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Abstract

We consider a model of social choice dealing with the problem of choosing a subset from a set of objects (e.g. candidate selection, membership, and qualification problems). Agents have *trichotomous preferences* for which objects are partitioned into three indifference classes, goods, bads, and nulls, or *dichotomous preferences* for which each object is either a good or a bad. We characterize plurality-like social choice rules on the basis of the three main axioms, known as *Pareto efficiency*, *anonymity*, and *independence*.

Keywords. Pareto efficiency; anonymity; independence; plurality; trichotomous preferences; dichotomous preferences.

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1 Introduction

We consider a model of social choice dealing with the following problems. There is a society consisting of at least two agents. The society needs to choose a subset from a set of objects. There is no constraint in the choice and any subset is a feasible alternative. Agents have simple preferences that are described by means of the three indifference classes of objects: goods, bads, and nulls. A social choice rule, briefly a *rule*, associates with each profile of preferences a single alternative. The main objective of this paper is to study rules that satisfy the three axioms of social choice, known as *Pareto efficiency*, *anonymity*, and *independence.*

Pareto efficiency requires that an alternative chosen by a rule should attain a state of maximal well-being of agents; that is, there should be no other alternative making an agent better off without making anyone else worse off. *Anonymity* requires that agents should be treated symmetrically. *Independence* (Kasher and Rubinstein 1997; Samet and Schmeidler 2003) requires that the decision on each object should be made independently of other objects, based only on who is in favor of this object and who is against it.

Independence enables society to simplify its decision process greatly. In order to make a decision on an object, the only information that society needs to collect is the set of agents for whom this object is a good and the set of agents for whom it is a bad. In the model with "separable preferences", the cost of this informational simplicity is enormous. As shown by Barberà, Sonnenschein, and Zhou (1991) and Ju (2004), *independence* is incompatible with *Pareto efficiency* and the minimal equity condition of "non-dictatorship".[1] This impossibility no longer applies for the following simple preferences in our model and, furthermore, the non-dictatorship condition can be strengthened to *anonymity*. A preference is *trichotomous* if it has an additive representation and objects are partitioned into three indifference classes: goods, bads, and nulls. Special examples are *dichotomous* preferences for which objects are partitioned into goods and bads as considered by Barberà, Maschler, and Shalev (2001).[2] Although domains

[1]When preferences are separable and linear, Barberà, Sonnenschein, and Zhou (1991) introduce a large family of rules satisfying *independence*, known as "voting by committees". They show that among these rules, only dictatorial rules satisfy *Pareto efficiency*. This is used to prove their impossibility result saying that only dictatorial rules satisfy *Pareto efficiency* and "strategy-proofness". Ju (2004, Section 6) shows that *strategy-proofness* in this result can be replaced with *independence* (which, in his paper, is referred to as the "issue-wise voting property").

[2]Barberà, Maschler, and Shalev (2001) consider dichotomous preferences for the study of strategic voting equilibria in a dynamic extension of our choice problem.

consisting of these preferences are substantially restricted compared to the entire domain of separable preferences, they are rich enough to have some interesting applications. For example:

1. *Qualification Problem* (Samet and Schmeidler 2003).[3] A society needs to decide which members are qualified for a certain activity. Assume that any two qualified (or unqualified) members have the same positive (negative) effect on society and that these individual effects can be aggregated by simple summation. The preference of each member can, then, be described by means of the information on who he/she thinks is qualified and who he/she thinks is unqualified. This type of preference is dichotomous. Naturally, there may be indifference between qualifying and disqualifying a member. In this case, each member's preference is trichotomous.

2. *Identity Problem* or *Dichotomous Aggregation Problem* (Kasher and Rubinstein 1997). Suppose that the definition of a collective such as a family, a nation, a religious group, etc. is obscure. Individuals in society N have different opinions about who belongs to the collective and who does not. A social choice specifies the collective as a subset of N, which can be interpreted as the aggregate opinion viewing each chosen member as belonging to the collective. An opinion can be described by an $|N|$-vector consisting of 0 and 1, where 0 in the i^{th} component means that individual i does not belong to the collective and 1 means that he/she does belong to the collective. Assume that individuals want the aggregate opinion to be as close as possible to their own. Then preferences over aggregate opinions coincide with dichotomous preferences, when the "closeness" is measured by some standard distance functions, including the Euclidean norm, L_1-norm, etc.

Our main results pertain to both trichotomous and dichotomous domains with "symmetry" between goods and bads (the utility from each good is identical to the disutility from each bad). We characterize rules satisfying *Pareto efficiency*, *anonymity*, and *independence* or rules satisfying the three axioms together with *neutrality* (which says that objects should be treated symmetrically). The plurality rule is one of these rules. It accepts each object if and only if the number of agents in favor of this object is greater than the number of agents against it. All other rules make the same decision as the plurality rule unless the number of agents in favor of an object is the same as the number of agents against it. These

[3]Samet and Schmeidler (2003) do not explicitly consider dichotomous preferences. However, we think that the dichotomous domain best fits their model and that on this domain, their two main axioms "monotonicity" and "independence" can be understood most naturally.

rules are called *semi-plurality rules*. When the number of agents is odd and preferences are dichotomous, the plurality rule is the only semi-plurality rule. *Pareto efficiency* plays a key role in pinning down the small family of semi-plurality rules among the great variety of rules satisfying *anonymity* and *independence*. Thus, our characterization shows the close logical relation between the two seemingly unrelated principles, *plurality* and *Pareto efficiency*.

We also show that a similar characterization holds even if we drop the assumption of symmetry between goods and bads. A characterization of some variants of semi-plurality rules is obtained on any trichotomous (or dichotomous) domain without symmetry. On each trichotomous domain, preferences have at most one indifference class of goods and at most one indifference class of bads. It is natural to ask whether the existence parts of our characterization results will continue to hold when we allow for more than one indifference class of goods or bads. We answer this question negatively under some weak assumptions on preference domains.

Trichotomous or dichotomous preferences can be considered as opinions in the identity problem studied by Kasher and Rubinstein (1997) and in the qualification problem studied by Samet and Schmeidler (2003). Both papers study social choice rules satisfying *anonymity* and *independence* as well as some other normative requirements.[4] However, *Pareto efficiency* is not one of them. Bogomolnaia, Moulin, and Stong (2002) study probabilistic social choice rules in the Arrovian social choice model. They focus on what they call dichotomous preferences and show existence of probabilistic rules satisfying *Pareto efficiency*, *anonymity*, and "strategy-proofness". Our dichotomous preferences may seem to be the natural counterpart to theirs. However, ours are not dichotomous in their sense. Any dichotomous preference in our model has more than two indifference classes of *alternatives*, although it has only two indifference classes of *objects*. Barberà, Sonnenschein, and Zhou (1991) consider separable linear preferences, which encompass a greater variety of preferences than our trichotomous preferences (but there is no inclusion between the two domains since none of our trichotomous preferences is linear). It follows from their main result that *independence* is a necessary condition of *strategy-proofness*.[5] This implication is also confirmed in the extended model of multi-dimensional public goods problems by Le Breton and Sen (1999).

