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# INTERTEMPORALLY NON-SEPARABLE MONETARY-ASSET RISK ADJUSTMENT AND AGGREGATION

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**Intertemporally non-separable monetary-asset risk adjustment and aggregation***

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**Summary.** Modern aggregation theory and index number theory were introduced into monetary economics by Barnett (1980). The widely used Divisia monetary aggregates were based upon that paper. A key result upon which the rest of the theory depended was Barnett's derivation of the user-cost price of monetary assets. To make that critical part of Barnett's results available prior to publication of that paper in the *Journal of Econometrics*, Barnett repeated that proof two years earlier in *Economics Letters*. Both papers have become seminal to the subsequent literature on monetary asset quantity and user cost aggregation. The extension of that literature to risk with intertemporally non-separable preferences now has become available in a working paper by Barnett and Wu (2004), and that paper will appear in volume 1, number 1 of the new journal, *Annals of Finance*. We are making available the key results from that paper below, without the proofs, which will be available in the longer paper.

**Keywords and Phrases:** User costs, Monetary Aggregation, Risk, Pricing kernel, CAPM

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## Introduction

Modern aggregation theory and index number theory were introduced into monetary economics by Barnett (1980). The widely used Divisia monetary aggregates were based upon that paper. A key result upon which the rest of the theory depended was Barnett's derivation of the user-cost price of monetary assets. To make that critical part of Barnett's (1980) results available prior to publication of the full paper in the *Journal of Econometrics,* Barnett (1978) repeated that part of the theory two years earlier in *Economics Letters*. Both papers have become seminal to the subsequent literature on monetary asset quantity and user cost aggregation.

The extension of that literature to risk with intertemporally non-separable preferences now has become available in a working paper by Barnett and Wu (2004), and that paper will appear in volume 1, number 1 of the new journal, *Annals of Finance*. That complete paper, in working paper form, is online in the Economics Working Paper Archive, at http://econwpa.wustl.edu/eprints/mac/papers/0406/0406009.abs. In this paper, we are making available the key results from the working paper, without the proofs, which will be available in the longer paper. We extend the monetary-asset user-cost risk adjustment of Barnett, Liu, and Jensen (1997) and their risk-adjusted Divisia monetary aggregates to the case of multiple non-monetary assets and intertemporal non-separability. Our model can generate potentially larger and more accurate CCAPM user-cost risk adjustments than those found in Barnett, Liu, and Jensen (1997).

Barnett (1978, 1980, 1997) produced the microeconomic theory of monetary aggregation under perfect certainty, derived the formula for the user cost of monetary assets, and originated the Divisia monetary aggregates to track the theory's quantity and price aggregator functions nonparametrically. The monetary aggregation theory was extended to risk by Barnett (1995) and Barnett, Hinich, and Yue (2000). In producing the Divisia index approximations to the theory's aggregator functions under risk, Barnett, Liu, and Jensen (1997) and Barnett and Liu (2000) showed that a risk adjustment term should be added to the certainty-equivalent user cost in a consumption-based capital asset pricing model (CCAPM). The risk adjustment depends upon the covariance between the rates of return on monetary assets and the growth rate of consumption. Using the components of the usual Federal Reserve System monetary aggregates, Barnett, Liu, and Jensen (1997) showed, however, that the CCAPM risk adjustment is slight and the gain from replacing the unadjusted Divisia index with the extended index is usually small. An overview of the relevant literature is provided in Barnett and Serletis (2000). As in the equity premium puzzle literature, the small risk adjustment is caused by the low contemporaneous covariance, under intertemporal separability, between the rate of return on the asset and consumption growth.

We also extend the model in Barnett, Liu, and Jensen (1997) to include multiple risky non-monetary assets, which are the assets that provide no liquidity services, other than their rates of return, and we show that any non-monetary asset can be used as the benchmark asset, when its rate of return is correctly adjusted.

