THE UNIVERSITY OF KANSAS
WORKING PAPERS SERIES IN
THEORETICAL AND APPLIED ECONOMICS

COALITIONAL MANIPULATION ON NETWORKS

Biung-Ghi Ju
University of Kansas

September 2004

THE UNIVERSITY OF KANSAS
WORKING PAPERS SERIES IN THEORETICAL AND APPLIED ECONOMICS
WORKING PAPER NUMBER 200410

Coalitional Manipulation on Networks

Biung-Ghi Ju[1]

August 21, 2004

[1]Department of Economics, The University of Kansas, 1300 Sunnyside Avenue, Lawrence, KS 66045, USA. E-mail: bgju@ku.edu. Tel.:+1-785-864-2860. Fax.:+1-785-864-5270.

Abstract

We consider an abstract model of division problems where each agent is identified by a characteristic vector. Agents are situated on a network (a non-directed graph) and any connected coalition can reallocate members' characteristics (e.g. reallocation of claims in bankruptcy problems). A *reallocation-proof* rule prevents any coalition from benefiting, in terms of its total award, through a reallocation. We offer a full characterization of *reallocation-proof* rules without any assumption on the network structure. This result yields a variety of useful corollaries for specific networks such as the complete network, trees, networks without a "bridge" etc. Our model has various special examples such as bankruptcy, surplus sharing, cost sharing, income redistribution, social choice with transferable utility, etc.

JEL Classification: C71, D30, D63, D71.
Keywords: Division problem; Coalitional manipulation; Network; Graph; Reallocation-proofness

1 Introduction

Division problems often take the following abstract form. There are a finite number of agents. Each agent is characterized by a vector in $\mathbb{R}^K_+$, where K is the set of characteristics. An amount of resource, a real number, has to be divided among these agents. A systematic method of division is described by a (division) *rule* associating with each division problem a vector of individual shares, or awards.

Suppose that agents are situated on a network (a non-directed graph), and any two adjacent agents can reallocate their characteristics. If two agents are not adjacent, they need other agents connecting them for a reallocation. Thus any "connected" coalition can manipulate members' characteristic vectors through a reallocation. For example, consider a company producing a set of products, K, through its local branches. The company has to divide its profit among branches based on their outputs. Branches are located on a transportation network (highway system). Suppose that two adjacent branches can reallocate their outputs costlessly and without being detected by other branches. Hence any two adjacent branches and also members of any connected coalition can freely reallocate their outputs. Depending on the rule being adopted, a coalition may or may not benefit, in terms of its total award, by reallocating outputs. What rules are robust to such coalitional manipulation? The main objective of this paper is to answer this question.

Our main result offers a full characterization of rules that have the robustness condition, called *reallocation-proofness*. This condition requires that no coalition should be able to raise its total award by a reallocation of characteristic vectors among its members. The main result is established without any assumption on the network structure.[1] It yields various characterization results depending on specific structures of the network. For example, when the graph is complete (any two nodes are directly linked), this result reduces to the result established by Ju, Miyagawa, and Sakai (2003, JMS below). We also consider other special cases such as trees and graphs without a "bridge".

Reallocation-proofness (also called "no advantageous reallocation", "strategy-proofness") on complete network has been studied by a number of authors in various specialized settings: O'Neill (1982), Moulin (1985a, 1985b, 1987), Chun (1988), Moulin and Shenker (1992), de Frutos (1999), Ching and Kakker (2001), Ju (2003), Moreno-Ternero (2004) etc. They consider models dealing with bankruptcy (or

[1]We assume *connectedness* of the network but our result can be applied easily for any disconnected network.

taxation), surplus sharing, social choice with transferable utility, and cost allocation. JMS (2003) consider the same abstract model as ours. Although the assumption of complete network is a fairly strong one, no earlier author, as far as we know, has investigated consequences of dropping it. Little is known about *reallocation-proofness* on incomplete networks and about whether results obtained for complete network still hold for incomplete networks. Our results are helpful for clarifying these issues.

We identify a condition for networks, called, *multi-node-connectivity*, under which all the earlier results for complete network continue to hold. This condition says that the graph cannot be disconnected after an elimination of a single node. It is clearly weaker than completeness. Moreover, we show that it is the maximal weakening of completeness under which *reallocation-proofness* is equivalent to *reallocation-proofness* under the assumption of complete network. In other words, *reallocation-proofness* on a network G is equivalent to *reallocation-proofness* on the complete network if and only if the network G has multi-node-connectivity. We also show that except for the case of linear networks, *reallocation-proofness* on an incomplete network together with "no award for null" (also called "dummy") imply *reallocation-proofness* on the complete network. Therefore, we are able to strengthen all the earlier results imposing the two axioms together on complete network continue to hold on incomplete and non-linear networks. On linear networks, the family of rules, we characterize, is larger than the family on the complete network. Without *no award for null*, *reallocation-proof* rules on incomplete networks may have very different representations from *reallocation-proof* rules on the complete network. In particular, on trees, *reallocation-proofness* no longer has the additivity implication reported by JMS (2003).

The rest of the paper is organized as follows. In Section 2, we define our model, network, coalition structure, axioms, and some important rules. In Section 3, we state and prove preliminary results. In Section 4, we state our main result. Some proofs are in Appendices A-B.

2 Definitions

2.1 Model

There is a finite set of *agents*, N. Each agent $i \in N$ is characterized by a vector $c_i \equiv (c_{ik})_{k \in K} \in \mathbb{R}^K_+$, where K denotes the set of issues. We refer to c_i as i's *characteristic vector*. Let $N = \{1, 2, \ldots, |N|\}$. Throughout, we assume $|N| \geq 3$. A profile of characteristic vectors of agents is denoted by $c \equiv (c_i)_{i \in N} \in \mathbb{R}^{N \times K}_+$, and the sum of these vectors is denoted by

$$\bar{c} \equiv (\bar{c}_k)_{k \in K} \equiv (\sum_{i \in N} c_{ik})_{k \in K} \in \mathbb{R}^K_+.$$

A *problem* is a pair $(c, E) \in \mathbb{R}^{N \times K}_+ \times \mathbb{R}_{++}$, where $c \in \mathbb{R}^{N \times K}_+$ is a profile of characteristic vectors and $E \in \mathbb{R}_{++}$ is an amount to be divided. For simplicity, we only consider problems such that $\bar{c}_k > 0$ for each $k \in K$. A *domain* is a non-empty set of problems and is denoted by $\mathcal{D}$. A division rule, or briefly, a *rule* over a domain $\mathcal{D}$ is a function f associating with each problem $(c, E) \in \mathcal{D}$ a vector of awards $f(c, E) \in \mathbb{R}^N$. A domain $\mathcal{D}$ is *rich* (JMS 2003) if, for each problem $(c, E) \in \mathcal{D}$ and each profile $\bar{c} \in \mathbb{R}^{N \times K}_+$ such that $\bar{c}' = \bar{c}$, we have $(\bar{c}', E) \in \mathcal{D}$. That is, $\mathcal{D}$ is rich if it is closed under reallocations of characteristic vectors. We restrict our attention to rich domains. Examples of rich domains are the set of bankruptcy problems in O'Neill (1982), the set of surplus sharing problems in Moulin (1987), the set of social choice problems with transferable utilities in Moulin (1985), the set of cost sharing problems in Moulin and Shenker (1992), etc.

We also use the following additional notation. For each $S \subseteq N$ and each $c \in \mathbb{R}^{N \times K}_+$,

$$\bar{c}_S \equiv (\bar{c}_{Sk})_{k \in K} \equiv (\sum_{i \in S} c_{ik})_{k \in K} \in \mathbb{R}^K_+.$$

Similarly, for each $S \subseteq N$ and each $x \in \mathbb{R}^N_+$,

$$\bar{x}_S \equiv \sum_{i \in S} x_i.$$

Given $x, y \in \mathbb{R}^M$, $x \geqq y$ means that $x_m \geq y_m$ for each m; $x \geq y$ means that $x \geqq y$ and $x \neq y$; and $x > y$ means that $x_m > y_m$ for each m.

2.2 Networks and Coalition Structures

Before defining "coalitional manipulation", we first need to explain possible coalition formations. We assume that agents form a coalition constrained by a network. The network is fixed throughout the paper. It is described by a (non-directed) *graph* consisting of a set of *nodes* N and a set of *edges* $D \equiv \{\{i, j\} : i, j \in N \text{ and } i \neq j\}$. Let $G \equiv (N, D)$. For simplicity, we sometimes denote an edge $\{i, j\} \in D$ by ij. Two nodes, i and j, are *adjacent* if $ij \in D$.

A graph $G \equiv (N, D)$ is *complete* if for each $i, j \in N$ with $i \neq j$, $ij \in D$. A *path* is a sequence of edges which are successively intersecting. A path is denoted simply by listing nodes that the path follows. A *line* is a path that never passes a node more than once. For each $h, i, j \in N$, we say i is *between* h *and* j if every path including h and j includes also i. A *cycle* is a path that passes more than two nodes and that passes one and only one node twice. With a slight abuse of terminology, we say that a graph is a *cycle* when the graph itself is a cycle. Similarly, we say that a graph is a *line*. A *spanning line* is a line containing all nodes in N. A *spanning cycle* is a cycle containing all nodes in N.

For each $S \subseteq N$, let $G_S \equiv (S, D_S \equiv \{ij \in D : i, j \in S\})$ be the *subgraph on* S. We say a subgraph G_S is *connected* if for any two nodes $i, j \in S$, there is a path in G_S from i to j. Note that when $S = \emptyset$ or a singleton, G_S is connected trivially. We say that S is *connected* when G_S is connected. Coalition S is *admissible* if S is connected. Let $\mathcal{C}(G)$ be the set of admissible coalitions, called, the *coalition structure* on G. For example, when G is *complete*, $\mathcal{C}(G)$ equals the set of all subsets of N, that is, 2^N, which is called the *unrestricted coalition structure*.

Throughout the paper, we assume that G is connected. However, our results are easily extended to the general case.[2]

A *tree* is a connected graph in which every two nodes have one and only one path from one to another. A node i in a tree is an *end node* if i is not between any two other nodes, that is, for all $h, j \in N \backslash \{i\}$, i is not between h and j. If G is a tree, by choosing any node $i^* \in N$ as a *root*, we can define the *directed tree with root* i^*, denoted by $G(i^*)$. In the directed tree $G(i^*)$, for each $i \in N$, let $s(i)$ be the set of *successors* of i, including i itself, and $s^0(i)$ the set of successors of i, not including i. Let $p(i)$ be the set of *predecessors*, including i itself, and $p^0(i)$ the set of predecessors of i, not including i. Let $sm(i)$ be the set of *immediate successors* of i and $pm(i)$ the *immediate predecessor* of i. Clearly, $j \in sm(i)$ if and only if $i = pm(j)$. It should be noted that all these functions, $s(\cdot)$, $s^0(\cdot)$,$sm(\cdot)$, $p(\cdot)$, $p^0(\cdot)$, and $pm(\cdot)$, depend on the choice of the root i^*.

An edge $ij \in D$ is called a *bridge* (also called an "isthmus" in Wilson 1979) if deleting ij from D results in a disconnected graph, that is, $(N, D\backslash\{i, j\})$ is not connected. A graph G is *multi-edge-connected* if it has no bridge.[3] Thus a multi-edge-connected graph remains connected after deleting any one of its edges. We next define graphs in which no single node plays a critical role in keeping the graph connected. A node $i \in N$ is called a *cutnode* if deleting i from G results

[2] Note that any (possibly disconnected) graph is partitioned into the unique family of maximal connected subgraphs. Our results can be applied for each of these maximal connected subgraphs.

[3] A graph is multi-edge-connected if and only if its degree of "edge-connectivity" (see p.10 of Diestel 2000 and p. 29 of Wilson 1979 for the definition) is greater than 1.

in a disconnected subgraph of G, that is, $G_{N\setminus\{i\}}$ is not connected. A graph G is *multi-node-connected* if it has no cutnode.[4] Thus a multi-node-connected graph stays connected after a deletion of any single node. Clearly, if G has a spanning cycle, G is multi-node-connected. There are, of course, multi-node-connected graphs that have no spanning cycle. No tree with at least three nodes is multi-node-connected.