[4]One of the main axioms in Samet and Schmeidler (2003) is the symmetry axiom that is weaker than the combination of *anonymity* and *neutrality*.

[5]More precisely, their results show that *strategy-proofness* and the full range condition together imply *independence*.

The rest of the paper is organized as follows. We introduce the model and define basic concepts in Section 2. We establish the main results in Section 3. In Section 4, we consider trichotomous domains without symmetry. In Section 5, we establish an impossibility result on non-trichotomous domains. We conclude with a few remarks in Section 6. Some proofs are collected in the appendix.

2 Model and Basic Concepts

There is a society with n members, $n \geq 2$. The society has to choose objects from a finite set A. There is no constraint in the choice. A social *alternative* is any subset of A. Then the power set of A, 2^A, describes the set of alternatives. Let $N \equiv \{1, \ldots, n\}$ be the set of the members, or agents. Assume that there are at least two objects in A, that is, $|A| \geq 2$.[6] Each agent is characterized by a *preference*, a complete and transitive binary relation over alternatives. For each preference R_i, we denote the associated strict and indifference relations by P_i and I_i, respectively.

A preference R_i is *additive* if there is a function $u_i \colon A \to \mathbb{R}$ such that for each $X, X' \subseteq A$, $X \; R_i \; X'$ if and only if $\sum_{x \in X} u_i(x) \geq \sum_{x' \in X'} u_i(x')$. Given an additive preference R_i represented by $u_i \colon A \to \mathbb{R}$, objects are partitioned into three categories. An object $x \in A$ is a *good* if $u_i(x) > 0$. It is a *bad* if $u_i(x) < 0$. It is a *null* if $u_i(x) = 0$. For each $X \subseteq A$, let $G(R_i, X)$ be the set of goods in X and $B(R_i, X)$ the set of bads in X. We write $G(R_i) \equiv G(R_i, A)$ and $B(R_i) \equiv B(R_i, A)$.

We focus on the following simple additive preferences, which are completely described by the information on goods, bads, and nulls. An additive preference R_i is *trichotomous* if objects are partitioned into three indifference classes, goods, bads, and nulls (any pair of goods are indifferent; any pair of bads are indifferent). Then, there exist two real numbers $u, v > 0$ such that for each $X, X' \subseteq A$, $X \; R_i \; X'$ if and only if $u|G(R_i, X)| - v|B(R_i, X)| \geq u|G(R_i, X')| - v|B(R_i, X')|$. We call u and v the *utility of goods* and the *disutility of bads*, respectively.[7] A preference is *dichotomous* if it is trichotomous and objects are either goods or bads (there is no null). When the utility and the disutility are equal, that is, $u = v$, we say that R_i is *symmetric.* Let $\mathcal{S}_{\text{Tri}}$ be the family of trichotomous preferences with symmetry and $\mathcal{S}_{\text{Di}}$ the family of dichotomous preferences with

[6] When there is only one object, our results do not apply. But in this case, it is trivial to characterize rules satisfying our axioms.

[7] Note that u and v may differ for various representations of a trichotomous preference but their ratio u/v is unique.

symmetry. For the time being, we will focus on such symmetric preferences. Other preferences will be considered in Section 4.

Let $\mathcal{D} \in \{\mathcal{S}^N_{\text{Tri}}, \mathcal{S}^N_{\text{Di}}\}$. A social choice rule on $\mathcal{D}$, or simply, a *rule,* is a function $\varphi\colon \mathcal{D} \to 2^A$ mapping each preference profile into a *single* alternative. We call $\mathcal{D}$ the *domain* of the rule. For each $i \in N$, let $\mathcal{D}_i \equiv \{R_i : R \in \mathcal{D}\}$ be the projection of $\mathcal{D}$ onto the i^{th} component.[8] With a slight abuse of terminology, we call both $\mathcal{S}^N_{\text{Tri}}$ and $\mathcal{S}_{\text{Tri}}$ (respectively, both $\mathcal{S}^N_{\text{Di}}$ and $\mathcal{S}_{\text{Di}}$) the *trichotomous* (resp. *dichotomous*) *domain with symmetry.*

Examples of Rules

The best-known rule is the *plurality rule* φ^{PL} defined as follows: for each $R \in \mathcal{D}$ and each $x \in A$, $x \in \varphi^{\text{PL}}(R)$ if and only if the number of agents for whom x is a good is greater than the number of agents for whom x is a bad. The plurality rule is a member of the following family of rules that are described by "power structures" between subgroups of N. Let $\mathfrak{C}^* \equiv \{(C_1, C_2) \in 2^N \times 2^N : C_1 \cap C_2 = \emptyset\}$ be the set of all pairs of disjoint subgroups of N. For each $x \in A$, a *power structure for* x, denoted by $\mathfrak{C}_x \subseteq \mathfrak{C}^*$, is a family of pairs of disjoint subgroups, which will be used to describe which subgroup is superior to another in deciding on x. A *profile of power structures* is a list $(\mathfrak{C}_x)_{x \in A}$ of power structures indexed by objects.[9] To define a family of rules represented by profiles of power structures, we use the following notation: for each $R \in \mathcal{D}$ and each $x \in A$,

$$\begin{aligned} N^G_x(R) &\equiv \{i \in N : x \text{ is a good for } R_i\}; \\ N^B_x(R) &\equiv \{i \in N : x \text{ is a bad for } R_i\}. \end{aligned}$$

We call $N^G_x(R)$ the *supporting group* and $N^B_x(R)$ the *opposing group.* A rule φ is *represented by a profile* $(\mathfrak{C}_x)_{x \in A}$ if for each $R \in \mathcal{D}$ and each $x \in A$, $x \in \varphi(R)$ if and only if $\left(N^G_x(R), N^B_x(R)\right) \in \mathfrak{C}_x$.

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[8] We use notation, R, R', $\bar{R}$, $\bar{R}'$, etc. for elements in $\mathcal{D}$. Following standard notational convention, we write i's component of R as R_i and we write i's component of R' as R'_i.

[9] Power structures are similar to "binary constitutions" by Ferejohn and Fishburn (1979) in the Arrovian social choice model. But the two concepts have crucial differences. First, binary constitutions are used to determine a social preference ordering between two alternatives. Second, an extra restriction, called "asymmetry", is required for the definition of a binary constitution. In the *linear* separable preference domain, power structures with some additional properties reduce to the "committee structures" of Barberà, Sonnenschein, and Zhou (1991). Power structures were introduced in the domain with separable preferences (not necessarily linear) by Ju (2003a).