## 2. Consumer's optimization problem

We assume that the representative consumer has an intertemporally non-separable general utility function, $U(\boldsymbol{m}_t, c_t, c_{t-1}, \cdots, c_{t-n})$, defined over current and past consumption and a vector of $L$ current-period monetary assets, $\boldsymbol{m}_t = (m_{1,t}, m_{2,t}, \cdots, m_{L,t})'$. The consumer's holdings of non-monetary assets are $\boldsymbol{k}_t = (k_{1,t}, k_{2,t}, ..., k_{K,t})'$. To ensure the existence of a monetary aggregate, we further assume that there exists a linearly homogenous aggregator function, $M(\cdot)$, such that U can be written in the form

$$U(\boldsymbol{m}_t, c_t, c_{t-1}, \cdots, c_{t-n}) = V(M(\boldsymbol{m}_t), c_t, c_{t-1}, \cdots, c_{t-n})\,. \tag{2.1}$$

Given initial wealth, $W_t$, the consumer seeks to maximize her expected lifetime utility function,

$$E_t \sum_{s=0}^{\infty} \beta^s U(\boldsymbol{m}_{t+s}, c_{t+s}, c_{t+s-1}, \cdots, c_{t+s-n})\,, \tag{2.2}$$

subject to the following budget constraints,

$$\begin{aligned} W_t &= p_t^* c_t + \sum_{i=1}^{L} p_t^* m_{i,t} + \sum_{j=1}^{K} p_t^* k_{j,t} \\ &= p_t^* c_t + p_t^* A_t \end{aligned} \tag{2.3}$$

and

$$W_{t+1} = \sum_{i=1}^{L} R_{i,t+1} p_t^* m_{i,t} + \sum_{j=1}^{K} \tilde{R}_{j,t+1} p_t^* k_{j,t} + Y_{t+1}\,, \tag{2.4}$$

where $\beta \in (0,1)$ is the consumer's subjective discount factor, $p_t^*$ is the true cost-of-living index, and $A_t = \sum_{i=1}^{L} m_{i,t} + \sum_{j=1}^{K} k_{j,t}$ is the real value of the asset portfolio. Non-monetary asset j provides gross rate of return, $\tilde{R}_{j,t+1}$. Monetary asset i, having quantity $m_{i,t}$, has a gross rate of return $R_{i,t+1}$. The consumer's income from any other sources, received at the beginning of period $t+1$, is $Y_{t+1}$. The consumer also is subject to the transversality condition

$$\lim_{s\to\infty} \beta^s p_t^* A_{t+s} = 0\,. \tag{2.5}$$

The first order conditions can be obtained as

$$\lambda_t = \beta E_t \left[ \lambda_{t+1} \tilde{R}_{j,t+1}\, p_t^* \big/ p_{t+1}^* \right] \tag{2.6}$$

and

$$\partial U_t \big/ \partial m_{i,t} = \lambda_t - \beta E_t \left[ \lambda_{t+1} R_{i,t+1}\, p_t^* \big/ p_{t+1}^* \right], \tag{2.7}$$

where $U_t = U(\boldsymbol{m}_t, c_t, c_{t-1,} \cdots, c_{t-n})$ and $\lambda_t = E_t(\partial U_t / \partial c_t + \beta\, \partial U_{t+1} / \partial c_t + \cdots + \beta^n\, \partial U_{t+n} / \partial c_t)$.

## 3. Risk-adjusted user cost of monetary assets

### 3.1. The theory

We define the contemporaneous real user-cost price of the services of monetary asset $i$ to be the ratio of the marginal utility of the monetary asset and the marginal utility of consumption, so that

$$\pi_{i,t} = \frac{\frac{\partial U_t}{\partial m_{i,t}}}{E_t\left( \frac{\partial U_t}{\partial c_t} + \beta\, \frac{\partial U_{t+1}}{\partial c_t} + \cdots + \beta^n\, \frac{\partial U_{t+n}}{\partial c_t} \right)} = \frac{\frac{\partial U_t}{\partial m_{i,t}}}{\lambda_t}. \tag{3.1}$$

We denote the vector of L monetary asset user costs by $\pi_t = (\pi_{1,t}, \pi_{2,t}, \cdots, \pi_{L,t})'$. With the user costs defined above, we can show that the solution value of the exact monetary aggregate, $M(\boldsymbol{m}_t)$, can be tracked accurately in continuous time by the generalized Divisia index, as proved in the perfect certainty special case by Barnett (1980).