2.3 Axioms

Our main objective is to study rules that are robust to coalitional manipulations through reallocations of characteristic vectors. Since coalition formation is constrained by a graph, such a robustness can be formalized by the requirement that the total amount allocated to each admissible coalition $S \in \mathcal{C}(G)$ should not be affected by any reallocation of c_i's within S. Formally:

Reallocation-Proofness. For each $(c, E) \in \mathcal{D}^N$, each $S \in \mathcal{C}(G)$, and each $c' \in \mathbb{R}_+^{N\times K}$, if $\bar{c}'_S = \bar{c}_S$ and $c'_{N\setminus S} = c_{N\setminus S}$,

$$\sum_{i\in S} f_i(c', c_{N\setminus S}, E) = \sum_{i\in S} f_i(c, E). \tag{1}$$

If the left-hand side of (1) is larger than the right-hand side, then coalition S with profile $(c_i)_{i\in S}$ can gain by reallocating their characteristic vectors to c'_S (and making appropriate side-payments). If the reverse inequality holds, then coalition S with profile $(c'_i)_{i\in S}$ can gain by the reverse reallocation. We also consider a weaker condition, by focusing on coalitions by pairs.

Pairwise Reallocation-Proofness. For each $(c, E) \in \mathcal{D}^N$, each $ij \in D$ (so $\{i, j\} \in \mathcal{C}(G)$) and each $c'_i, c'_j \in \mathbb{R}_+^K$, if $c'_i + c'_j = c_i + c_j$,

$$f_i(c'_i, c'_j, c_{N\setminus\{i,j\}}, E) + f_j(c'_i, c'_j, c_{N\setminus\{i,j\}}, E) = f_i(c, E) + f_j(c, E).$$

The next axiom is a useful implication of *reallocation-proofness* (see Lemma 2). It says that any admissible coalition cannot change, through a reallocation of characteristic vectors, the shares of others, without affecting its own aggregate share. This axiom is similar, in spirit, to "non-bossiness" in economic environments introduced by Satterthwaite and Sonnenschein (1981).

[4]A graph is multi-node-connected if and only if its degree of "connectivity" (see p.10 of Diestel 2000; p. 29 of Wilson 1979 for the definition) is greater than 1.

Non-Bossiness. For each $(c, E) \in \mathcal{D}^N$, each $S \in \mathcal{C}(G)$, and each $c' \in \mathbb{R}_+^{N \times K}$, if $\bar{c}'_S = \bar{c}_S$, $c'_{N \setminus S} = c_{N \setminus S}$, and $\sum_{i \in S} f_i(c', E) = \sum_{i \in S} f_i(c, E)$,

$$f_{N \setminus S}(c', E) = f_{N \setminus S}(c, E). \tag{2}$$

The next axiom is the pairwise version of *non-bossiness*.

Pairwise Non-Bossiness. For each $(c, E) \in \mathcal{D}^N$, each $ij \in D$, and each $c'_i, c'_j \in \mathbb{R}_+^K$, if $c'_i + c'_j = c_i + c_j$ and $f_i(c'_i, c'_j, c_{N \setminus \{i,j\}}, E) + f_j(c'_i, c'_j, c_{N \setminus \{i,j\}}, E) = f_i(c, E) + f_j(c, E)$,

$$f_{N \setminus \{i,j\}}\left(c'_i, c'_j, c_{N \setminus \{i,j\}}, E\right) = f_{N \setminus \{i,j\}}(c, E).$$

In the context of bankruptcy problems, there is a large family of *non-bossy* rules, known as "parametric rules".

In some of our results, we characterize rules satisfying some combinations of the following axioms as well as *reallocation-proofness*.

The next axiom says that awards should add up to the amount to divide:

Efficiency. For each $(c, E) \in \mathcal{D}$, $\sum_{i \in N} f_i(c, E) = E$.

For each problem $(c, E) \in \mathcal{D}$, let $\mathcal{D}(\bar{c}, E) \equiv \{(c', E) \in \mathbb{R}_+^{N \times K} \times \mathbb{R}_{++} : \bar{c}' = \bar{c}\}$. Note that on the compact set $\mathcal{D}(\bar{c}, E)$, each agent's characteristic vector is both bounded above and below. Then, it is appealing to require that each agent should not get unlimited reward or unlimited loss on the set $\mathcal{D}(\bar{c}, E)$. The next axiom states an even weaker condition that at least one agent's award should be bounded above or below on $\mathcal{D}(\bar{c}, E)$.

One-Sided Boundedness. For each $(c, E) \in \mathcal{D}$, there exists $i \in N$ such that $f_i(\,\cdot\,, E)$ is bounded from either above or below over $\mathcal{D}(\bar{c}, E)$.

This axiom is implied by each of the following two axioms. The first one requires awards to be non-negative:

Non-Negativity. For each $(c, E) \in \mathcal{D}$ and each $i \in N$, $f_i(c, E) \geq 0$.

The next axiom considered by Moulin (1985a) says that no agent can increase its award by transferring part of its characteristic vector to other agents:

No Transfer Paradox.[5] For each $(c, E) \in \mathcal{D}$, each $c' \in \mathbb{R}_+^{N \times K}$, each $i, j \in N$ with $\{i, j\} \in D$, and each $t \in [0, c_i] \equiv [0, c_{i1}] \times \cdots \times [0, c_{iK}]. \subseteq \mathbb{R}_+^K$,

$$f_i(c_i - t, c_j + t, c_{-\{i,j\}}, E) \leq f_i(c_i, c_j, c_{-\{i,j\}}, E).$$

[5] Since we focus on pairs $\{i, j\}$ that are edges on the graph G, our axiom is weaker than the axiom in Moulin (1985a).

The next axiom says that no amount should be awarded to agents with the zero characteristic vector:

No Award for Null. For each $(c, E) \in \mathcal{D}$ and each $i \in N$, if $c_i = 0$, then $f_i(c, E) = 0$.

2.4 Examples of Division Rules

For the case when characteristic vectors are single-dimensional (i.e., $|K| = 1$), one of the simplest and best-known rules is proportional rule, which divides the total amount proportionally to the single characteristic. JMS (2003) extend the definition of proportional rule to the case of multi-dimensional characteristics. A *weight function* is a function mapping each $(\bar{c}, E) \in \mathbb{R}^K_{++} \times \mathbb{R}_{++}$ into a weight vector in $\Delta^{|K|-1}$, $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \Delta^{|K|-1}$.

Definition 1 (Proportional Rules, $|K| \geq 1$). A rule f is a *proportional rule* if there exists a weight function W such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = \sum_{k \in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E) E.$$[6]

We use P^W to denote the proportional rule associated with W.

Note that P^W first applies the proportional rule to each single-dimensional sub-problem (c^k, E), where $c^k \equiv (c_{ik})_{i \in N}$, and then takes the weighted average of the solutions to the sub-problems using the vector of weights $W(\bar{c}, E)$. The weights depend on the problem being considered but depend only on $(\bar{c}, E)$. Proportional rules are *efficient* since $\sum_{k \in K} W_k(\bar{c}, E) = 1$. Proportional rules also satisfy all other axioms defined in Section 2.3. It is evident that, if $|K| = 1$, Definition 1 reduces to the standard definition of proportional rule in the case of $|K| = 1$.

We now define *generalized proportional rules*, introduced by JMS (2003). These rules are characterized by two functions $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$, and i's award is given by the sum of the following two terms. The first term is $A_i(\bar{c}, E)$, which is independent of i's characteristic vector but may treat i differently from others. The second term is proportional to i's characteristic vector and treats agents symmetrically. On the other hand, the second term may treat issues asymmetrically, and the degree of importance attached to each issue $k \in K$ is given by $W_k(\bar{c}, E)$. Formally,

[6]The right-hand side is well-defined since we rule out problems for which $\bar{c}_k = 0$ for some $k \in K$.

Definition 2 (Generalized Proportional Rules). There exist two functions $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = A_i(\bar{c}, E) + \sum_{k \in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E)E. \tag{3}$$

Note that W is not required to be a weight function, i.e., neither $W_k(\bar{c}, E) \geq 0$ nor $\sum_{k\in K} W_k(\bar{c}, E) = 1$ is required. Proportional rules are special cases where $A_i = 0$ and W is a weight function. Since, given $(\bar{c}, E)$, the second term of (3) is linear in c_{ik}, generalized proportional rules satisfy *reallocation-proofness* and *one-sided boundedness*. These rules do not necessarily satisfy other axioms in Section 2.3. Necessary and sufficient conditions for (A, W) to satisfy each of those axioms are offered by JMS (2003).

3 Preliminary Results

In this section, we consider three special cases of connected graphs: multi-node-connected graphs, multi-edge-connected graphs, and trees. In each of the three cases, we offer a full characterization of *reallocation-proof* rules and necessary and sufficient conditions for additional axioms in Section 2.3.

We first establish two useful lemmas. The first lemma shows that any reallocation of characteristic vectors among agents in a connected coalition can be described by successive reallocations among edges in this coalition.

Lemma 1. *If S is connected and $c, c' \in \mathbb{R}^{N\times K}_+$ are such that $\bar{c}'_S = \bar{c}_S$ and $c'_{N\setminus S} = c_{N\setminus S}$, then c' can be reached from c through successive reallocations of characteristic vectors among edges in S, that is, there exist a number r and $S_1, \cdots, S_r \in D_S$ and $c^1, c^2, \cdots, c^r \in \mathbb{R}^{N\times K}_+$ such that $\bar{c}^1_{S_1} = \bar{c}_{S_1}$, $c^1_{N\setminus S_1} = c_{N\setminus S_1}$, $c^r = c'$, and for each $m = 2, \cdots, r$, $\bar{c}^m_{S_m} = \bar{c}^{m-1}_{S_m}$ and $c^m_{N\setminus S_m} = c^{m-1}_{N\setminus S_m}$.*

Proof. Let S and $c, c' \in \mathbb{R}^{N\times K}_+$ be given as above. The formal proof is tedious and so skipped. Below we only give the basic idea. Pick an agent, say 1, in S. For any $i \in S$, since S is connected, there is a path from i to 1, denoted by p_i, and we can transfer all i's characteristics in c_i to 1's through successive pairwise reallocations along this path. Then we end up with $c'' \in \mathbb{R}^{N\times K}_+$ such that $c''_1 \equiv \bar{c}_S$, $c''_{S\setminus\{1\}} = 0$, and $c''_{N\setminus S} = c_{N\setminus S}$. Now we do the reverse changes, that is, for each $i \in S$, we use path p_i to increase i's vector from 0 to c'_i and decrease 1's vector from $\bar{c}_S$ to $\bar{c}_S - c'_i$. Throughout this procedure, we always have non-negative characteristic vectors for all agents and the constant sum of characteristic vectors of agents in

S. Since there is no change made in the characteristic vectors of agents in $N \backslash S$, the final outcome is c'. ∎

We now establish logical relation among *reallocation-proofness*, *non-bossiness*, and their pairwise versions.

Lemma 2. *Assume that G is a connected graph.*
(i) *Reallocation-proofness implies non-bossiness.*
(ii) *Reallocation-proofness is equivalent to the combination of pairwise reallocation-proofness and pairwise non-bossiness.*

Proof. To prove part (i), let f be a rule satisfying *reallocation-proofness*. Let $S \subseteq N$ be a connected coalition on G and $S \neq N$. Let $(c, E) \in \mathcal{D}$ and $c' \in \mathbb{R}_+^{N \times K}$ be such that $\bar{c}_S = \bar{c}'_S$ and $c_{N \backslash S} = c'_{N \backslash S}$. Let $x \equiv f(c, E)$ and $x' \equiv f(c', E)$. By *reallocation-proofness*, $\bar{x}_S = \bar{x}'_S$. Since G is a connected graph, there exists a node $i_1 \in N \backslash S$ that is adjacent to a node in S. Let $S_1 \equiv S \cup \{i_1\}$. Then S_1 is also connected and $c_{i_1} = c'_{i_1}$. Hence $\bar{c}_{S_1} (= \bar{c}_S + c_{i_1}) = \bar{c}'_{S_1} (= \bar{c}'_S + c'_{i_1})$ and so by *reallocation-proofness*, $\bar{x}_S + x_{i_1} = \bar{x}'_S + x'_{i_1}$. Since $\bar{x}_S = \bar{x}'_S$, $x_{i_1} = x'_{i_1}$. Suppose by induction that $k \leq |N \backslash S|$ and $i_1, \cdots, i_k \in N \backslash S$ are such that $S_k \equiv S \cup \{i_1, \cdots, i_k\}$ is connected, $\bar{c}_{S_k} = \bar{c}'_{S_k}$, and $x_{\{i_1, \cdots, i_k\}} = x'_{\{i_1, \cdots, i_k\}}$. If $N \backslash S_k = \emptyset$, we are done. If not, then since G is a connected graph, there exists a node $i_{k+1} \in N \backslash S_k$ that is adjacent to a node in S_k. Let $S_{k+1} \equiv S_k \cup \{i_{k+1}\}$. Then S_{k+1} is connected and since $\bar{c}_{S_k} = \bar{c}'_{S_k}$ and $c_{i_{k+1}} = c'_{i_{k+1}}$, $\bar{c}_{S_{k+1}} = \bar{c}'_{S_{k+1}}$. Hence by *reallocation-proofness*, $\bar{x}_{S_{k+1}} = \bar{x}'_{S_{k+1}}$. Since $\bar{x}_{S_k} (= \bar{x}_S + x_{i_1} + \cdots + x_{i_k}) = \bar{x}'_{S_k} (= \bar{x}'_S + x'_{i_1} + \cdots + x'_{i_k})$, $x_{k+1} = x'_{k+1}$. Therefore, $x_{\{i_1, \cdots, i_{k+1}\}} = x'_{\{i_1, \cdots, i_{k+1}\}}$. Since N is finite, the iteration will end after a finite number of steps and, at the end, we obtain $x_{N \backslash S} = x'_{N \backslash S}$.