Axioms

We are interested in rules satisfying the following requirements, or axioms. The most crucial one in this paper is *Pareto efficiency*. It says that there should be no other alternative making an agent better off without making anyone else worse off.[10] Formally:

Pareto Efficiency. For each $R \in \mathcal{D}$, there is no $X \subseteq A$ such that for each $i \in N$, $X \; R_i \; \varphi(R)$, and for some $j \in N$, $X \; P_j \; \varphi(R)$.

The next axiom requires that the decision on each object should depend only on agents' evaluations of *this object*. This axiom is considered by Rubinstein and Fishburn (1986), Kasher and Rubinstein (1997), and Samet and Schmeidler (2003).[11]

Independence. For each $x \in A$ and each $R, R' \in \mathcal{D}$ with $N_x^G(R) = N_x^G(R')$ and $N_x^B(R) = N_x^B(R')$, $x \in \varphi(R)$ if and only if $x \in \varphi(R')$.

It should be noted that every rule satisfying *independence* is represented by a profile of power structures and the converse also holds.

We next state two standard axioms requiring symmetric treatment for both agents and objects. For each permutation on N, $\pi : N \to N$, and each $R \in \mathcal{D}$, let R^π be such that for each $i \in N$, $R_i^\pi \equiv R_{\pi(i)}$. Let $\rho\colon A \to A$ be a permutation on A. For each $R_i \in \mathcal{D}_i$, let $\rho R_i \in \mathcal{D}_i$ be the preference such that for each $X, X' \subseteq A$, $X \; \rho R_i \; X'$ if and only if $\rho(X) \; R_i \; \rho(X')$. For each $R \in \mathcal{D}$, let $\rho R \equiv (\rho R_i)_{i \in N}$.

The next axiom states that names of agents should not matter.

Anonymity. For each $R \in \mathcal{D}$ and each permutation π on N, $\varphi(R) = \varphi(R^\pi)$.

Note that the plurality rule is represented by the profile $(\mathfrak{C}_x)_{x \in A}$ such that for each $x \in A$, $\mathfrak{C}_x \equiv \{(C_1, C_2) \in \mathfrak{C}^* : |C_1| > |C_2|\}$. Whether a pair (C_1, C_2) is in the power structure for the plurality rule depends only on the number of agents in the first group C_1 and the number of agents in the second group C_2. Because of this property, the plurality rule satisfies *anonymity*. There are various other rules with the same property. To define them, let $\mathcal{I}^* \equiv \{(t_1, t_2) \in \mathbb{Z}_+^2 : t_1 + t_2 \leq n\}$ be the set of possible pairs of cardinalities of two disjoint groups, where $\mathbb{Z}_+$ denotes the set of non-negative integers. A profile $(\mathfrak{C}_x)_{x \in A}$ satisfies power-anonymity,

[10] In stating an axiom, we only state the condition on the generic rule φ required by the axiom.

[11] Our axiom is stated differently from the independence axiom in these papers, in which each object is either a good or a bad. We generalize their axiom in order to deal with nulls. On the dichotomous domain, our axiom coincide with theirs.

briefly *P-anonymity*, if for each $x \in A$, $\mathfrak{C}_x$ can be described by a set $\mathcal{I}_x \subseteq \mathcal{I}^*$ as follows: $(C_1, C_2) \in \mathfrak{C}_x \Leftrightarrow (|C_1|, |C_2|) \in \mathcal{I}_x$. We call $\mathcal{I}_x$ the *index set for x*. It is easy to show that every rule represented by a profile of index sets $(\mathcal{I}_x)_{x\in A}$ satisfies *anonymity*.

The next axiom states that names of objects should not matter.

Neutrality. For each $R \in \mathcal{D}$ and each permutation ρ on A, $\varphi(\rho R) = \rho(\varphi(R))$.

A profile $(\mathfrak{C}_x)_{x\in A}$ satisfies power-neutrality, briefly *P-neutrality*, if for each $x, y \in A$, $\mathfrak{C}_x = \mathfrak{C}_y$. It is easy to show that every rule represented by a P-neutral profile of power structures satisfies *neutrality*. When a rule is represented by a P-neutral profile of power structures, we say that it is *represented by a single power structure*. Similarly, we say that a rule is *represented by a single index set*.

3 Main Results

Imposing *Pareto efficiency*, *independence*, and *anonymity*, we characterize a family of rules that are very close to the plurality rule. Each of these rules makes the same decision on each object as does plurality rule, unless the supporting group has the same number of agents as the opposing group has. Formally:

Definition (Semi-Plurality Rules). A *semi-plurality rule* is a rule represented by a profile of index sets $(\mathcal{I}_x)_{x\in A}$ such that for each $x \in A$ and each $(t_1, t_2) \in \mathcal{I}^*$, (i) if $t_1 > t_2$, $(t_1, t_2) \in \mathcal{I}_x$; (ii) if $t_1 < t_2$, $(t_1, t_2) \notin \mathcal{I}_x$.

It is shown in the earlier version of this article, Ju (2003b; Lemma 2), that every semi-plurality rule maximizes the sum of individual utility functions.[12] Therefore, every semi-plurality rule satisfies *Pareto efficiency*. We show that semi-plurality rules are the only rules satisfying *Pareto efficiency*, *independence*, and *anonymity*.

Theorem 1. *A rule on the trichotomous domain with symmetry $\mathcal{S}^N_{Tri}$ satisfies Pareto efficiency, independence, and anonymity if and only if it is a semi-plurality rule.*

Independence of the three axioms is easily established. Note that the theorem pertains to the trichotomous domain $\mathcal{S}^N_{\text{Tri}}$. On the dichotomous domain $\mathcal{S}^N_{\text{Di}}$, there exist rules that satisfy the three axioms, but that are not semi-plurality rules, as

[12]Adopting the normalization $u = v = 1$, $R_i \in \mathcal{S}_{\text{Tri}}$ is represented by $U_i : 2^A \to \mathbb{R}$ such that for each $X \in 2^A$, $U_i(X) \equiv |G(R_i, X)| - |B(R_i, X)|$.

shown by Example 1 below. We prove the theorem in Appendix A. Here, instead, we add *neutrality* and provide a proof of the following result, Theorem 2. On the trichotomous domain $\mathcal{S}^N_{\text{Tri}}$, it can be obtained as a corollary to Theorem 1. Moreover, it also holds over the dichotomous domain $\mathcal{S}^N_{\text{Di}}$.