**Proposition 1.** *Let* $s_{i,t} = \frac{\pi_{i,t} m_{i,t}}{\sum_{l=1}^{L} \pi_{l,t} m_{l,t}}$ *be the user-cost-evaluated expenditure share. Under the weak-separability assumption, (2.1), we have for any linearly homogenous monetary aggregator function,* $M(\cdot)$*, that*

$$d \log M_t = \sum_{i=1}^{L} s_{i,t} d \log m_{i,t} , \tag{3.2}$$

*where* $M_t = M(\boldsymbol{m}_t)$.

The exact price aggregate dual, $\Pi_\tau = \Pi(\pi_t)$, to the monetary quantity aggregator function, $M_t = M(\boldsymbol{m}_t)$, is easily computed from factor reversal, $\Pi(\pi_t) M(\boldsymbol{m}_t) = \sum_{i=1}^{L} \pi_{it} m_{it}$ , so that

$$\Pi(\pi_t) = \frac{\sum_{i=1}^{L} \pi_{it} m_{it}}{M(\boldsymbol{m}_t)} . \tag{3.3}$$

In continuous time, the user cost price dual can be tracked without error by the Divisia user cost price index

$$d \log \Pi_t = \sum_{i=1}^{L} s_{i,t} d \log \pi_{i,t} . \tag{3.4}$$

To get a more convenient expression for the user cost, $\pi_{i,t}$, we define the pricing kernel to be

$$Q_{t+1} = \beta \lambda_{t+1} / \lambda_t . \tag{3.5}$$

Let $\tilde{r}_{j,t+1} = \tilde{R}_{j,t+1} p_t^* / p_{t+1}^*$ be the real gross rate of return on non-monetary asset, $k_{j,t}$, and let $r_{i,t+1} = R_{i,t+1} p_t^* / p_{t+t}^*$ be the real gross rate of return on monetary asset, $m_{i,t}$. We can prove the following.

**Proposition 2.** *Given the real rate of return,* $r_{i,t+1}$*, on a monetary asset and the real rate of return,* $\tilde{r}_{j,t+1}$*, on an arbitrary non-monetary asset, the risk-adjusted real user-cost price of the services of the monetary asset can be obtained as*

$$\pi_{i,t} = \frac{(1+\omega_{i,t}) E_t \tilde{r}_{j,t+1} - (1+\omega_{j,t}) E_t r_{i,t+1}}{E_t \tilde{r}_{j,t+1}} , \tag{3.6}$$

*where*

$$\omega_{i,t} = -Cov_t(Q_{t+1}, r_{i,t+1}) \tag{3.7}$$

*and*

$$\omega_{j,t} = -Cov_t(Q_{t+1}, \tilde{r}_{j,t+1}) . \tag{3.8}$$

**Corollary 1.** *Under uncertainty we can choose any non-monetary asset as the "benchmark" asset, when computing the risk-adjusted user-cost prices of the services of monetary assets.*

Notice that Proposition 2 doesn't require existence of a risk-free non-monetary asset (in real-terms). To see the intuition of Proposition 2, assume that one of the non-monetary assets is risk-free with *gross* real interest rate of $r_t^f$ at time $t$. Further, as proved by Barnett (1978), the certainty-equivalent user cost, $\pi_{i,t}^e$, of a monetary asset $m_{i,t}$ is

$$\pi_{i,t}^e = \frac{r_t^f - E_t r_{i,t+1}}{r_t^f} . \tag{3.9}$$

We can prove the following:

$$\pi_{i,t} = \frac{r_t^f - E_t r_{i,t+1}}{r_t^f} + \omega_{i,t} = \pi_{i,t}^e + \omega_{i,t} , \tag{3.10}$$

where $\omega_{i,t} = -Cov_t(Q_{t+1}, r_{i,t+1})$. Therefore, $\pi_{i,t}$ could be larger or smaller than the certainty-equivalent user cost, $\pi_{i,t}^e$, depending on the sign of the covariance between $r_{i,t+1}$ and $Q_{t+1}$.