By part (i), *reallocation-proofness* implies both *pairwise reallocation-proofness* and *pairwise non-bossiness*. To prove the converse, let f be a rule satisfying *pairwise reallocation-proofness* and *pairwise non-bossiness*. Let $S \subseteq N$ be connected. Let $(c, E), (c', E) \in \mathcal{D}$ be such that $\bar{c}_S = \bar{c}'_S$ and $c_{N \backslash S} = c'_{N \backslash S}$. We only have to show $\sum_{i \in S} f_i(c, E) = \sum_{i \in S} f_i(c', E)$ and $f_{N \backslash S}(c, E) = f_{N \backslash S}(c', E)$.

By Lemma 1, there exist a number r, $S_1, S_2, \cdots, S_r \in D_S$, and $c^1, c^2, \cdots, c^r \in \mathbb{R}_+^{N \times K}$ such that $\bar{c}^1_{S_1} = \bar{c}_{S_1}$, $c^1_{N \backslash S_1} = c_{N \backslash S_1}$, $c^r = c'$, and for each $m = 2, \cdots, r$, $\bar{c}^m_{S_m} = \bar{c}^{m-1}_{S_m}$ and $c^m_{N \backslash S_m} = c^{m-1}_{N \backslash S_m}$. By richness of $\mathcal{D}$, $(c^1, E), \cdots, (c^r, E) \in \mathcal{D}$. For each $m = 1, \cdots, r-1$, let $x^m \equiv f(c^m, E)$. Let $x \equiv f(c, E)$ and $x' \equiv f(c', E)$. Since $\bar{c}^1_{S_1} = \bar{c}_{S_1}$, then by *pairwise reallocation-proofness*, $\bar{x}^1_{S_1} = \bar{x}_{S_1}$. By *pairwise non-bossiness*, $x^1_{N \backslash S_1} = x_{N \backslash S_1}$. Since $S_1 \subseteq S$, then $\bar{x}^1_S = \bar{x}_S$ and $x^1_{N \backslash S} = x_{N \backslash S}$. For each $m = 2, \cdots, r$, since $\bar{c}^m_{S_m} = \bar{c}^{m-1}_{S_m}$, then by *pairwise reallocation-proofness*,

$\bar{x}^m_{S_m} = \bar{x}^{m-1}_{S_m}$ and by *pairwise non-bossiness*, $x^m_{N\backslash S_m} = x^{m-1}_{N\backslash S_m}$. Since $S_m \subseteq S$, then $\bar{x}^m_S = \bar{x}^{m-1}_S$ and $x^m_{N\backslash S} = x^{m-1}_{N\backslash S}$. This shows $\bar{x}'_S = \bar{x}_S$ and $x'_{N\backslash S} = x_{N\backslash S}$. ∎

Remark 1. (i) *Reallocation-proofness* implies *non-bossiness* if and only if the graph is connected.
(ii) Even if the graph is connected, *pairwise reallocation-proofness* does not imply *pairwise non-bossiness*.

By Lemma 2, *reallocation-proofness* in all our results can be replaced with the combination of *pairwise reallocation-proofness* and *pairwise non-bossiness*. Also, by virtue of Lemma 2, in order to check *reallocation-proofness*, we only need to consider edges, instead of considering all connected coalitions, and check the two pairwise axioms.

3.1 Multi-Node-Connected Graphs

We start with multi-node-connected graphs. In the next lemma, we show that when G is multi-node-connected, *reallocation-proofness* under coalition structure $\mathcal{C}(G)$ is equivalent to *reallocation-proofness* under the unrestricted coalition structure.

Lemma 3. *Given a connected graph $G \equiv (N, D)$, let f be a rule satisfying reallocation-proofness. For each $T \subseteq N$, if no node in $N\backslash T$ is a cutnode, then for each $(c, E), (c', E) \in \mathcal{D}$ with $\bar{c}_T = \bar{c}'_T$ and $c_{N\backslash T} = c'_{N\backslash T}$,*

$$\begin{aligned} \sum_{i\in T} f_i(c, E) &= \sum_{i\in T} f_i(c', E), \\ f_{N\backslash T}(c, E) &= f_{N\backslash T}(c', E). \end{aligned}$$

Therefore, if G is multi-node-connected, then reallocation-proofness under $\mathcal{C}(G)$ is equivalent to reallocation-proofness under the unrestricted coalition structure.

Proof. Let $G \equiv (N, D)$ be a connected graph. Let f be a rule satisfying *reallocation-proofness* under $\mathcal{C}(G)$. Then by Lemma 2, f satisfies *non-bossiness*. Let $T \subseteq N$. Assume that no node in $N\backslash T$ is a cutnode. Let $(c, E), (c', E) \in \mathcal{D}$ be such that $\bar{c}_T = \bar{c}'_T$ and $c_{N\backslash T} = c'_{N\backslash T}$. Let $x \equiv f(c, E)$ and $x' \equiv f(c', E)$. We only have to show $\bar{x}_T = \bar{x}'_T$ and $x_{N\backslash T} = x'_{N\backslash T}$. Since N is connected, by *reallocation-proofness*,

$$\bar{x}_N = \bar{x}'_N. \tag{4}$$

For each $i \in N\backslash T$, since i is not a cutnode, $N\backslash\{i\}$ is connected. Since $\bar{c}_{N\backslash\{i\}} = \bar{c}'_{N\backslash\{i\}}$, then by *reallocation-proofness* and *non-bossiness*, $x_i = x'_i$. Hence $x_{N\backslash T} = x'_{N\backslash T}$. Combining this with (4), we obtain $\bar{x}_T = \bar{x}'_T$. ∎

This lemma allows us to strengthen all results established for the complete graph case by JMS (2003). First is their characterization of *reallocation-proof* rules.

Proposition 1. *Assume that G is a multi-node-connected graph. Then a rule f on a rich domain $\mathcal{D}$ satisfies* reallocation-proofness *if and only if there exist two functions $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $\hat{W}\colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,*

$$f_i(c, E) = A_i(\bar{c}, E) + \sum_{k \in K} \hat{W}_k(c_{ik}, \bar{c}, E),$$

and $\hat{W}(\cdot, \bar{c}, E)$ is additive.

Proof. By Lemma 3, the result is obtained directly from Theorem 1 in JMS (2003). ∎

We will show later that multi-node-connectivity of G is a necessary and sufficient condition for equivalence between *reallocation-proofness* under $\mathcal{C}(G)$ and *reallocation-proofness* under the unrestricted coalition structure.

Next are necessary and sufficient conditions for additional axioms, described in terms of the two functions $A(\cdot)$ and $\hat{W}(\cdot)$.

Proposition 2. *Assume that G is a multi-node-connected graph. Let f be a* reallocation-proof *rule represented by $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $\hat{W}\colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ as in part* (i) *of Proposition 1. Then f satisfies*
(i) Efficiency *if and only if for each $(c, E) \in \mathcal{D}$,*

$$\sum_{i \in N} A_i(\bar{c}, E) + \sum_{k \in K} \hat{W}_k(\bar{c}_k, \bar{c}, E) = E.$$

(ii) No award for nulls *if and only if for each $(c, E) \in \mathcal{D}$ and each $i \in N$,*

$$A_i(\bar{c}, E) = 0.$$

(iii) Non-negativity *if and only if f satisfies* one-sided boundedness *and, for each $(c, E) \in \mathcal{D}$,*

$$A_i(\bar{c}, E) \geq 0 \text{ for each } i \in N,$$
$$\min_{j \in N} A_j(\bar{c}, E) + \sum_{k \in K} \min\{0, \hat{W}_k(\bar{c}_k, \bar{c}, E)\} \geq 0.$$

(iv) No transfer paradox *if and only if for each $(c, E) \in \mathcal{D}$ and each $k \in K$, $\hat{W}_k(\cdot, \bar{c}, E)$ is non-decreasing.*

Proof. The four conditions are established using Proposition 1 and the same arguments used in the proof of Proposition 1 by JMS (2003). ∎

The following results obtained by JMS (2003) for complete graphs are also extended to multi-node-connected graphs.

Proposition 3. *Assume that G is a multi-node-connected graph.*
(i) *A rule on a rich domain satisfies* reallocation-proofness *and* one-sided boundedness *if and only if it is a generalized proportional rule.*
(ii) *A rule on a rich domain satisfies* pairwise reallocation-proofness, no award for null, *and* non-negativity (*or* no transfer paradox) *if and only if it is a proportional rule.*

Proof. The two characterizations are established using Propositions 1 and 2, and the same arguments used in the proofs of Theorems 2 and 3 by JMS (2003). ∎

3.2 Multi-Edge-Connected Graphs

In this section, we consider multi-edge-connected graphs.

Let G be a multi-edge-connected graph. Let $S \subseteq N$. Subgraph G_S is *maximally multi-node-connected on G* if there is no greater multi-node-connected subgraph, that is, there is no $S' \subseteq N$ such that $S' \supsetneqq S$ and $G_{S'}$ is multi-node-connected. In the next lemma, we show that each multi-edge-connected graph is composed of maximal multi-node-connected subgraphs connected with each other by cutnodes.

Lemma 4. *Assume that $G \equiv (N, D)$ is a multi-edge-connected graph.*
(i) *The set of nodes N is uniquely divided into a finite number of subsets $N_1, \cdots, N_M$ with $\cup_{m=1}^{M} N_m = N$ such that for each $m = 1, \cdots, M$, $|N_m| \geq 3$ and G_{N_m} is a maximal multi-node-connected subgraph on G.*
(ii) *There is no cycle of successively intersecting sets among $N_1, \cdots, N_M$, that is, there is no $r \geq 3$ and no $N_{m_1}, \cdots, N_{m_r} \in \{N_1, \cdots, N_M\}$ such that $N_{m_1} \cap N_{m_2} \neq \emptyset, \cdots, N_{m_{r-1}} \cap N_{m_r} \neq \emptyset$, and $N_{m_1} = N_{m_r}$.*

The proof is left for readers [`see Omitted Proofs, Section C.1`].