Theorem 2. *A rule on the trichotomous (or dichotomous) domain with symmetry $\mathcal{S}^N_{Tri}$ (or $\mathcal{S}^N_{Di}$) satisfies Pareto efficiency, independence, anonymity, and neutrality if and only if it is a semi-plurality rule represented by a single index set.*

Proof. It is easy to show that every semi-plurality rule represented by a single index set satisfies the listed axioms. To prove the converse, let φ be a rule satisfying the four axioms. Then, by *independence* and *anonymity*, φ can be represented by a profile of index sets $(\mathcal{I}_x)_{x\in A}$. By *neutrality*, all objects have the same index sets. Let $\mathcal{I}$ be the common index set. We need to show that $\{(t_1,t_2)\in\mathcal{I}^*: t_1>t_2\}\subseteq\mathcal{I}\subseteq\{(t_1,t_2)\in\mathcal{I}^*: t_1\geq t_2\}$. We show below the first inclusion. The proof of the second inclusion is similar and so is omitted.

Suppose by contradiction that there exists $(t_1,t_2)\in\mathcal{I}^*$ such that $t_1>t_2$ and $(t_1,t_2)\notin\mathcal{I}$. Pick two objects $a,b\in A$ and a set of t_1+t_2 agents, $\bar{N}\equiv\{1,\ldots,t_1+t_2\}$. We now construct a preference profile R^*, in which all objects other than a and b are bads for all agents, a and b are nulls for all agents outside $\bar{N}$, and there are t_1 agents in $\bar{N}$ for whom a is a good and t_2 agents in $\bar{N}$ for whom a is a bad, and similarly for b. See Table 1 for an illustration.

Construction of R^*. For each $i=1,\ldots,t_2$, let R^*_i be such that $G(R^*_i)\equiv\{a\}$ and $B(R^*_i)\equiv A\backslash\{a\}$. For each $i=t_2+1,\ldots,t_1$, let R^*_i be such that $G(R^*_i)\equiv\{a,b\}$ and $B(R^*_i)\equiv A\backslash\{a,b\}$. For each $i=t_1+1,\ldots,t_1+t_2$, let R^*_i be such that $G(R^*_i)\equiv\{b\}$ and $B(R^*_i)\equiv A\backslash\{b\}$. For each $i\notin\bar{N}$, let $G(R^*_i)=B(R^*_i)\equiv\emptyset$.

Note that such a construction is admissible with the trichotomous domain $\mathcal{S}^N_{\text{Tri}}$. However, it is admissible over the dichotomous domain $\mathcal{S}^N_{\text{Di}}$ if and only if $t_1+t_2=n$. Since the case $t_1+t_2=n$ is the only case we need to be concerned with on the dichotomous domain $\mathcal{S}^N_{\text{Di}}$, the proof is still valid for $\mathcal{S}^N_{\text{Di}}$.

By construction, $N^G_a(R^*)=\{1,\ldots,t_1\}$, $N^B_a(R^*)=\{t_1+1,\ldots,t_1+t_2\}$, $N^G_b(R^*)=\{t_2+1,\ldots,t_1+t_2\}$, and $N^B_b(R^*)=\{t_1,\ldots,t_2\}$. Since $(t_1,t_2)\notin\mathcal{I}$, $a,b\notin\varphi(R^*)$. Since all other objects are bads for agents in $\bar{N}$ and nulls for agents outside $\bar{N}$, then by *Pareto efficiency*, none of them is chosen. Therefore, $\varphi(R^*)=\emptyset$. Note that for each $i\in\{1,\ldots,t_2\}\cup\{t_1+1,\ldots,t_1+t_2\}$, $\{a,b\}\ I^*_i\ \emptyset$ and for each $i\in\{t_2+1,\ldots,t_1\}$, $\{a,b\}\ P^*_i\ \emptyset$. Since $t_2<t_1$, $\{a,b\}$ Pareto dominates $\emptyset=\varphi(R^*)$, contradicting *Pareto efficiency*. ∎

	R_1^*	$\cdots$	$R_{t_2}^*$	$R_{t_2+1}^*$	$\cdots$	$R_{t_1}^*$	$R_{t_1+1}^*$	$\cdots$	$R_{t_1+t_2}^*$
$G(R_i^*)$	$\{a\}$	$\cdots$	$\{a\}$	$\{a,b\}$	$\cdots$	$\{a,b\}$	$\{b\}$	$\cdots$	$\{b\}$
$B(R_i^*)$	$A\backslash\{a\}$	$\cdots$	$A\backslash\{a\}$	$A\backslash\{a,b\}$	$\cdots$	$A\backslash\{a,b\}$	$A\backslash\{b\}$	$\cdots$	$A\backslash\{b\}$

Table 1: Construction of R^* in Proof of Theorem 2, when $t_1 > t_2$. Note that $\{a,b\}$ Pareto dominates $\emptyset$.

Note that in this proof, we make use of a Pareto improvement that does not make all agents better off. Thus, one may wonder whether more variety of rules will emerge, relaxing *Pareto efficiency* to the axiom saying that there should be no other alternative that makes all agents better off.

Weak Pareto Efficiency. For each $R \in \mathcal{D}$, there is no $X \subseteq A$ such that for each $i \in N$, $X\ P_i\ \varphi(R)$.

When there are at least as many objects as agents, substituting *weak Pareto efficiency* for *Pareto efficiency*, we obtain the same result as in Theorem 2.

Theorem 3. *Assume $|A| \geq |N|$. Then, a rule on the trichotomous (or dichotomous) domain with symmetry $\mathcal{S}_{Tri}^N$ (or $\mathcal{S}_{Di}^N$) satisfies weak Pareto efficiency, independence, anonymity, and neutrality if and only if it is a semi-plurality rule represented by a single index set.*

Proof. We only consider the trichotomous domain $\mathcal{S}_{\text{Tri}}^N$ below. The same proof applies on the dichotomous domain $\mathcal{S}_{\text{Di}}^N$. Let φ be a rule satisfying the listed axioms. Then by *independence* and *anonymity*, φ can be represented by a profile of index sets $(\mathcal{I}_x)_{x\in A}$. By *neutrality*, all objects have the same index sets. Let $\mathcal{I}$ be the common index set. We need to show that $\{(t_1,t_2) \in \mathcal{I}^* : t_1 > t_2\} \subseteq \mathcal{I} \subseteq \{(t_1,t_2) \in \mathcal{I}^* : t_1 \geq t_2\}$. We show below the first inclusion. The proof of the second inclusion is similar. Suppose by contradiction that there exists $(t_1,t_2) \notin \mathcal{I}$ such that $t_1 > t_2$. Pick n objects, which is possible because $|A| \geq |N|(= n)$. Let $\bar{A} \equiv \{a_1,\ldots,a_n\}$ be the set of these objects. In what follows, we will construct a preference profile R^* such that for each $a \in \bar{A}$, there are t_1 agents for whom a is a good and t_2 agents for whom a is a bad; for each $a \in A\backslash\bar{A}$, a is a bad for all agents; for each $i \in N$, $|G(R_i^*,\bar{A})| = t_1$ and $|B(R_i^*,\bar{A})| = t_2$. See Table 2 for an illustration in the special case of $(t_1,t_2) = (3,1)$ and $n = 5$.