### 3.2. Approximation to the theory

All of the consumption-based asset pricing models require us to make explicit assumptions about investors' utility functions. An alternative approach, which is commonly practiced in finance, is to approximate $Q_{t+1}$ by some simple function of observable macroeconomic factors that are believed to be closely related to investors' marginal utility growth. For example, the well-known CAPM [Sharpe (1964) and Lintner (1965)] approximates $Q_{t+1}$ by a linear function of the rate of return on the market portfolio. We show that there also exists a similar CAPM-type relationship among user costs of risky monetary assets, under the assumption that $Q_{t+1}$ is a linear function of the rate of return on a well-diversified wealth portfolio

Specifically, define $r_{A,t+1}$ to be the share-weighted rate of return on the consumer's asset portfolio, including both the monetary assets, $m_{i,t}$ ($i = 1,\cdots,L$), and the non-monetary assets, $k_{j,t}$ ($j = 1,\cdots,K$). Then the traditional CAPM approximation to $Q_{t+1}$ mentioned above is of the form $Q_{t+1} = a_t - b_t r_{A,t+1}$, where $a_t$, and $b_t$ can be time dependent. Econometric methodology for possible extensions of the specification can be found in Barnett and Binner (2004).

Let $\phi_{i,t}$ and $\varphi_{j,t}$ denote the share of $m_{i,t}$ and $k_{j,t}$, respectively, in the portfolio's stock value, so that

$$\phi_{i,t} = \frac{m_{i,t}}{\sum_{l=1}^{L} m_{l,t} + \sum_{j=1}^{K} k_{j,t}} = \frac{m_{i,t}}{A_t} \tag{3.11}$$

and

$$\varphi_{j,t} = \frac{k_{j,t}}{\sum_{l=1}^{L} m_{l,t} + \sum_{i=1}^{K} k_{i,t}} = \frac{k_{j,t}}{A_t}. \tag{3.12}$$

Then, by construction, $r_{A,t+1} = \sum_{i=1}^{L} \phi_{i,t} r_{i,t+1} + \sum_{j=1}^{K} \varphi_{j,t} \tilde{r}_{j,t+1}$, where

$$\sum_{i=1}^{L} \phi_{i,t} + \sum_{j=1}^{K} \varphi_{j,t} = 1. \tag{3.13}$$

Let $\Pi_{A,t} = \sum_{i=1}^{L} \phi_{i,t} \pi_{i,t} + \sum_{j=1}^{K} \varphi_{i,t} \tilde{\pi}_{j,t}$, where $\tilde{\pi}_{j,t}$ is the user cost of non-monetary asset j. We define $\Pi_{A,t}$ to be the user cost of the consumer's asset wealth portfolio. But the user cost, $\tilde{\pi}_{j,t}$, of every non-monetary asset is simply 0. Hence equivalently $\Pi_{A,t} = \sum_{i=1}^{L} \phi_{i,t} \pi_{i,t}$. The reason is that consumers do not pay a price, in terms of foregone interest, for the monetary services of non-monetary assets, since they provide no monetary services and provide only their investment rate of return. We can show that our definition of $\Pi_{A,t}$ is consistent with Fisher's factor reversal test, as follows.

**Result**: *The pair ( $A_t$, $\Pi_{A,t}$ ) satisfies factor reversal, defined by:*

$$\Pi_{A,t} A_t = \sum_{i=1}^{L} \pi_{i,t} m_{i,t} + \sum_{j=1}^{K} \tilde{\pi}_{j,t} k_{j,t}. \tag{3.14}$$

Observe that the wealth portfolio is different from the monetary services aggregate, $M(\boldsymbol{m}_t)$. The portfolio weights in the asset wealth stock are the market-value-based shares, while the growth rate weights in the monetary services flow aggregate are the user-cost-evaluated shares.