By Lemma 4, N has the unique family of subsets $N_1, \cdots, N_M$ such that for each $m \in \{1, \cdots, M\}$, $|N_m| \geq 3$ and G_{N_m} is a maximal multi-node-connected subgraph. In this case, we say that *multi-edge-connected graph G is composed of maximal multi-node-connected subgraphs $G_{N_1}, \cdots, G_{N_M}$*. Let $\mathcal{N}^* (G) \equiv \{N_1, \cdots, N_M\}$

and $\mathcal{R}^*(G) \equiv \{G_{N_1}, \cdots, G_{N_M}\}$. For each $m \in \{1, \cdots, M\}$, let

$$C(N_m) \equiv \{i \in N_m : i \text{ is a cutnode on } G\}$$

be the set of cutnodes in N_m on graph G. For each $m \in \{1, \cdots, M\}$ and each $i \in N_m$, let

$$S(i, N_m) \equiv \{j \in N \backslash [N_m \backslash \{i\}] : i \text{ is between } j \text{ and any node in } N_m\}$$

be the set of nodes outside $N_m \backslash \{i\}$ that can be connected with any node in N_m only through i. Note $i \in S(i, N_m)$. Note also that $S(i, N_m)$ is not a singleton if and only if $i \in C(N_m)$. For example, if G is composed of two multi-node-connected subgraphs G_{N_1} and G_{N_2} and the cutnode is $\hat{\imath}$, then $S(\hat{\imath}, N_1) = N_2$, $S(\hat{\imath}, N_2) = N_1$, and $C(N_1) = C(N_2) = \{\hat{\imath}\}$. Another example is depicted in Figure 1. For each $i \in N$, let

$$\mathfrak{m}(i) \equiv \{m \in \{1, \ldots, M\} : i \in N_m\}$$

be the set of indices of maximal multi-node-connected subgraphs containing i. Then for each $i \in N$, $S(i, N_m) \backslash \{i\} = \cup_{m' \in \mathfrak{m}(i) \backslash \{m\}} \cup_{j \in N_{m'} \backslash \{i\}} S(j, N_{m'})$ (see Figure 1).

Proposition 4. *Assume that $G \equiv (N, D)$ is a multi-edge-connected graph and that G is composed of M maximal multi-node-connected subgraphs $G_{N_1}, \cdots, G_{N_M}$: that is, $\mathcal{R}^*(G) \equiv \{G_{N_1}, \cdots, G_{N_M}\}$. Then a rule on a rich domain $\mathcal{D}$ satisfies reallocation-proofness if and only if there exists a list of functions $(A^m : \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{N_m}, \hat{W}^m : \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K)_{m \in \{1, \cdots, M\}}$ such that for each $(c, E) \in \mathcal{D}$, each $m \in \{1, \cdots, M\}$, and each $i \in N_m$,*

$$f_i(c, E) = \begin{cases} A_i^m(\bar{c}, E) + \sum_{k \in K} \hat{W}_k^m\left(\bar{c}_{S(i,N_m)k}, \bar{c}, E\right) \\ -\sum_{m' \in \mathfrak{m}(i) \backslash \{m\}} \sum_{j \in N_{m'} \backslash \{i\}} A_j^{m'}(\bar{c}, E) \\ -\sum_{m' \in \mathfrak{m}(i) \backslash \{m\}} \sum_{k \in K} \hat{W}_k^{m'}\left(\sum_{j \in N_{m'} \backslash \{i\}} \bar{c}_{S(j, N_{m'})k}, \bar{c}, E\right), \end{cases} \text{[7]} \quad (5)$$

where for each $m, m' \in \{1, \cdots, M\}$, $\hat{W}^m(\cdot, \bar{c}, E)$ is additive and

$$\sum_{i \in N_m} A_i^m(\bar{c}, E) + \sum_{k \in K} \hat{W}_k^m(\bar{c}_k, \bar{c}, E) = \sum_{i \in N_{m'}} A_i^{m'}(\bar{c}, E) + \sum_{k \in K} \hat{W}_k^{m'}(\bar{c}_k, \bar{c}, E). \quad (6)$$

[7] If $i \notin C(N_m)$, then $\mathfrak{m}(i) \backslash \{m\} = \emptyset$ and so

$$f_i(c, E) = A_i^m(\bar{c}, E) + \sum_{k \in K} \hat{W}_k^m(c_{ik}, \bar{c}, E).$$

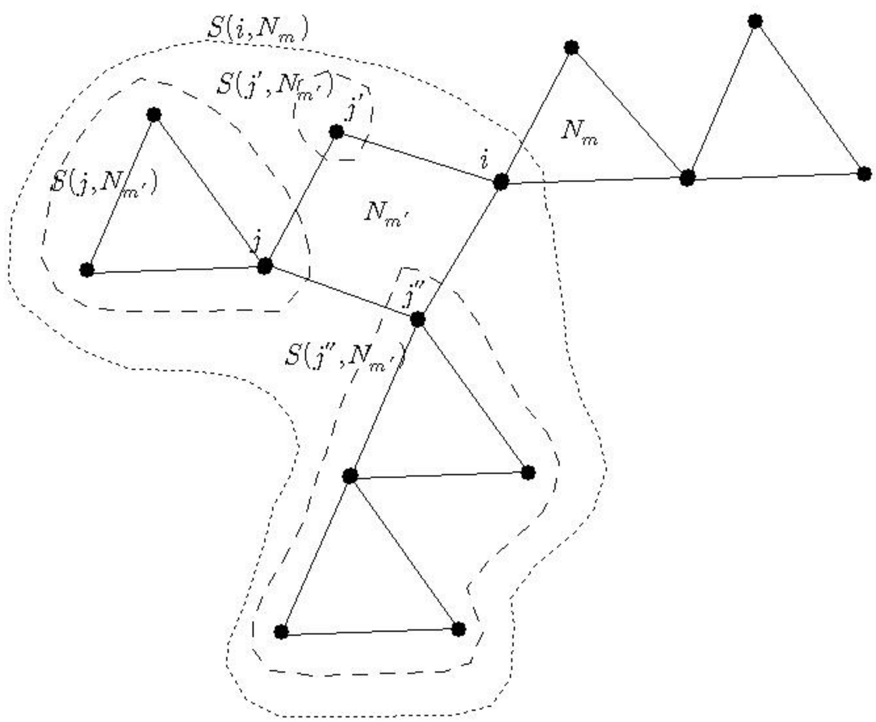

Figure 1: The multi-edge-connected graph is composed of six maximal multi-node-connected subgraphs, two of which are N_m and $N_{m'}$. Note that $i \in C(N_m)$, $j, j'' \in C(N_{m'})$, and $j' \notin C(N_{m'})$. Note also that $N_{m'} = \{j, j', j'', i\}$ and $S(i, N_m) \backslash \{i\} = S(j, N_{m'}) \cup S(j', N_{m'}) \cup S(j'', N_{m'})$.

The proof is in Appendix A.

Remark 2. Note that when G is a multi-node-connected graph, $M = 1$ and Proposition 4 reduces to Proposition 1.

We next establish necessary and sufficient conditions for the four additional axioms, *efficiency*, *no award for nulls*, *non-negativity*, and *no transfer paradox*.

Proposition 5. *Assume that $G \equiv (N, D)$ is a multi-edge-connected graph and that G is composed of M maximal multi-node-connected subgraphs $G_{N_1}, \cdots, G_{N_M}$. Let f be a reallocation-proof rule represented by a list of functions $(A^m : \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{N_m}, \hat{W}^m : \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K)_{m \in \{1,\cdots,M\}}$ as in Proposition 4. Then f satisfies*
(i) *Efficiency if and only if for each $(c, E) \in \mathcal{D}$ and each $m \in \{1, \ldots, M\}$,*

$$\sum_{i \in N} A^m_i (\bar{c}, E) + \sum_{k \in K} \hat{W}^m_k (\bar{c}_k, \bar{c}, E) = E.$$

(ii) *No award for nulls if and only if for each $(c, E) \in \mathcal{D}$, each $i \in N$, and each $m, m' \in \{1, \ldots, M\}$,*

$$\begin{aligned} A^m_i (\bar{c}, E) &= 0; \\ \hat{W}^m (\cdot, \bar{c}, E) &= \hat{W}^{m'} (\cdot, \bar{c}, E). \end{aligned}$$

Thus by additivity of $\hat{W}^m (\cdot, \bar{c}, E)$, f is a rule characterized in part (ii) *of Proposition 2.*
(iii) *Non-negativity if and only if f satisfies one-sided boundedness and, for each $(c, E) \in \mathcal{D}$ and each $m \in \{1, \ldots, M\}$,*

$$\begin{aligned} &A^m_i (\bar{c}, E) \geq 0 \text{ for each } i \in N, \\ &\min_{j \in N} A^m_j (\bar{c}, E) + \sum_{k \in K} \min\{0, \hat{W}^m_k(\bar{c}_k, \bar{c}, E)\} \geq 0, \end{aligned}$$

and for each $i \in C(N_m)$,

$$\begin{aligned} &A^m_i (\bar{c}, E) + \textstyle\sum_{k \in K} \hat{W}^m_k \left(\bar{c}_{S(i,N_m)k}, \bar{c}, E\right) \geq \\ &\textstyle\sum_{m' \in \mathfrak{m}(i) \setminus \{m\}} \sum_{j \in N_{m'} \setminus \{i\}} A^{m'}_j (\bar{c}, E) + \sum_{m' \in \mathfrak{m}(i) \setminus \{m\}} \sum_{k \in K} \hat{W}^{m'}_k \left(\sum_{j \in N_{m'} \setminus \{i\}} \bar{c}_{S(j,N_{m'})k}, \bar{c}, E \right). \end{aligned}$$

(iv) *No transfer paradox if and only if for each $(c, E) \in \mathcal{D}$, each $k \in K$, and each $m \in \{1, \ldots, M\}$, $\hat{W}^m_k (\cdot, \bar{c}, E)$ is non-decreasing.*

The proof is in Appendix A.

Remark 3. Part (ii) shows that under *no award for nulls*, *reallocation-proofness* under $\mathcal{C}(G)$ is equivalent to *reallocation-proofness* under the unrestricted coalition structure.

Examples of rules that are in the family characterized in Proposition 4 but not in the family characterized in Proposition 1 are easily provided by using different functions $A^m(\cdot)$ and $\hat{W}^m(\cdot)$ for different m's.

3.3 Trees

In this section, we consider trees.

The next result is a characterization of *reallocation-proof* rules for trees.

Proposition 6. *Assume that G is a tree. Then a rule f on a rich domain $\mathcal{D}$ satisfies reallocation-proofness if and only if f is represented by a function $T\colon \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ such that for each $(c, E) \in \mathcal{D}$ and each $i \in N$,*

$$f_i(c,E) = T_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) - \sum_{j \in sm(i)} T_j\left(\bar{c}_{s(j)}, \bar{c}, E\right), \text{[8]} \tag{7}$$

where $s(\cdot)$ and $sm(\cdot)$ are defined on a directed tree $G(i^*)$ with root $i^* \in N$.

Proof. Let $G \equiv (N, D)$ be a tree. Fix $i^* \in N$ and consider the directed tree with root i^*, $G(i^*)$. The proof of *reallocation-proofness* of rules with the stated representation will be provided in the proof of Theorem, Section B. Before proving the converse, note that we can rewrite (7) equivalently as follows: for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c,E) = T_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) - \sum_{j \in s^0(i)} f_j(c,E). \tag{$\star$}$$

Thus, for each $(c, E) \in \mathcal{D}$ and each $i \in N$, $T_i\left(\bar{c}_{s(i)}, \bar{c}, E\right)$ is the total award for agent i and i's successors, that is,

$$T_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) = \sum_{j \in s(i)} f_j(c,E). \tag{$\star\star$}$$

Let f be a *reallocation-proof* rule. Then by Lemma 2, it also satisfies *non-bossiness*. For each $i \in N$, define T as follows: for each $i \in N$ and each $(x, y, E) \in$

[8] Throughout, we use the notational convention that any summation over the empty set is zero; formally, for any function $g(\cdot)$, if $X = \emptyset$, $\sum_{x \in X} g(x) = 0$. Thus when $sm(i) = \emptyset$, (7) reduces to $f_i(c,E) = T_i(c_i, \bar{c}, E)$.