Construction of R^*. Let $[0] \equiv n$. For each $k \in \{1,\ldots,n\}$, let $[k] \equiv k$, $[n+k] \equiv k$, and $[-k] \equiv [n-k]$. For each $k \in \{1,\ldots,n\}$, let a_k be a good for each agent $i = [k], [k+1], \ldots, [k+t_1-1]$ and a bad for each agent $i =$

	R_1^*	R_2^*	R_3^*	R_4^*	R_5^*
$G\left(R_i^*, \bar{A}\right)$	$\{a_1, a_4, a_5\}$	$\{a_1, a_2, a_5\}$	$\{a_1, a_2, a_3\}$	$\{a_2, a_3, a_4\}$	$\{a_3, a_4, a_5\}$
$B\left(R_i^*, \bar{A}\right)$	$\{a_3\}$	$\{a_4\}$	$\{a_5\}$	$\{a_1\}$	$\{a_2\}$

Table 2: Construction of R^* in Proof of Theorem 3, when $n = 5$, $(t_1, t_2) = (3, 1)$ and $\bar{A} = \{a_1, \cdots, a_5\}$. Note that for all $i \in \{1, \cdots, 5\}$, $\bar{A}$ contains more goods than bads and so $\bar{A}\ P_i^*\ \emptyset$.

$[k + t_1], \ldots, [k + t_1 + t_2 - 1]$.

Note that the construction is possible because $t_1 + t_2 \leq |N| \leq |A|$. For agent 1, there are t_1 goods, namely $a_{[1]} = a_1$, $a_{[1-1]} = a_n$, …, $a_{[1-(t_1-1)]} = a_{n-(t_1-2)}$ and there are t_2 bads, namely $a_{[1-t_1]} = a_{n-(t_1-1)}$, $a_{[1-(t_1+1)]} = a_{n-t_1}$, …, $a_{[1-(t_1+t_2-1)]} = a_{n-(t_1+t_2-2)}$. In general, for each agent $k \in N$, there are t_1 goods, namely $a_{[k]}$, $a_{[k-1]}$, …, $a_{[k-(t_1-1)]}$, and there are t_2 bads, namely $a_{[k-t_1]}$, $a_{[k-(t_1+1)]}$, …, $a_{[k-(t_1+t_2-1)]}$. Since all objects in $A \backslash \bar{A}$ are bads for all agents, then by *Pareto efficiency*, none of these objects is chosen by R^*. Since $(t_1, t_2) \notin \mathcal{I}$, no object in $\bar{A}$ is chosen either. Thus, $\varphi(R^*) = \emptyset$. For each $i \in N$, let U_i be the representation of R_i such that for each $X \in 2^A$, $U_i(X) \equiv |G(R_i, X)| - |B(R_i, X)|$. Then for each $i \in N$, $U_i\left(\bar{A}\right) = t_1 - t_2 > 0 = U_i(\varphi(R^*))$, contradicting *weak Pareto efficiency* of φ. ∎

Independence of the four axioms in each of Theorems 2 and 3 is easily established. In particular, without *neutrality*, there do exist rules that are not semi-plurality rules but satisfy all of the remaining axioms. Here is an example.

Example 1. Fix $a \in A$. Let $\hat{\varphi}$ be defined as follows: for each $R \in \mathcal{D}$, $a \in \hat{\varphi}(R) \Leftrightarrow |N_a^G(R)| > |N_a^B(R)| - 2$ and for each $x \neq a$, $x \in \hat{\varphi}(R) \Leftrightarrow |N_a^G(R)| > |N_a^B(R)|$. We show that on the dichotomous domain $\mathcal{S}_{\mathrm{Di}}^N$ with an odd number of agents, $\hat{\varphi}$ is *Pareto efficient*. We use the fact, shown in the earlier version of this paper, Ju (2003b), that the plurality rule maximizes the sum of individual utilities. Let $R \in \mathcal{S}_{\mathrm{Di}}^N$ be such that $\hat{\varphi}(R) \neq \varphi^{\mathrm{PL}}(R)$. Then $|N_a^G(R)| = |N_a^B(R)| - 1$ or $|N_a^G(R)| = |N_a^B(R)|$. Thus $\hat{\varphi}(R) = \varphi^{\mathrm{PL}}(R) \cup \{a\}$ and $a \notin \varphi^{\mathrm{PL}}(R)$. Since n is odd, $|N_a^G(R)| = |N_a^B(R)| - 1$. Then $\sum_i U_i(\hat{\varphi}(R)) = \sum_i U_i(\varphi^{\mathrm{PL}}(R)) - 1$, where $U_i(\cdot)$ is the representation of R_i defined in the proof of Theorem 3. Therefore, if an alternative X improves upon $\hat{\varphi}(R)$, there can be only one agent, say, agent 1, whose welfare improves only by 1 unit and all others' welfares stay constant (otherwise, X will have a higher sum of individual utilities than $\varphi^{\mathrm{PL}}(R)$, contradicting the maximization of the sum of utilities by $\varphi^{\mathrm{PL}}(R)$). So both X and $\varphi^{\mathrm{PL}}(R)$ have the maximum sum of individual utilities. This implies that X contains $\varphi^{\mathrm{PL}}(R)$ because every object chosen by the plurality rule will add a positive amount to the sum of individual utilities (see Lemma 2 in Ju 2003b, for the detail). On the other hand, if $x \notin \varphi^{\mathrm{PL}}(R)$ is such that $|N_x^G(R)| < |N_x^B(R)|$, x will add a negative number to the sum of individual utilities. Therefore, $\varphi^{\mathrm{PL}}(R) \subseteq X$ and for each $x \in X \backslash \varphi^{\mathrm{PL}}(R)$, $|N_x^G(R)| = |N_x^B(R)|$. Since, over the dichotomous domain with an odd number of agents, the supporting and opposing groups for any object cannot have the same size, then $X \backslash \varphi^{\mathrm{PL}}(R) = \emptyset$. Therefore, $X = \varphi^{\mathrm{PL}}(R)$. Since X Pareto dominates $\hat{\varphi}(R) = \varphi^{\mathrm{PL}}(R) \cup \{a\}$, then $N_a^B(R) = \emptyset$, contradicting $|N_a^G(R)| = |N_a^B(R)| - 1$.

When *Pareto efficiency* is dropped, there are a variety of rules satisfying the remaining axioms in Theorem 1. Examples are all the rules represented by profiles of index sets. Thus, *Pareto efficiency* plays an important role in pinning down the very small family of semi-plurality rules.

4 Trichotomous or Dichotomous Domains without Symmetry

We, so far, assumed symmetry between goods and bads. In this section, we consider "trichotomous and dichotomous domains without symmetry".

