term of (16),

$$1 = f_j(c') - f_j(c) = \sum_{k\in K} (c_{jk} + 1 - c_{jk})\rho_k(\bar{c}) = \sum_{k\in K} \rho_k(\bar{c}).$$

Proof of Corollary 7. Let f be a rule satisfying the axioms in the corollary. By Theorem 4, f is a proportional rule with some weight function W. Define ρ by $\rho_k(\bar{c}) = W_k(\bar{c})\bar{c}_{\max}/\bar{c}_k$. Then $f_i(c) = \sum_{k\in K} \rho_k(\bar{c})c_{ik}$.

We first show that ρ satisfies (15). Let $d \in \mathbb{R}^K_{++}$ and $\lambda > 0$. Pick $j \in N$ and $h \in K$ arbitrarily, and let $c \in \mathbb{R}^{N\times K}_+$ be such that $\bar{c} = d$, $c_{jh} > 0$, and $c_{jk} = 0$ for all $k \in K \setminus \{h\}$. Let $\ell \in N \setminus \{j\}$ and $c' \equiv (c_\ell + \lambda\mathbf{1}, c_{-\ell})$. By *translation invariance*, $f_j(c') = f_j(c)$. Since $f_j(c) = \rho_h(\bar{c})c_{jh}$ and $f_j(c') = \rho_h(\bar{c} + \lambda\mathbf{1})c_{jh}$, we obtain $\rho_h(\bar{c} + \lambda\mathbf{1}) = \rho_h(\bar{c})$.

The same argument in the proof of Corollary 6 shows that $\sum_{k\in K} \rho_k(d) = 1$ for all d.

It remains to show that $\rho(\bar{c})$ satisfies (17). By *efficiency*, $\bar{c}_{\max} = \sum_{i\in N} f_i(c) = \sum_{k\in K} \rho_k(\bar{c})\bar{c}_k$. Since $\rho_k(\bar{c}) \geq 0$ and $\sum_{k\in K} \rho_k(\bar{c}) = 1$, the equality holds if and only if $\rho(\bar{c})$ satisfies (17). □

Generalized richness. We here define a generalized condition of richness that enables us to extend our results to division problems that involve negative values. Intuitively, instead of using 0 as the origin, we can have any number b_{ik} as the relevant origin for agent i's claim in issue k.

Definition 6. A domain $\mathscr{D} \subseteq \mathbb{R}^{N\times K} \times \mathbb{R}$ satisfies *generalized richness* if for each $(d,E) \in \mathbb{R}^K \times \mathbb{R}$ with $\mathscr{D}(d,E) \neq \emptyset$, there exists $b = (b_i)_{i\in N} \in \mathbb{R}^{N\times K}$ such that if a set $X \subseteq \mathbb{R}^K$ is defined by

$$X \equiv \Big\{ \sum_{i\in S} c_i - \sum_{i\in S} b_i : (c,E) \in \mathscr{D}(d,E) \text{ and } \emptyset \neq S \subseteq N \Big\},$$

then
(i) $(0,\ldots,0) \in X$,
(ii) For each $x \in X$ and each $k \in K$, $(0,\ldots,0,x_k,0,\ldots,0) \in X$,
(iii) For each pair $x,y \in X$ with $x+y \in X$ and each pair of disjoint subsets $S,T \subseteq N$ with $S\cup T \subsetneq N$, there exists $(c,E) \in \mathscr{D}(d,E)$ such that

$$\sum_{i\in S} c_i - \sum_{i\in S} b_i = x, \quad \sum_{i\in T} c_i - \sum_{i\in T} b_i = y,$$

(iv) For each $S \subseteq N$ and each pair $(c,E),(c',E) \in \mathscr{D}(d,E)$, if $\sum_{i\in S} c'_i = \sum_{i\in S} c_i$, then $((c'_S, c_{N\setminus S}),E) \in \mathscr{D}$.

Generalized richness is indeed weaker than richness; richness implies that generalized richness can be satisfied by setting $b = 0$ for any (d,E). On the other hand, there are a number of domains that satisfy generalized richness but not richness: e.g., $\mathbb{R}^{N\times K} \times \mathbb{R}$ (setting $b = 0$), $\mathbb{R}_-^{N\times K} \times \mathbb{R}_-$ (setting $b = 0$), and for any $b \in \mathbb{R}^{K\times N}$, $\{(c,E) \in \mathbb{R}^{N\times K} \times \mathbb{R} : c \geq b\}$ and $\{(c,E) \in \mathbb{R}^{N\times K} \times \mathbb{R} : c \leq b\}$.

For any domain that satisfies generalized richness, Theorem 1 extends and characterizes rules of the following form:

$$f_i(c,E) = A_i(\bar{c},E) + \sum_{k\in K} \hat{W}_k(c_{ik} - b_{ik}, \bar{c}, E),$$

where b_{ik} depends on $(\bar{c},E)$ and $\hat{W}_k$ is additive in $c_{ik} - b_{ik}$ for a given $(\bar{c},E)$. Since the proof of this extension is essentially the same as before, we omit it.[20]

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[20] Here are two important changes in the proof: define $A_i(d,E) \equiv f_i(c,E)$ with $c \in \mathscr{C}$ and $c_i = b_i$; for the definition of $w(S,x)$, replace $\prod_{k=1}^{K}[0,d_k]$ with X and $\sum_{i\in S} c_i = x$ with $\sum_{i\in S}(c_i - b_i) = x$.

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