$\mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++}$ with $x \le y$,

$$T_i(x, y, E) \equiv \sum_{j \in s(i)} f_j(c, E),$$

for some $(c, E) \in \mathcal{D}$ with $\bar{c}_{s(i)} = x$ and $\bar{c} = y$. For all other $(x, y, E) \in \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++}$, set $T_i(x, y, E)$ arbitrarily. Then $(\star)$ follows directly from the definition of T and we obtain (7). Therefore, we only have to show that T is well-defined. Let $c, c' \in \mathbb{R}^{N \times K}_+$ be such that $\bar{c}_{s(i)} = \bar{c}'_{s(i)} = x$ and $\bar{c} = \bar{c}' = y$. Let $x \equiv f(c, E)$, $x' \equiv f(c', E)$, and $x'' \equiv f(c_{s(i)}, c'_{N \setminus s(i)}, E)$. Since $N \setminus s(i)$ is connected, then by *reallocation-proofness* and *non-bossiness*, $x_{s(i)} = x''_{s(i)}$ (and $\bar{x}_{N \setminus s(i)} = \bar{x}''_{N \setminus s(i)}$). Since $s(i)$ is also connected, then by *reallocation-proofness* and *non-bossiness*, $\bar{x}''_{s(i)} = \bar{x}'_{s(i)}$ (and $x''_{N \setminus s(i)} = x'_{N \setminus s(i)}$). Therefore, $\bar{x}_{s(i)} = \bar{x}'_{s(i)}$. ∎

Note that there is no restriction on $T(\cdot)$. Examples of rules without additivity property are easily constructed and this shows a clear contrast with the results on multi-node-connected graphs and multi-edge-connected graphs. Although the domain of T_i is stated as $\mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++}$ in Proposition 6, only its subset $\{(x, y, E) \in \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} : \text{for some } (c, E) \in \mathcal{D},\ \bar{c}_{s(i)} = x \text{ and } \bar{c} = y\}$ matters.[9] What values T_i takes outside this subset is not relevant to our result and in (7). In what follows we will say that T or T_i has a certain property, when it has the property only over this subset. Generalized proportional rules are members of this family: when f is a generalized proportional rule associated with (A, W), for each $(c, E) \in \mathcal{D}$ and each $i \in N$, let

$$T_i(c, E) \equiv \sum_{j \in s(i)} A_j(\bar{c}, E) + \sum_{k \in K} \frac{\bar{c}_{s(i)k}}{\bar{c}_k} W_k(\bar{c}, E) E.$$

Proposition 7. *Assume that G is a tree. Let f be a reallocation-proof rule represented by $T \colon \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ as in Proposition 6, where $s(\cdot)$ and $sm(\cdot)$ be defined on a directed tree $G(i^*)$ with root $i^* \in N$. Then f satisfies*
(i) *Efficiency if and only if $T_{i^*}(\bar{c}, \bar{c}, E) = E$ for each $(c, E) \in \mathcal{D}$.*
(ii-1) *Assume that $G(i^*)$ has a node $i \neq i^*$ with at least two immediate successors (that is, G is a non-linear tree). Then f satisfies no award for null if and only if $T_1 = \cdots = T_N \equiv T_0$ and for each $(c, E) \in \mathcal{D}$, $T_0(\cdot, \bar{c}, E)$ is additive.*
Hence, for each $(c, E) \in \mathcal{D}$, $T_0(0, \bar{c}, E) = 0$ and $T_0(\cdot, \bar{c}, E)$ can be decomposed into K functions as follows:

$$\begin{aligned} f_i(c, E) &= T_0(c_i, \bar{c}, E) \\ &= \sum_{k \in K} \hat{W}_k(c_{ik}, \bar{c}, E), \end{aligned}$$

[9] In particular, for $T_{i^*}(\cdot, \bar{c}, E)$, only one value $T_{i^*}(\bar{c}, \bar{c}, E)$ matters.

where $\hat{W}_k(c_{ik}, \bar{c}, E) \equiv T_0(c_{ik}\mathbf{u}_k, \bar{c}, E)$, *denoting the* k^{th} *unit vector of* $\mathbb{R}^K$ *by* $\mathbf{u}_k$, *and so* $\hat{W}_k(\cdot, \bar{c}, E)$ *is additive.*
(ii-2) *When* G *is a line,* f *satisfies no award for null if and only if for each* $(\bar{c}, E) \in \mathbb{R}^K_{++} \times \mathbb{R}_{++}$

$$T_1 = T_2 = \cdots = T_N \equiv T_0,$$
$$T_0(0, \bar{c}, E) = 0.$$

(iii) *Non-negativity if and only if for each* $i \in N$, *each* $x, y \in \mathbb{R}^K_+$, *each* $E \in \mathbb{R}_{++}$, *and each* $(a_j)_{j \in sm(i)} \in \mathbb{R}^{sm(i) \times K}_+$ *with* $0 \le \sum_{j \in sm(i)} a_j \le x \le y$,

$$T_i(x, y, E) \ge \sum_{j \in sm(i)} T_j(a_j, y, E).$$[10]

(iv) *No transfer paradox if and only if* $T_i(\cdot, \bar{c}, E)$ *is non-decreasing for each* $i \in N$ *and each* $(c, E) \in \mathcal{D}$.
Thus, if f *satisfies no award for null, then non-negativity is equivalent to no transfer paradox.*

Proof. **(i):** This follows from $s(i^*) = N$ and the fact that for each (c, E) and each $i \in N$, $T_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) = \sum_{j \in s(i)} f_j(c, E)$.

(ii-1): Let f satisfy *no award for null*. Then by (7), for each $i \in N$ and each $(c, E) \in \mathcal{D}$ with $c_i = 0$,

$$T_i\left(\sum_{j \in sm(i)} \bar{c}_{s(j)}, \bar{c}, E\right) = \sum_{j \in sm(i)} T_j\left(\bar{c}_{s(j)}, \bar{c}, E\right). \tag{8}$$

Thus for each $i \in N$, each $(x_j)_{j \in sm(i)} \in \mathbb{R}^{sm(i) \times K}_+$, and each $(y, E) \in \mathbb{R}^K_{++} \times \mathbb{R}_{++}$, if there is $(c, E) \in \mathcal{D}$ such that $c_i = 0$, $\bar{c}_{s(j)} = x_j$ for each $j \in sm(i)$, and $\bar{c} = y$, then

$$T_i\left(\bar{x}_{sm(i)}, y, E\right) = \sum_{j \in sm(i)} T_j(x_j, y, E). \tag{$\star$}$$

By *no award for null* and ($\star\star$) in the proof of Proposition 6, for each $i \in N$ and each $(c, E) \in \mathcal{D}$, if all successors of i have the zero characteristic vector, then they all receive nothing and so $\sum_{j \in s(i)} f_j(c, E) = 0$. Hence, for each $(y, E) \in \mathbb{R}^K_{++} \times \mathbb{R}^K_{++}$,

$$T_i(0, y, E) = 0. \tag{$\star\star$}$$

Let $i \in N$ and $j \in sm(i)$. Let $(c, E) \in \mathcal{D}$ be such that $c_i = 0$ and for each $h \in s(i) \backslash \{j\}$, $c_h = 0$. Then by (8) and ($\star\star$), $T_i(c_j, \bar{c}, E) = T_j(c_j, \bar{c}, E)$. Since

[10] Thus, $T_i(x, y, E) \ge 0$, if $sm(i) = \emptyset$.

this holds for each c_j with $0 \leq c_j \leq \bar{c}$, $T_i = T_j$. Using this and the tree structure of G, we show $T_1 = \cdots = T_N$. Let T_0 be the common function. For each $(c, E) \in \mathcal{D}$, if there is a node $i \in N \backslash \{i^*\}$ with at least two immediate successors, we obtain additivity of $T_0(\cdot, \bar{c}, E)$ from $(\star)$ (note that if $(\star)$ holds for $i = i^*$, then we can only obtain the limited additivity of $T_0(\cdot, \bar{c}, E)$ saying that for each $x, x' \in \mathbb{R}^K_+$, if $x + x' = \bar{c}$, $T_0(x, \bar{c}, E) + T_0(x', \bar{c}, E) = T_0(x + x', \bar{c}, E)$). Using additivity of $T_0(\cdot, \bar{c}, E)$ and (7) in Proposition 6, we show $f_i(c, E) = T_0(c_i, \bar{c}, E)$.

The converse follows easily from the fact that $T_0(0, \bar{c}, E) = 0$ and $f_i(c, E) = T_0(c_i, \bar{c}, E)$ for each $(c, E) \in \mathcal{D}$.

(ii-2): This is easily proven using part (ii)-1.

(iii): This part follows directly from (7).

(iv): Assume that f satisfies *no transfer paradox*. Let i be a terminal node, that is, $s^0(i) = \emptyset$. Then for each $(c, E) \in \mathcal{D}$, since $f_i(c, E) = T_i(c_i, \bar{c}, E)$, $T_i(\cdot, \bar{c}, E)$ is non-decreasing. Let j be such that for each $i \in s^0(j)$, $s^0(i) = \emptyset$. Then $f_j(c, E) = T_j\left(\bar{c}_{s(j)}, \bar{c}, E\right) - \sum_{i \in sm(j)} T_i(c_i, \bar{c}, E)$ and for each $i \in sm(j)$, $T_i(\cdot, \bar{c}, E)$ is non-decreasing. Consider transferring $t \in [0, c_i]$ from $h \in p^0(j)$ to j. Then by *no transfer paradox*, j's award should not decrease. Thus $T_j\left(\bar{c}_{s(j)} + t, \bar{c}, E\right) - \sum_{i \in sm(j)} T_i(c_i, \bar{c}, E) \geq T_j\left(\bar{c}_{s(j)}, \bar{c}, E\right) - \sum_{i \in sm(j)} T_i(c_i, \bar{c}, E)$. Hence, $T_j\left(\bar{c}_{s(j)} + t, \bar{c}, E\right) \geq T_j\left(\bar{c}_{s(j)}, \bar{c}, E\right)$. This shows that $T_j(\cdot, \bar{c}, E)$ is non-decreasing. Proceeding backward, we complete our proof. The converse is shown easily. ∎

Remark 4. When G is a non-linear tree, adding *no award for null*, we obtain a subfamily of rules that are characterized Proposition 1 and that have $A_i(\cdot) = 0$ for each $i \in N$. Thus, given *no award for null*, *reallocation-proofness* on a tree is equivalent to *reallocation-proofness* on a complete graph. Therefore, all earlier characterization results based on *reallocation-proofness* on a complete graph and *no award for null* continue to hold on a tree.

Lines

We obtain the following corollaries for lines.

Corollary 1. *Assume that G is a line. A rule f on a rich domain $\mathcal{D}$ satisfies reallocation-proofness if and only if f is represented by a function $T\colon \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ such that for each $(c, E) \in \mathcal{D}$ and each $i \in N$,*

$$f_i(c, E) = \begin{cases} T_i\left(\bar{c}_{s(i)}, \bar{c}, E\right), & \text{if } sm(i) = \emptyset; \\ T_i\left(\bar{c}_{s(i)}, \bar{c}, E\right) - T_{sm(i)}\left(\bar{c}_{s(sm(i))}, \bar{c}, E\right), & \text{if } sm(i) \neq \emptyset,^{11} \end{cases} \tag{9}$$

where, for an end node $i^* \in N$, $s(\cdot)$ and $sm(\cdot)$ are defined on the directed line $G(i^*)$.

Combining *reallocation-proofness*, *efficiency*, and *no award for null*, we obtain:

Corollary 2. *Assume that G is a line. A rule f on a rich domain $\mathcal{D}$ satisfies* reallocation-proofness, efficiency, *and* no award for null *if and only if f is represented by a function $T_0 \colon \mathbb{R}^K_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ such that for each $(c, E) \in \mathcal{D}$ and each $i \in N$, $T_0(0, \bar{c}, E) = 0$, $T_0(\bar{c}, \bar{c}, E) = E$, and*

$$f_i(c, E) = \begin{cases} T_0(c_i, \bar{c}, E), & \text{if } sm(i) = \emptyset, \\ T_0\left(\bar{c}_{s(i)}, \bar{c}, E\right) - T_0\left(\bar{c}_{s(sm(i))}, \bar{c}, E\right), & \text{if } sm(i) \neq \emptyset, \end{cases}$$

where, for an end node $i^* \in N$, $s(\cdot)$ and $sm(\cdot)$ are defined on the directed line $G(i^*)$.

Proposition 7 (parts 2.1 and 2.2) shows that when *no award for null* is imposed, there is a remarkable difference between the linear tree case and the non-linear tree case. As shown in Corollary 2, in the case of linear tree, there are rules that are not necessarily a member of the family of rules characterized in Proposition 1 but that satisfy *reallocation-proofness* and *no award for null*. When G is a non-linear tree, only those rules characterized in Proposition 1 satisfy the two axioms.

4 Theorem

We now consider the most general case when G is a connected graph.

The next lemma says that every connected graph is uniquely decomposed into a family of maximal multi-edge-connected subgraphs.