Assume that all goods and all bads have fixed utility $u > 0$ and disutility $v > 0$ (up to normalization), respectively. Let $\mathcal{R}_{\mathrm{Tri}(u,v)}$ be the set of trichotomous preferences associated with the utility of goods, u, and the disutility of bads, v. When $u \neq v$, we call $\mathcal{R}_{\mathrm{Tri}(u,v)}$ a *trichotomous domain without symmetry* (between goods and bads). Similarly, let $\mathcal{R}_{\mathrm{Di}(u,v)}$ be the set of dichotomous preferences in $\mathcal{R}_{\mathrm{Tri}(u,v)}$, called a *dichotomous domain without symmetry*, when $u \neq v$. If $u = v$, $\mathcal{R}_{\mathrm{Tri}(u,v)} \equiv \mathcal{S}_{\mathrm{Tri}}$ and $\mathcal{R}_{\mathrm{Di}(u,v)} = \mathcal{S}_{\mathrm{Di}}$.

On the trichotomous domain $\mathcal{R}_{\mathrm{Tri}(u,v)}$, the following modification of the plurality rule is important. The *modified plurality rule*, denoted by φ^{MPL}, is defined as follows: for each $R \in \mathcal{R}_{(u,v)}^N$ and each $a \in A$, $a \in \varphi^{\mathrm{MPL}}(R)$ if and only if $u|N_a^G(R)| > v|N_a^B(R)|$. A *modified semi-plurality rule* is a P-anonymous rule represented by a profile $(\mathcal{I}_x)_{x\in A}$ of index sets such that for each $x \in A$ and each $(t_1, t_2) \in \mathcal{I}^*$, (i) if $ut_1 > vt_2$, $(t_1, t_2) \in \mathcal{I}_x$; (ii) if $ut_1 < vt_2$, $(t_1, t_2) \notin \mathcal{I}_x$. Note that when $\{(t_1, t_2) \in \mathcal{I}^* : ut_1 = vt_2\} = \emptyset$, the modified plurality rule is the only modified semi-plurality rule. When u/v is sufficiently close to 1, modified semi-plurality rules coincide with semi-plurality rules. When u/v is sufficiently small, modified semi-plurality rules are associated with the index set $\mathcal{I} \equiv \{(t_1, t_2) \in \mathcal{I}^* : t_2 = 0\}$. Thus, they accept each object if and only if no one is against the object. Applying the same argument as for the trichotomous domain with symmetry $\mathcal{S}^N_{\text{Tri}}$, we extend Theorem 3 to any trichotomous domain with or without symmetry. The proof of Theorem 4 is similar to the proof of Theorem 3, and so is omitted.[13]

Theorem 4. *Assume $|A| \geq |N|$. A rule on a trichotomous (or dichotomous) domain with or without symmetry satisfies* weak Pareto efficiency *(or* Pareto efficiency*),* independence*,* anonymity*, and* neutrality *if and only if it is a modified semi-plurality rule represented by a single index set.*

5 Non-Trichotomous Domains: An Impossibility Result

On each trichotomous domain, preferences have at most one indifference class of goods and at most one indifference class of bads. It is natural to ask whether our results, or at least the existence parts, will continue to hold when we allow for more than one indifference classes of goods or bads. To address this question, we consider domains of additive preferences, in which each preference may have multiple indifference classes of goods, associated with a finite number of utility levels, and also multiple indifference classes of bads, associated with a finite number of disutility levels. Formally:

Definition (Domains with fixed classifications of goods and bads). Let K and L be two natural numbers. Consider K positive numbers, $u_1, \ldots, u_K$, and L positive numbers, $v_1, \ldots, v_L$. Let $\mathbf{u} \equiv (u_1, \ldots, u_K)$ and $\mathbf{v} \equiv (v_1, \ldots, v_L)$. Let $\mathcal{R}_{(\mathbf{u},\mathbf{v})}$ be the domain of additive preferences in which all goods are associated with utilities in $\{u_1, \ldots, u_K\}$ and all bads are associated with disutilities in $\{v_1, \ldots, v_L\}$. Thus, for each preference in $\mathcal{R}_{(\mathbf{u},\mathbf{v})}$, there are at most K indifference classes of goods and at most L indifference classes of bads (and one indifference class of nulls).

Trichotomous domains are special cases with $K = L = 1$. In what follows, we will consider non-trichotomous domains with $K \geq 2$ or $L \geq 2$. Under three additional, yet weak, conditions on domains, we show that no rule on any such non-trichotomous domain can satisfy *Pareto efficiency*, *independence*, and *anonymity*.

[13]The proof is available upon request.

Theorem 5. *Consider the non-trichotomous domain $\mathcal{R}_{(\mathbf{u},\mathbf{v})}$, where $\mathbf{u} \in \mathbb{R}^K_{++}$ and $\mathbf{v} \in \mathbb{R}^L_{++}$. Assume $|A| \geq |N|$, $\max\{u_1, \ldots, u_K\} > \min\{v_1, \ldots, v_L\}$, and $\min\{u_1, \ldots, u_K\} < \max\{v_1, \ldots, v_L\}$. Then there is no rule on $\mathcal{R}^N_{(\mathbf{u},\mathbf{v})}$ satisfying Pareto efficiency, independence, and anonymity.*

Proof. Let $\mathcal{R}_{(\mathbf{u},\mathbf{v})}$ be a domain satisfying the assumptions. Suppose by contradiction that a rule φ on $\mathcal{R}^N_{(\mathbf{u},\mathbf{v})}$ satisfies the three axioms. Assume $K \geq 2$ (the same argument applies when $L \geq 2$). Assume without loss of generality that $u_1 < \ldots < u_K$ and $v_1 < \ldots < v_L$. Then $u_K > v_1$ and $u_1 < v_L$.

By *independence*, φ is represented by a profile $(\mathfrak{C}_x)_{x \in A}$. Now we show that for each $x, y \in A$, $\mathfrak{C}_x \backslash \{(\emptyset, \emptyset)\} = \mathfrak{C}_y \backslash \{(\emptyset, \emptyset)\}$. Let $x, y \in A$ and $(C_1, C_2) \in \mathfrak{C}_x \backslash \{(\emptyset, \emptyset)\}$. Suppose $(C_1, C_2) \notin \mathfrak{C}_y$. Let $R_0 \in \mathcal{R}_{(\mathbf{u},\mathbf{v})}$ be such that both x and y are goods with the utilities u_1 and u_K, respectively and that all other objects are bads. Let $R'_0 \in \mathcal{R}_{(\mathbf{u},\mathbf{v})}$ be such that both x and y are nulls and all other objects are bads. For each $i \in C_1 \cup C_2$, let $R_i \equiv R_0$. For each $j \notin C_1 \cup C_2$, let $R_j \equiv R'_0$. Since all objects other than x and y are bads, then by *Pareto efficiency*, $\varphi(R) \subseteq \{x, y\}$. Since $(N^G_x(R), N^B_x(R)) = (N^G_y(R), N^B_y(R)) \equiv (C_1, C_2)$ and $(C_1, C_2) \in \mathfrak{C}_x \backslash \mathfrak{C}_y$, then $\varphi(R) \equiv \{x\}$. Since everyone weakly prefers $\{y\}$ to $\{x\}$ and all agents in $C_1 \cup C_2$ prefer $\{y\}$ to $\{x\}$, then $\{y\}$ Pareto dominates $\{x\}$, contradicting *Pareto efficiency*. Therefore, $\mathfrak{C}_x \backslash \{(\emptyset, \emptyset)\} \subseteq \mathfrak{C}_y \backslash \{(\emptyset, \emptyset)\}$. The proof of the reverse inclusion is similar.