Suppose one of the non-monetary assets is (locally) risk-free with gross real interest rate $r_t^f$, and let $r_t^f$ =E $\tilde{r}_{j,t+1}$ for all j. It follows that the certainty equivalent user cost of the asset wealth portfolio is $\Pi_{A,t}^e = \frac{r_t^f - E_t r_{A,t+1}}{r_t^f}$. We can prove the following proposition.

**Proposition 3.** *If one of the non-monetary assets is (locally) risk-free with gross real interest rate* $r_t^f$*, and if* $Q_{t+1} = a_t - b_t r_{A,t+1}$*, where* $r_{A,t+1}$ *is the gross real rate of return on the consumer's wealth portfolio, then the user cost of any monetary asset* $i$ *is given by*

$$\pi_{i,t} - \pi_{i,t}^e = \beta_{i,t} (\Pi_{A,t} - \Pi_{A,t}^e), \tag{3.15}$$

*where* $\pi_{i,t}$ *and* $\Pi_{A,t}$ *are the user costs of asset* $i$ *and of the asset wealth portfolio, respectively, and* $\pi^e_{i,t} = \frac{r_t^f - E_t r_{i,t+1}}{r_t^f}$ *and* $\Pi^e_{A,t} = \frac{r_t^f - E_t r_{A,t+1}}{r_t^f}$ *are the certainty-equivalent user costs of asset* $i$ *and the asset wealth portfolio, respectively. The "beta" of asset* $i$ *in equation (3.15)) is given by*

$$\beta_{i,t} = \frac{Cov_t(r_{A,t+1}, r_{i,t+1})}{Var_t(r_{A,t+1})}. \tag{3.16}$$

Proposition 3 is very similar to the standard CAPM formula for asset returns. In CAPM theory, the expected excess rate of return, $E_t r_{i,t+1} - r_t^f$, on an individual asset is determined by its covariance with the excess rate of return on the market portfolio, $E_t r_{M,t+1} - r_t^f$, in accordance with

$$E_t r_{i,t+1} - r_t^f = \beta_{i,t}(E_t r_{M,t+1} - r_t^f), \tag{3.17}$$

where $\beta_{i,t} = \frac{Cov_t(r_{i,t+1} - r_t^f, r_{M,t+1} - r_t^f)}{Var_t(r_{M,t+1} - r_t^f)}$.

This result implies that asset *i*'s risk premium depends on its market portfolio risk exposure, which is measured by the *beta* of this asset.

## 4. Concluding remarks

Simple sum monetary aggregates treat monetary assets with different rates of return as perfect substitutes. Barnett (1978, 1980) showed that the Divisia index, with user cost prices, is a more appropriate measure for monetary services, and derived the formula for the user cost of monetary asset services in the absence of uncertainty. Barnett, Liu, and Jensen (1997) extended the Divisia monetary quantity index to the case of uncertain returns and risk aversion. For risky monetary assets, however, the magnitude of the risk adjustment to the certainty equivalent user cost is unclear. Using a standard time-separable power utility function, Barnett, Liu, and Jensen (1997) showed that the difference between the unadjusted Divisia index and the index extended for risk is usually small. However, this result could be a consequence of the same problem that causes the equity premium puzzle in the asset pricing literature. The consumption-based asset pricing model with more general utility functions, most notably those that are intertemporally non-separable, can reproduce the large and time-varying risk premium observed in the data [Campbell and Cochrane (1999)]. We believe that similarly extended asset pricing models will provide larger and *more accurate* CCAPM adjustment to the user costs of monetary assets than those found in Barnett, Liu, and Jensen (1997). The current paper extends the basic result in Barnett, Liu, and Jensen (1997) in that manner. The proofs and more discussion will be available in the full paper, now in working paper form, and to appear in vol. 1, no. 1. of the *Annals of Finance* as Barnett and Wu (2004).

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