Lemma 5. *Assume that $G \equiv (N, D)$ is a connected graph.*
(i) *The set of nodes N is uniquely partitioned into a finite number of subsets $N_1, \cdots, N_L$ such that for each $l = 1, \cdots, L$, $|N_l| = 1$ or $|N_l| \geq 3$ and G_{N_l} is a maximal multi-edge-connected subgraph on G.*
(ii) *There is no cycle of sets among $N_1, \cdots, N_L$, which are successively connected by* bridges; *that is, there is no $r \geq 3$ and no $N_{l_1}, \cdots, N_{l_r} \in \{N_1, \cdots, N_L\}$ such that $N_{l_1} = N_{l_r}$ and for two sequences of nodes, $i_1 \in N_{l_1}, \cdots, i_{r-1} \in N_{r-1}$ and $j_2 \in N_{l_2}, \cdots, j_r \in N_r$, we have $i_1 j_2, i_2 j_3, \cdots, i_{r-1} j_r \in D$.*

The proof is left for readers [`see Omitted Proofs, Section C.1`].

By Lemma 5, N is partitioned into maximal multi-edge-connected subgraphs and these subgraphs are located with a tree structure. Formally:

Definition 3 (Tree of Maximal Multi-Edge-Connected Subgraphs). Given a connected graph $G \equiv (N, D)$, let N be partitioned into $N_1, \cdots, N_L$ such that for each $l = 1, \cdots, L$, G_{N_l} is a maximal multi-edge-connected subgraph. We now define a graph $\mathcal{G}$ of which nodes are composed of these subgraphs. Formally, let $\mathcal{N} \equiv \{N_1, \cdots, N_L\}$ be the set of nodes. For each $l, l' \in \{1, \cdots, L\}$, $\{N_l, N_{l'}\}$ is an edge of $\mathcal{G}$ if there is an edge of the original graph G, which connects N_l and $N_{l'}$, that is, for some $i \in N_l$ and $i' \in N_{l'}$, $ii' \in D$. Denote the set of edges of $\mathcal{G}$ by $\mathcal{E}$. Then $\mathcal{G} \equiv (\mathcal{N}, \mathcal{E})$ is a tree because of part (ii) of Lemma 5.

Let $\mathcal{R} \equiv \{G_{N_1}, \cdots, G_{N_L}\}$ be the set of maximal multi-edge-connected subgraphs on G. By Lemma 4, for each $l = 1, \cdots, L$, N_l is again divided into a finite number $M_l \in \mathbb{N}$ of subsets, denoted by $N_{l1}, \cdots, N_{lM_l}$, such that for each $m = 1, \cdots, M_l$, $G_{N_{lm}}$ is a maximal multi-node-connected subgraph on G_{N_l}.

Next we define a family of rules of which representations have the mixed feature of both rules in Proposition 4 and Proposition 6. We use the following notation. Let $N_{l^*} \in \mathcal{N}$. Let $\mathcal{G}(N_{l^*})$ be the directed tree with root N_{l^*}. We use the same notation as in Section 3.3 for the set of successors $s(\cdot)$, the set of immediate successors $sm(\cdot)$, the set of predecessors $p(\cdot)$, and immediate predecessor $pm(\cdot)$ on $\mathcal{G}(N_{l^*})$. We also use notation $s^0(\cdot)$ and $p^0(\cdot)$ as used in Section 3.3. For each $l \in \{1, \cdots, L\}$, let $\cup s(N_l)$ be the union of all $N_{l'} \in \mathcal{N}$ that succeeds N_l on $\mathcal{G}(N_{l^*})$, that is,

$$\cup s(N_l) \equiv \bigcup_{N_{l'} \in s(N_l)} N_{l'}.$$

Similarly, let

$$\cup s^o(N_l) \equiv \bigcup_{N_{l'} \in s^o(N_l)} N_{l'}.$$

For each $l \in \{1, \cdots, L\}$ and each $m \in \{1, \cdots, M_l\}$, let

$$\begin{aligned} C(N_l) &\equiv \{j \in N_l : j \text{ is a cutnode on } G\}; \\ C(N_{lm}, G_{N_l}) &\equiv \{j \in N_{lm} : j \text{ is a cutnode on } G_{N_l}\}. \end{aligned}$$

Then $C(N_{lm}, G_{N_l}) \subseteq C(N_l)$ but the reverse inclusion does not hold. For example, in Figure 2, $j \in C(N_l)$ but $j \notin C(N_{lm}, G_{N_l})$, and $i \in C(N_l) \cap C(N_{lm}, G_{N_l})$. Let $C^*(N_l)$ be the set of all cutnodes $i \in N_l$ on G, which belongs to a bridge connecting N_l to an immediate successor of N_l, that is,

$$C^*(N_l) \equiv \{i \in C(N_l) : \text{for some } N_{l'} \in sm(N_l) \text{ and some } j \in N_{l'},\ ij \in D\}.$$

Let $C^* \equiv \cup_{l=1}^{L} C^*(N_l)$. Let $D^*(N_l)$ be the set of all cutnodes $i \in N_l$ on G, which belongs to a bridge connecting N_l to an immediate predecessor of N_l, that is,

$$D^*(N_l) \equiv \{i \in C(N_l) : \text{for some } j \in pm(N_l),\ ij \in D\}.$$

Let $D^* \equiv \cup_{l=1}^{L} D^*(N_l)$. For example, in Figure 2, $i \in C^*(N_l)$ and $j \in D^*(N_l)$.

For each $l \in \{1, \cdots, L\}$ and each $i \in N_l$, let $sm(N_l; i)$ be the set of immediate successors of N_l "originating from i", that is,

$$sm(N_l; i) \equiv \{N_{l'} \in sm(N_l) : \text{for some } j \in N_{l'},\ ij \in D\}.$$

Let $s(N_l; i)$ be the set of successors of N_l originating from i, that is,

$$s(N_l; i) \equiv \{N_l\} \cup \{N_{l''} : \text{for some } N_{l'} \in sm(N_l; i),\ N_{l''} \in s(N_{l'})\}.$$

Let $s^0(N_l; i)$ be the set of strict successors of N_l originating from i, that is,

$$s^0(N_l; i) \equiv \{N_{l''} : \text{for some } N_{l'} \in sm(N_l; i),\ N_{l''} \in s(N_{l'})\}.$$

Note that if $i \notin C^*(N_l)$, $sm(N_l; i) = s^0(N_l; i) = \emptyset$. Let $\cup s^0(N_l; i)$ be the union of all sets in $s^0(N_l; i)$. For each $i \in N_{lm}$, let

$$S(i, N_{lm}) \equiv \{j \in N_l \backslash [N_{lm} \backslash \{i\}] : i \text{ is between } j \text{ and each node in } N_{lm} \text{ on } G_{N_l}\}.$$

Let $\sigma(i, N_{lm})$ be the set of all agents "succeeding i and N_{lm}", that is,

$$\sigma(i, N_{lm}) \equiv \cup_{j \in S(i, N_{lm})} \cup s^0(N_l; j) \cup \{j\}.$$

See Figure 2 for an illustration of $\sigma(\cdot)$. It should be noted that $S(i, N_{lm})$ is defined on the subgraph G_{N_l} and $i \in S(i, N_{lm})$, and that $S(i, N_{lm})$ is not a singleton if and only if $i \in C(N_{lm}, G_{N_l})$ (that is, $S(i, N_{lm}) = \{i\}$ if and only if $i \notin C(N_{lm}, G_{N_l})$). Also when $i \notin C^*(N_l) \cup C(N_{lm}, G_{N_l})$, $\sigma(i, N_{lm}) = \{i\}$.

The family of rules to be defined next are represented by three lists of functions. First is the list of functions $T \equiv (T_l \colon \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R})_{l=1,\ldots,L}$ determining the total award of all agents in the successors of each $N_l \in \mathcal{N}$, $\cup s(N_l)$; more precisely, the total award of all agents in $\cup s(N_l)$ is given by $T_l(\cdot)$ as a function of the sum of characteristic vectors of these agents, $\bar{c}$, and E. Second and third are the following list of functions

$$\left(\left(A^{lm} \colon \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^{N_{lm}}, \hat{W}^{lm} \colon \mathbb{R}_+ \times \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^K\right)_{m=1}^{M_l}\right)_{l=1}^{L},$$

determining the total award of each agent $i \in N_{lm}$ and agents succeeding i and N_{lm}; more precisely, for each $(c, E) \in \mathcal{D}$, each $l \in \{1, \ldots, L\}$, each $m \in \{1, \ldots, M_l\}$, and each $i \in N_{lm}$, the total award of all agents in $\sigma(i, N_{lm})$ is given by $A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(\bar{c}_{\sigma(i,N_{lm})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right)$. Thus to obtain a formula describing i's award, we need to subtract from this amount the total awards

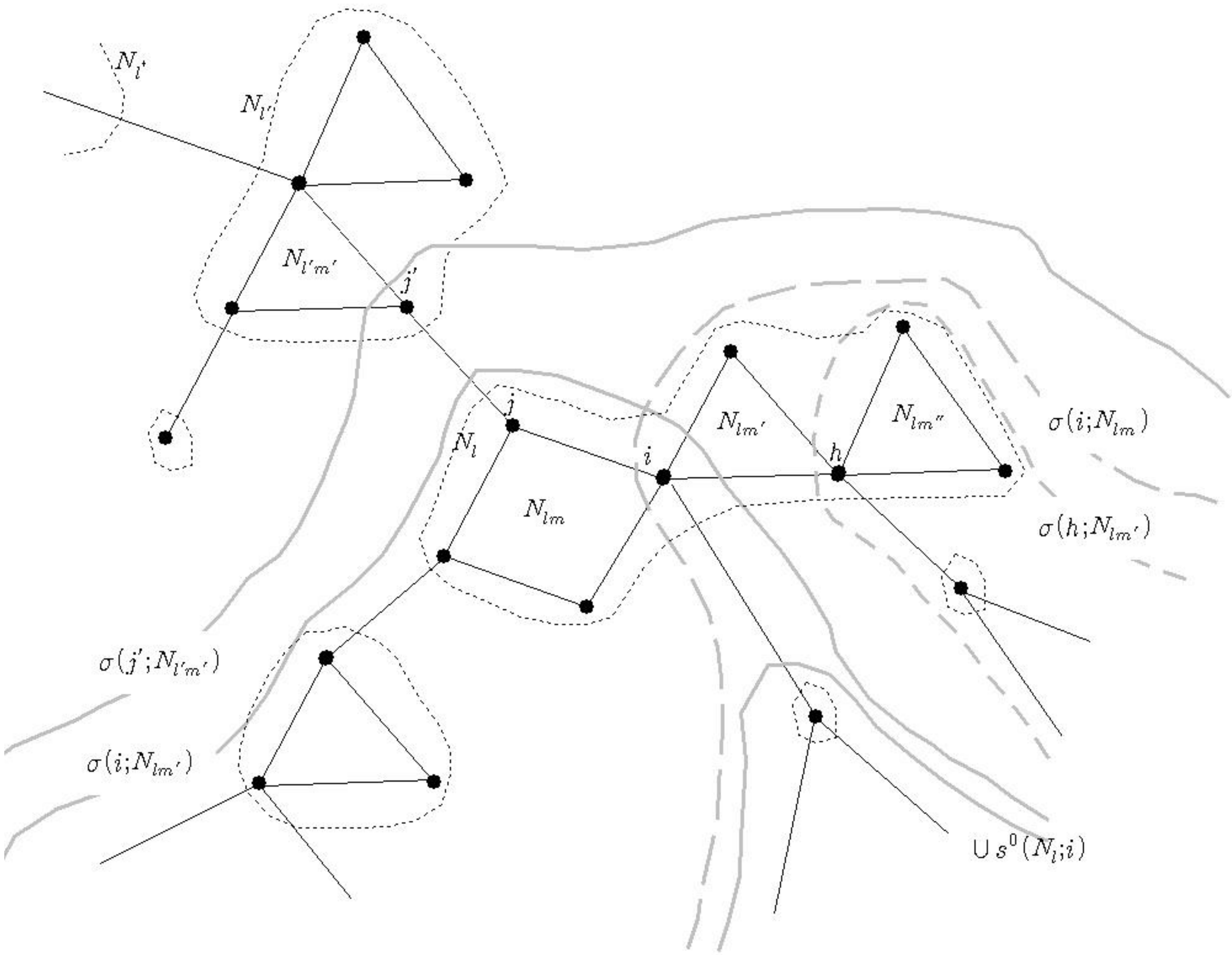

Figure 2: The tree of maximal multi-edge-connected subgraphs of a connected graph. Maximal multi-edge-connected subgraphs are indicated by black dotted circles such as N_l and $N_{l'}$. Note that the maximal multi-edge-connected subgraph N_l is composed of three maximal multi-node-connected sugraphs N_{lm}, $N_{lm'}$, and $N_{lm''}$.

all agents succeeding $j \in N_l \backslash \{i\}$ and the total award of all agents in successors of N_l originating from i, as described by (10) below.