By *anonymity*, $(\mathfrak{C}_x)_{x \in A}$ is described by a profile of index sets $(\mathcal{I}_x)_{x \in A}$. By the property of $(\mathfrak{C}_x)_{x \in A}$ shown in the previous paragraph, for each $x, y \in A$, $\mathcal{I}_x \backslash \{(0, 0)\} = \mathcal{I}_y \backslash \{(0, 0)\}$. Note that $(0, 0)$ may be in $\mathcal{I}_x$ and not in $\mathcal{I}_y$. Thus, φ may not satisfy *neutrality*. However, φ is "almost neutral" in the sense that all index sets are equal to each other except for the zero index pair $(0, 0)$, and this is enough for the proofs of Theorems 2 and 3. Therefore, φ is a modified semi-plurality rule on $\mathcal{R}^N_{\text{Tri}(u_K, v_1)} \subseteq \mathcal{R}^N_{(\mathbf{u},\mathbf{v})}$ and also on $\mathcal{R}^N_{\text{Tri}(u_1, v_L)} \subseteq \mathcal{R}^N_{(\mathbf{u},\mathbf{v})}$.

Let $t_0 \in \{1, \ldots, n\}$. Since $u_K > v_1$, $u_K t_0 > v_1 t_0$. Since φ is a modified semi-plurality rule on $\mathcal{R}^N_{\text{Tri}(u_K, v_1)}$, then $(t_0, t_0) \in \mathcal{I}_x$ for each $x \in A$. On the other hand, since $u_1 < v_L$, $u_1 t_0 < v_L t_0$. Since φ is a modified semi-plurality rule on $\mathcal{R}^N_{\text{Tri}(u_1, v_L)}$, then $(t_0, t_0) \notin \mathcal{I}_x$ for each $x \in A$, contradicting the previous conclusion. ∎

6 Concluding Remarks

We have extended Theorem 3 to any trichotomous or dichotomous domain *without* symmetry (Theorem 4). However, we leave it for future research whether Theorems 1 and 2 can also be established without the symmetry assumption.

In Section 5, we investigated non-trichotomous domains and studied whether there exist rules over a non-trichotomous domain satisfying our three main axioms, *Pareto efficiency*, *independence*, and *anonymity*. As shown by Ju (2004), these three axioms are not compatible on the domain of additive preferences.[14] Therefore, the additive domain can be considered as an upper bound for the possibility result. On the other hand, our results show that the trichotomous domains in this paper are lower bounds. A natural question is: are there domains between the upper and lower bounds on which the possibility result prevails? We have a partial answer to this question. Theorem 5 says that if we are interested in domains satisfying the three richness conditions stated in the theorem, then we cannot find any non-trichotomous domain in which *Pareto efficiency*, *independence*, and *anonymity* are compatible. Theorem 5 does not offer a characterization of maximal domains in which these three axioms are compatible, but it shows that the gap between maximal domains and our trichotomous domains is quite thin. Although the three richness conditions, we think, are mild, the consequences of dropping these conditions are left for future study.

A Proof of Theorem 1

Throughout this section, we consider the trichotomous domain $\mathcal{S}^N_{\text{Tri}}$. We begin with two useful lemmas.

Lemma 1. *Let φ be a rule over the trichotomous domain with symmetry $\mathcal{S}^N_{Tri}$. If φ satisfies Pareto efficiency and is represented by a profile of power structures $(\mathfrak{C}_x)_{x\in A}$, then $(\mathfrak{C}_x)_{x\in A}$ has the following properties. For each $x, y \in A$ with $x \neq y$ and each $(C_1, C_2), (C_1', C_2') \in \mathfrak{C}^*$,*
(i) if $(C_1, C_2) \in \mathfrak{C}_x$, $C_1' \subseteq C_2$, $C_2' \supseteq C_1$, and $(C_2\backslash C_1') \cup (C_2'\backslash C_1) \neq \emptyset$, then $(C_1', C_2') \notin \mathfrak{C}_y$;
(ii) if $(C_1, C_2) \notin \mathfrak{C}_x$, $C_1' \supseteq C_2$, $C_2' \subseteq C_1$, and $(C_1'\backslash C_2) \cup (C_1\backslash C_2') \neq \emptyset$, then $(C_1', C_2') \in \mathfrak{C}_y$.

Proof. Let φ be represented by $(\mathfrak{C}_x)_{x\in A}$. To prove part (i), let $(C_1, C_2) \in \mathfrak{C}_x$ and $(C_1', C_2') \in \mathfrak{C}^*$ be such that $C_1' \subseteq C_2$, $C_2' \supseteq C_1$, and $(C_2\backslash C_1') \cup (C_2'\backslash C_1) \neq \emptyset$. Suppose by contradiction that $(C_1', C_2') \in \mathfrak{C}_y$. Let R^* be such that $N^G_x(R^*) = C_1$, $N^B_x(R^*) = C_2$, $N^G_y(R^*) = C_1'$, $N^B_y(R^*) = C_2'$, and for each $z \neq x, y$, $N^B_z(R^*) = N$. See Table 3 for an illustration. Since all objects other than x and y are bads, then by *Pareto efficiency*, they are rejected. Since $(C_1, C_2) \in \mathfrak{C}_x$ and

[14] See Section 6 in Ju (2004).

	R_1^*	R_2^*	R_3^*	R_4^*	R_5^*
$G(R_i^*, \{x,y\})$	$\{x\}$	$\{x\}$	$\emptyset$	$\emptyset$	$\{y\}$
$B(R_i^*, \{x,y\})$	$\{y\}$	$\{y\}$	$\{y\}$	$\emptyset$	$\{x\}$

Table 3: Construction of R^* in the proof of Lemma 1, when $C_1 = \{1,2\}$, $C_2 = \{5\}$, $C_1' = \{5\}$, and $C_2' = \{1,2,3\}$. Note that $\emptyset$ Pareto dominates $\{x,y\}$.