The following two conditions are required for *reallocation-proofness*.

AD (additivity): For each $l \in \{1, \cdots, L\}$, each $m \in \{1, \cdots, M_l\}$, and each $(c, E) \in \mathcal{D}$, $\hat{W}^{lm}\left(\cdot, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right)$ is additive.

CONS (constancy): For each $l \in \{1, \cdots, L\}$, each $m \in \{1, \cdots, M_l\}$, and each $i \in N_{lm}$, if there is $j \in D^* \cap N_l$ "preceding i and N_{lm}", that is, $j \notin S(i, N_{lm})$ (see Figure 2), then for each $(c, E) \in \mathcal{D}$, $A_i^{lm}(\cdot, \bar{c}, E)$ is constant and for each $k \in K$ and each $\alpha \in \mathbb{R}_+$, $\hat{W}_k^{lm}(\alpha, \cdot, \bar{c}, E)$ is constant.

For each $l \in \{1, \ldots, L\}$ and each $i \in N_l$, let

$$\mathfrak{m}(i) \equiv \{m \in \{1, \ldots, M_l\} : i \in N_{lm}\}$$

be the set of the second indices of all multi-node-connected subgraphs to which i belongs.

Definition 4 (TAW-family). A rule f is in the *TAW-family* if f is represented by a list of functions,

$$T \colon \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^L;$$
$$((A^{lm} \colon \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^{N_{lm}}, \hat{W}^{lm} \colon \mathbb{R}_+ \times \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^K)_{m=1}^{M_l})_{l=1}^{L},$$

satisfying AD and CONS, as follows: for each $(c, E) \in \mathcal{D}$, each $l \in \{1, \cdots, L\}$, each $m \in \{1, \cdots, M_l\}$, and each $i \in N_{lm}$,

$$f_i(c, E) = \begin{cases} A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(\bar{c}_{\sigma(i, N_{lm})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\ - \sum_{m' \in \mathfrak{m}(i) \backslash \{m\}} \sum_{j \in N_{lm'} \backslash \{i\}} A_j^{lm'}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\ - \sum_{m' \in \mathfrak{m}(i) \backslash \{m\}} \sum_{k \in K} \hat{W}_k^{lm'}\left(\sum_{j \in N_{lm'} \backslash \{i\}} \bar{c}_{\sigma(j, N_{lm'})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\ - \sum_{l' : N_{l'} \in sm(N_l; i)} T_{l'}\left(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right), \end{cases} \tag{10}$$

where for each $l \in \{1, \cdots, L\}$, each $m \in \{1, \cdots, M_l\}$, and each $(c, E) \in \mathcal{D}$,

$$\sum_{i \in N_{lm}} A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(\bar{c}_{\cup s(N_l)k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) = T_l\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \tag{11}$$

and, $s(\cdot)$, $sm(\cdot)$, $C^*(\cdot)$, and $\sigma(\cdot)$ are defined on the directed graph $\mathcal{G}(N_{l^*})$.[12]

[12]Condition (11) implies that for each $m, m' \in \{1, \ldots, L_l\}$,

$$\begin{aligned} & \sum_{i \in N_{lm}} A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(\bar{c}_{\cup s(N_l)k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\ = & \sum_{i \in N_{lm'}} A_i^{lm'}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm'}\left(\bar{c}_{\cup s(N_l)k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right), \end{aligned}$$

The first line in the right-hand side of (10) describes the total award of all agents in $\sigma(i, N_{lm})$ (see Figure 2). The second and third lines account for the total awards all agents succeeding each $j \in N_l \backslash \{i\}$. The fourth line accounts for the total award of all agents in successors of N_l originating from i. Depending on how each agent i is located on the graph, (10) may reduce to a simpler formula. If $i \in N_l \backslash (C^*(N_l) \cup C(N_l, G_{N_l}))$, $\mathfrak{m}(i) \backslash \{m\} = \emptyset$ and $sm(N_l; i) = \emptyset$. Thus (10) reduces to

$$f_i(c, E) = A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}(c_{ik}, \bar{c}_{\cup s(N_l)}, \bar{c}, E), \tag{12}$$

If $i \in C(N_l, G_{N_l}) \backslash C^*(N_l)$, $sm(N_l; i) = \emptyset$. Thus (10) reduces to

$$f_i(c, E) = \begin{cases} A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(\bar{c}_{\sigma(i, N_{lm})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\ - \sum_{m' \in \mathfrak{m}(i) \backslash \{m\}} \sum_{j \in N_{lm'} \backslash \{i\}} A_j^{lm'}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\ - \sum_{m' \in \mathfrak{m}(i) \backslash \{m\}} \sum_{k \in K} \hat{W}_k^{lm'}\left(\sum_{j \in N_{lm'} \backslash \{i\}} \bar{c}_{\sigma(j, N_{lm'})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right). \end{cases} \tag{13}$$

If $i \in C^*(N_l) \cap (N_l \backslash C(N_l, G_{N_l}))$, $\mathfrak{m}(i) \backslash \{m\} = \emptyset$. Thus (10) reduces to

$$f_i(c, E) = \begin{cases} A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}(\bar{c}_{\sigma(i, N_{lm}), k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E) \\ - \sum_{l' : N_{l'} \in sm(N_l; i)} T_{l'}\left(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right). \end{cases} \tag{14}$$

Now we are ready to state our main result.

Theorem. *Given a connected graph, a rule on a rich domain satisfies* reallocation-proofness *if and only if it is a member of TAW-family.*

The proof is in Appendix B.

We next establish necessary and sufficient conditions for *efficiency*, *no award for nulls*, *non-negativity*, and *no transfer paradox*.

Proposition 8. *Given a connected graph, let f be a reallocation-proof rule represented by the following functions*

$T \colon \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^L$;
$((A^{lm} \colon \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^{N_{lm}}, \hat{W}^{lm} \colon \mathbb{R}_+ \times \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^K)_{m=1}^{M_l})_{l=1}^L$

as in Definition 4. Then f satisfies
(i) *Efficiency if and only if for each $(c, E) \in \mathcal{D}$, $T_{l^*}(\bar{c}, \bar{c}, E) = E$;*

(ii) *Assume that* $L \geq 2$ *and there is* $l \in \{1, \ldots, L\}$ *such that* $|N_l| \geq 3$ *(otherwise, Propositions 5 or 7 apply). Then* f *satisfies* no award for nulls *if and only if for each* $(c, E) \in \mathcal{D}$ *and each* $i \in N$,

$$\begin{aligned} T_0(\cdot) &\equiv T_1(\cdot) = \cdots = T_L(\cdot), \\ f_i(c, E) &= T_0(c_i, \bar{c}, E), \end{aligned}$$

and $T_0(\cdot, \bar{c}, E)$ *is additive (so* $T_0(0, \bar{c}, E) = 0$*). Thus, for each* $(c, E) \in \mathcal{D}$, $T_0(\cdot, \bar{c}, E)$ *can be decomposed into* K *functions and we have*

$$\begin{aligned} f_i(c, E) &= T_0(c_i, \bar{c}, E) \\ &= \sum_{k \in K} \hat{W}_k(c_{ik}, \bar{c}, E), \end{aligned}$$

where $\hat{W}_k(c_{ik}, \bar{c}, E) \equiv T_0(c_{ik}\mathbf{u}_k, \bar{c}, E)$*, denoting the* k^{th} *unit vector of* $\mathbb{R}^K$ *by* $\mathbf{u}_k$*, and for each* $l \in \{1, \ldots, L\}$ *and each* $m \in \{1, \ldots, M_l\}$*,*

$$\hat{W}^{lm}\left(\cdot, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) = \hat{W}(\cdot, \bar{c}, E).$$

(iii) *Non-negativity if and only if* f *satisfies* one-sided boundedness *and, for each* $(c, E) \in \mathcal{D}$*, each* $l \in \{1, \ldots, L\}$*, each* $m \in \{1, \ldots, M_l\}$*, and each* $i \in N_{lm}$*,*

$$\begin{aligned} & A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \geq 0, \\ & \min_{j \in N_{lm}} A_j^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \min\{0, \hat{W}_k^{lm}(\bar{c}_{\cup s(N_l)k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E)\} \geq 0, \\ & A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(\bar{c}_{S(i, N_{lm})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \geq \\ & \sum_{m' \in \mathfrak{m}(i) \setminus \{m\}} \sum_{j \in N_{lm'} \setminus \{i\}} A_j^{lm'}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{m' \in \mathfrak{m}(i) \setminus \{m\}} \sum_{k \in K} \hat{W}_k^{lm'}\left(\sum_{j \in N_{lm'} \setminus \{i\}} \bar{c}_{S(j, N_{lm'})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right), \end{aligned}$$

for each $x, y \in \mathbb{R}_+^K$ *and each* $\left(a_{N_{l'}}\right)_{N_{l'} \in sm(N_l)} \in \mathbb{R}_+^{sm(N_l) \times K}$ *with* $0 \leq \sum_{N_{l'} \in sm(N_l)} a_{N_{l'}} \leq x \leq y$*,*

$$T_l(x, y, E) \geq \sum_{N_{l'} \in sm(N_l)} T_{l'}(a_{N_{l'}}, y, E).$$

(iv) *No transfer paradox if and only if for each* $(c, E) \in \mathcal{D}$*, each* $l \in \{1, \ldots, L\}$*, each* $m \in \{1, \ldots, M_l\}$*, and each* $k \in K$*,* $\hat{W}_k^{lm}\left(\cdot, \bar{c}_{s(N_l)}, \bar{c}, E\right)$ *and* $T_l(\cdot, \bar{c}, E)$ *are non-decreasing.*

The proof is in Appendix B.

Remark 5. Combining the necessary and sufficient conditions for *no award for nulls* in Propositions 5, 7, and 8, we obtain the following relations: if G is not a line, then for each rule f satisfying *no award for nulls*, f satisfies *reallocation-proofness* under $\mathcal{C}(G)$ if and only if f satisfies *reallocation-proofness* under the unrestricted coalition structure.

By Lemma 2, we may replace *reallocation-proofness* in Theorem 4 with the combination of *pairwise reallocation-proofness* and *pairwise non-bossiness.*

Corollary 3. *Assume that* G *is a connected graph. Then a rule on a rich domain satisfies* pairwise reallocation-proofness *and* pairwise non-bossiness *if and only if it is a member of the TAW-family.*

It follows from Theorem 4 and Propositions 1-4 that:

Corollary 4. *Assume that* G *is a connected graph. Then the following two statements are equivalent*:
(i) *Graph* G *is multi-node-connected*;
(ii) Reallocation-proofness *under* $\mathcal{C}(G)$ *is equivalent to* reallocation-proofness *under the unrestricted coalition structure.*

A Proofs of Propositions 4 and 5

In this section, we prove Propositions 4 and 5.

Proof of Proposition 4. Let $G \equiv (N, D)$, $N_1, \cdots, N_M$, and $G_{N_1}, \cdots, G_{N_M}$ be given as in the proposition. It will follow from our proof of Theorem in Appendix B that every rule with the stated representation is *reallocation-proof.* To prove the converse, let f be a rule satisfying *reallocation-proofness*. Then by Lemma 2, f satisfies *non-bossiness*. Let $m \in \{1, \ldots, M\}$. Consider N_m and multi-node-connected subgraph G_{N_m}. Let $\mathcal{D}_{N_m} \equiv \{(d, E) \in \mathbb{R}_+^{N_m \times K} \times \mathbb{R}_{++}$: for some $(c, E) \in \mathcal{D}$, $c_{N_m \setminus C(N_m)} = d_{N_m \setminus C(N_m)}$ and for each $i \in C(N_m)$, $\bar{c}_{S(i,N_m)} = d_i\}$. Let $g \colon \mathcal{D}_{N_m} \to \mathbb{R}^{N_m}$ be defined as follows: for each $(d, E) \in \mathcal{D}_{N_m}$,

$$g_i(d, E) \equiv \sum_{j \in S(i,N_m)} f_j(c, E),$$

where $(c, E) \in \mathcal{D}$ is such that $c_{N_m \setminus C(N_m)} = d_{N_m \setminus C(N_m)}$ and for each $i \in C(N_m)$, $\bar{c}_{S(i,N_m)} = d_i$. To show that g is well-defined, let c, c' be such that $c_{N_m \setminus C(N_m)} = c'_{N_m \setminus C(N_m)} = d_{N_m \setminus C(N_m)}$ and for each $i \in C(N_m)$, $\bar{c}_{S(i,N_m)} = \bar{c}'_{S(i,N_m)} = d_i$. For each $i \in N_m$, if coalition $S(i, N_m)$ changes $c_{S(i,N_m)}$ to $c'_{S(i,N_m)}$, then since $S(i, N_m)$

is connected, by *reallocation-proofness* and *non-bossiness*, the total award of $S(i, N_m)$ remains constant and the awards of all others also remain constant. After making these changes for all agents in N_m, we finally get c'. And for each $i \in N_m$,

$$\sum_{j \in S(i,N_m)} f_j(c, E) = \sum_{j \in S(i,N_m)} f_j(c', E).$$

This shows that g is well-defined.