$(C_1', C_2') \in \mathfrak{C}_y$, $\varphi(R^*) = \{x,y\}$. For each $i \in C_1'$, $i \in C_2$ and so $\emptyset\ I_i^*\ \{x,y\}$. For each $i \in C_1$, $i \in C_2'$ and so $\emptyset\ I_i^*\ \{x,y\}$. For each $i \in (C_2 \backslash C_1') \cup (C_2' \backslash C_1)$, either $x \in B(R_i^*)$ and $y \notin G(R_i^*)$ or $x \notin G(R_i^*)$ and $y \in B(R_i^*)$. Therefore, $\emptyset\ P_i^*\ \{x,y\}$. For each $i \notin C_2 \cup C_2'$, both x and y are nulls and so $\emptyset\ I_i^*\ \{x,y\}$. Hence $\emptyset$ Pareto dominates $\varphi(R^*)$, contradicting *Pareto efficiency*. Part (ii) can be proven similarly. ∎

The following is a direct consequence of Lemma 1 for rules represented by profiles of index sets.

Lemma 2. *Let φ be a rule over the trichotomous domain with symmetry $\mathcal{S}_{Tri}^N$. If φ satisfies* Pareto efficiency *and is represented by a profile of index sets $(\mathcal{I}_x)_{x \in A}$, then $(\mathcal{I}_x)_{x \in A}$ has the following properties. For each $x, y \in A$ with $x \neq y$ and each $(t_1, t_2), (t_1', t_2') \in \mathcal{I}^*$,*
(i) if $(t_1, t_2) \in \mathcal{I}_x$, $t_1' \leq t_2$, $t_2' \geq t_1$, and at least one of the two inequalities is strict, then $(t_1', t_2') \notin \mathcal{I}_y$;
(ii) if $(t_1, t_2) \notin \mathcal{I}_x$, $t_1' \geq t_2$, $t_2' \leq t_1$, and at least one of the two inequalities is strict, then $(t_1', t_2') \in \mathcal{I}_y$.

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We only prove the non-trivial direction.[15] Let φ be a rule satisfying *Pareto efficiency*, *independence*, and *anonymity*. Then, by *independence* and *anonymity*, φ is represented by a profile of index sets $(\mathcal{I}_x)_{x \in A}$. We only have to show that for each $x \in A$, $\{(t_1, t_2) \in \mathcal{I}^* : t_1 > t_2\} \subseteq \mathcal{I}_x \subseteq \{(t_1, t_2) \in \mathcal{I}^* : t_1 \geq t_2\}$.

Let $x \in A$. To prove the first inclusion, let $(t_1, t_2) \in \mathcal{I}^*$ be such that $t_1 > t_2$. Suppose by contradiction that $(t_1, t_2) \notin \mathcal{I}_x$. Then let S_1, S_2, S_3 be the partition of N such that $|S_1| = t_1$, $|S_2| = t_2$, and $|S_3| = n - t_1 - t_2$. Let $R \in \mathcal{S}_{\text{Tri}}^N$ be such that (i) for each $i \in S_1$, $G(R_i) = \{x\}$ and $B(R_i) = A \backslash \{x\}$; (ii) for each $i \in S_2$,

[15] The current proof is due to an anonymous referee. It is simpler than the original proof, which can be found in the earlier version Ju (2003b).

$G(R_i) = \emptyset$ and $B(R_i) = A$; (iii) for each $i \in S_3$, $G(R_i) = \emptyset$ and $B(R_i) = A\backslash\{x\}$. Then $|N_x^G(R)| = t_1$ and $|N_x^B(R)| = t_2$. Since $(t_1, t_2) \notin \mathcal{I}_x$, $x \notin \varphi(R)$. Let $y \in A\backslash\{x\}$ and $i \in S_1$ (note that since $t_1 \geq 1$, $S_1 \neq \emptyset$). Let $R' \in \mathcal{S}_{\text{Tri}}^N$ be such that (i) for each $j \in S_1\backslash\{i\}$, $G(R'_j) = \{x, y\}$ and $B(R'_j) = A\backslash\{x, y\}$; (ii) $G(R'_i) = \{x\}$ and $B(R'_i) = A\backslash\{x, y\}$; (iii) for each $j \in S_2$, $G(R'_j) = \emptyset$ and $B(R'_j) = A$; (iv) for each $j \in S_3$, $G(R'_i) = \emptyset$ and $B(R'_i) = A\backslash\{x, y\}$. Note that $|N_x^G(R')| = t_1$ and $|N_x^B(R')| = t_2$. Thus $x \notin \varphi(R')$. Then by *Pareto efficiency*, $\varphi(R') = \{y\}$ or $\emptyset$. Since $\{y\}$ is Pareto dominated by $\{x\}$, $\varphi(R') = \emptyset$. Hence $(|N_y^G(R')|, |N_y^B(R')|) = (t_1 - 1, t_2) \notin \mathcal{I}_y$. Since $t_1 > t_2$ and $t_2 \leq t_1 - 1$, then by (ii) of Lemma 2, $(t_1, t_2) \in \mathcal{I}_x$. This contradicts the initial assumption.

To prove the second inclusion, let $(t_1, t_2) \in \mathcal{I}^*$ be such that $t_1 < t_2$. Suppose by contradiction that $(t_1, t_2) \in \mathcal{I}_x$. Then using the same preference R constructed above, $x \in \varphi(R)$. Let $y \in A\backslash\{x\}$ and $i \in S_2$ (note that since $t_2 \geq 1$, $S_2 \neq \emptyset$). Let $R'' \in \mathcal{S}_{\text{Tri}}^N$ be such that (i) for each $j \in S_1$, $G(R''_j) = \{x, y\}$ and $B(R''_j) = A\backslash\{x, y\}$; (ii) for each $j \in S_2\backslash\{i\}$, $G(R''_j) = \emptyset$ and $B(R''_j) = A$; (iii) $G(R''_i) = \emptyset$ and $B(R''_i) = A\backslash\{y\}$; (iv) for each $j \in S_3$, $G(R''_j) = \emptyset$ and $B(R''_j) = A\backslash\{x, y\}$. Note that $|N_x^G(R'')| = t_1$ and $|N_x^B(R'')| = t_2$. Thus $x \in \varphi(R'')$. Then by *Pareto efficiency*, $\varphi(R'') = \{x\}$ or $\{x, y\}$. Since $\{y\}$ Pareto dominates $\{x\}$, then $\varphi(R'') = \{x, y\}$. Hence $(|N_y^G(R'')|, |N_y^B(R'')|) = (t_1, t_2 - 1) \in \mathcal{I}_y$. Since $t_1 \leq t_2 - 1$ and $t_2 > t_1$, then by (i) of Lemma 2, $(t_1, t_2) \notin \mathcal{I}_x$. This contradicts the initial assumption. ∎

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