We now show that g is a rule over $\mathcal{D}_{N_m}$ satisfying *pairwise reallocation-proofness* and *pairwise non-bossiness* under $\mathcal{C}(G_{N_m})$ and, therefore, satisfying *reallocation-proofness* under $\mathcal{C}(G_{N_m})$. Let $i^*, j^* \in N_m \backslash C(N_m)$ be such that $i^* j^* \in D_{N_m}$. Then it follows from *pairwise reallocation-proofness* and *pairwise non-bossiness* of f and the definition of g that this pair $\{i^*, j^*\}$ cannot change their total award or awards of others by any reallocation of characteristic vectors among the pair. Now consider a pair $\{i^*, j^*\}$ that is an edge in D_{N_m} and $i^* \in C(N_m)$. Let $(d, E), (d', E) \in \mathcal{D}_{N_m}$ be such that $d_{N_m \backslash \{i^*, j^*\}} = d'_{N_m \backslash \{i^*, j^*\}}$ and $d_{i^*} + d_{j^*} = d'_{i^*} + d'_{j^*}$. Let $c \in \mathcal{D}$ be such that $c_{N_m \backslash C(N_m)} = d_{N_m \backslash C(N_m)}$ and for each $i \in C(N_m)$, $\bar{c}_{S(i,N_m)} = d_i$. Without loss of generality, suppose $j^* \notin C(N_m)$ (a similar argument applies when $j^* \in C(N_m)$). Let c' be such that $\bar{c}'_{S(i^*,N_m)} = d'_{i^*}$ and $c'_{j^*} = d'_{j^*}$ and for each $i \notin S(i^*, N_m) \cup \{j^*\}$, $c'_i = c_i$. Then $\bar{c}'_{S(i^*,N_m)} + c'_{j^*} = \bar{c}_{S(i^*,N_m)} + c_{j^*}$ and $c'_{N \backslash (S(i^*,N_m) \cup \{j^*\})} = c_{N \backslash (S(i^*,N_m) \cup \{j^*\})}$. Since $i^* j^*$ is an edge, $S(i^*, N_m) \cup \{j^*\}$ is connected. Thus by *reallocation-proofness* and *non-bossiness* of f,

$$\begin{aligned} \sum_{i \in S(i^*,N_m) \cup \{j^*\}} f_i(c', E) &= \sum_{i \in S(i^*,N_m) \cup \{j^*\}} f_i(c, E); \\ f_{N \backslash (S(i^*,N_m) \cup \{j^*\})}(c', E) &= f_{N \backslash (S(i^*,N_m) \cup \{j^*\})}(c, E). \end{aligned}$$

Therefore,

$$\begin{aligned} g_{i^*}(d', E) + g_{j^*}(d', E) &= g_{i^*}(d, E) + g_{j^*}(d, E); \\ g_{N \backslash \{i^*, j^*\}}(c', E) &= g_{N \backslash \{i^*, j^*\}}(c, E). \end{aligned}$$

This shows that g satisfies *pairwise reallocation-proofness* and *pairwise non-bossiness* under $\mathcal{C}(G_{N_m})$.

Since G_{N_m} is multi-node-connected and $|N_m| \geq 3$, then applying Proposition 1, we conclude that there exist $A^m \colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{N_m}$ and $\hat{W}^m \colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that for each $(d, E) \in \mathcal{D}_{N_m}$ and each $i \in N_m$,

$$g_i(d, E) = A^m_i(\bar{d}, E) + \sum_{k \in K} \hat{W}^m_k(d_{ik}, \bar{d}, E),$$

and $\hat{W}^m\left(\cdot,\bar{d},E\right)$ is additive. Therefore, for each $(c,E)\in\mathcal{D}$,

$$\sum_{j\in S(i,N_m)} f_j\left(c,E\right)=A_i^m\left(\bar{c},E\right)+\sum_{k\in K}\hat{W}_k^m\left(\bar{c}_{S(i,N_m)k},\bar{c},E\right), \tag{$\dagger$}$$

and $\hat{W}^m\left(\cdot,\bar{c},E\right)$ is additive.[13] Thus for each $i\in N_m$,

$$f_i\left(c,E\right)=A_i^m\left(\bar{c},E\right)+\sum_{k\in K}\hat{W}_k^m\left(\bar{c}_{S(i,N_m)k},\bar{c},E\right)-\sum_{j\in S(i,N_l)\backslash\{i\}} f_j\left(c,E\right).$$

Since $S\left(i,N_m\right)\backslash\{i\}=\cup_{m'\in\mathfrak{m}(i)\backslash\{m\}}\cup_{j\in N_{m'}\backslash\{i\}}S\left(j,N_{m'}\right)$ (see Figure 1),

$$\sum_{j\in S(i,N_l)\backslash\{i\}} f_j\left(c,E\right)=\sum_{m'\in\mathfrak{m}(i)\backslash\{m\}}\sum_{j\in N_{m'}\backslash\{i\}}\sum_{h\in S(j,N_{m'})} f_h\left(c,E\right).$$

Using ($\dagger$) for each $m'\in\mathfrak{m}\left(i\right)\backslash\{m\}$ and each $j\in N_{m'}\backslash\{i\}$,

$$f_i\left(c,E\right)=\begin{cases} A_i^m\left(\bar{c},E\right)+\sum\limits_{k\in K}\hat{W}_k^m\left(\bar{c}_{S(i,N_m)k},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i)\backslash\{m\}}\sum\limits_{j\in N_{m'}\backslash\{i\}}\left(A_j^{m'}\left(\bar{c},E\right)+\sum\limits_{k\in K}\hat{W}_k^{m'}\left(\bar{c}_{S(j,N_{m'})k},\bar{c},E\right)\right).\end{cases}$$

Finally, using additivity of $\hat{W}^{m'}\left(\cdot,\bar{c},E\right)$,

$$f_i\left(c,E\right)=\begin{cases} A_i^m\left(\bar{c},E\right)+\sum\limits_{k\in K}\hat{W}_k^m\left(\bar{c}_{S(i,N_m)k},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i)\backslash\{m\}}\sum\limits_{j\in N_{m'}\backslash\{i\}}A_j^{m'}\left(\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i)\backslash\{m\}}\sum\limits_{k\in K}\hat{W}_k^{m'}\left(\sum\limits_{j\in N_{m'}\backslash\{i\}}\bar{c}_{S(j,N_{m'})k},\bar{c},E\right).\end{cases}$$

Applying the same argument for each $m'\in\mathfrak{m}\left(i\right)\backslash\{m'\}$,

$$f_i\left(c,E\right)=\begin{cases} A_i^{m'}\left(\bar{c},E\right)+\sum\limits_{k\in K}\hat{W}_k^{m'}\left(\bar{c}_{S(i,N_{m'})k},\bar{c},E\right) \\ -\sum\limits_{m''\in\mathfrak{m}(i)\backslash\{m'\}}\sum\limits_{j\in N_{m''}\backslash\{i\}}A_j^{m''}\left(\bar{c},E\right) \\ -\sum\limits_{m''\in\mathfrak{m}(i)\backslash\{m'\}}\sum\limits_{k\in K}\hat{W}_k^{m''}\left(\sum\limits_{j\in N_{m''}\backslash\{i\}}\bar{c}_{S(j,N_{m''})k},\bar{c},E\right).\end{cases}$$

Equating the two expressions for $f_i\left(c,E\right)$ and using additivity of $\hat{W}^m\left(\cdot,\bar{c},E\right)$ and $\hat{W}^{m'}\left(\cdot,\bar{c},E\right)$,

$$\begin{aligned} & A_i^m\left(\bar{c},E\right)+\sum_{j\in N_m\backslash\{i\}}A_j^m\left(\bar{c},E\right)+\sum_{k\in K}\hat{W}_k^m\left(\bar{c}_{S(i,N_m)k}+\sum_{j\in N_m\backslash\{i\}}\bar{c}_{S(j,N_m)k},\bar{c},E\right) \\ = \; & A_i^{m'}\left(\bar{c},E\right)+\sum_{j\in N_m\backslash\{i\}}A_j^{m'}\left(\bar{c},E\right)+\sum_{k\in K}\hat{W}_k^{m'}\left(\bar{c}_{S(i,N_{m'})k}+\sum_{j\in N_{m'}\backslash\{i\}}\bar{c}_{S(j,N_{m'})k},\bar{c},E\right).\end{aligned}$$

[13]Note that when $i\in N_m\backslash C(N_m)$, $f_i\left(c,E\right)=A_i^m(\bar{c},E)+\sum_{k\in K}\hat{W}_k^m(c_{ik},\bar{c},E)$.

Since $S(i, N_m) \cup \left[\cup_{j \in N_m \setminus \{i\}} S(j, N_m)\right] = S(i, N_{m'}) \cup \left[\cup_{j \in N_{m'} \setminus \{i\}} S(j, N_{m'})\right] = N$, we obtain (6). ∎

Proof of Proposition 5. Note that for each $m \in \{1, \ldots, M\}$, $g\colon \mathcal{D}_{N_m} \to \mathbb{R}^{N_m}$ defined in the proof of Proposition 4 inherits any of the additional four properties of f, namely, *efficiency*, *no award for nulls*, *non-negativity*, and *no transfer paradox*. Therefore, applying Proposition 2 for the two functions $A^m\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^{N_m}$ and $\hat{W}^m\colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ that represent g, we easily obtain all the conditions stated in Proposition 5 except for the second condition for *no award for nulls*. This condition is verified below.

Let $m, m' \in \{1, \ldots, M\}$ be such that $N_m \cap N_{m'} \neq \emptyset$ (if there is no such pair, then $M = 1$ and we are done). Without loss of generality, let $1 \in N_m \cap N_{m'}$. Then 1 is a cutnode. Since $N_{m'}$ is multi-node-connected, then there is $i \in N_{m'} \setminus \{1\}$. Let $(y, E) \in \mathbb{R}^K_{++} \times \mathbb{R}_{++}$. Let $(c, E) \in \mathcal{D}$ be such that $\bar{c} = y$ and for each $j \in S(1, N_m) \setminus \{i\}$, $c_j = 0$ (thus $c_1 = 0$ and except for i, all nodes that succeed 1 have the zero characteristic vector). Then $\bar{c}_{S(1,N_m)} = c_i = \bar{c}_{S(i,N_{m'})}$. By *no award for nulls*, $f_1(c, E) = 0$, and for each $j \in S(1, N_m) \setminus \{i\}$, $f_j(c, E) = 0$. Then, applying (‡) in the proof of Proposition 4 twice with regard to N_m and $N_{m'}$, we obtain

$$\begin{aligned} f_i(c, E) &= \sum_{k \in K} \hat{W}^m_k(c_{ik}, y, E) \\ &= \sum_{k \in K} \hat{W}^{m'}_k(c_{ik}, y, E). \end{aligned}$$

Since this holds for each c with $\bar{c} = y$, then for each $k \in K$, letting $c_{ik'} = 0$ for each $k' \neq k$ and using the fact that by additivity, $\hat{W}^m_{k'}(0, y, E) = 0$, we obtain:

$$\hat{W}^m_k(c_{ik}, y, E) = \hat{W}^{m'}_k(c_{ik}, y, E).$$

This shows the second condition for *no award for nulls*. ∎

B Proofs of Theorem and Proposition 8

Proof of Theorem. The proof is composed of two steps corresponding to "if" part and "only if" part of the theorem.

Step 1. Every rule in TAW-family is reallocation-proof.

Let $\{i, j\} \in D$. There are two cases.

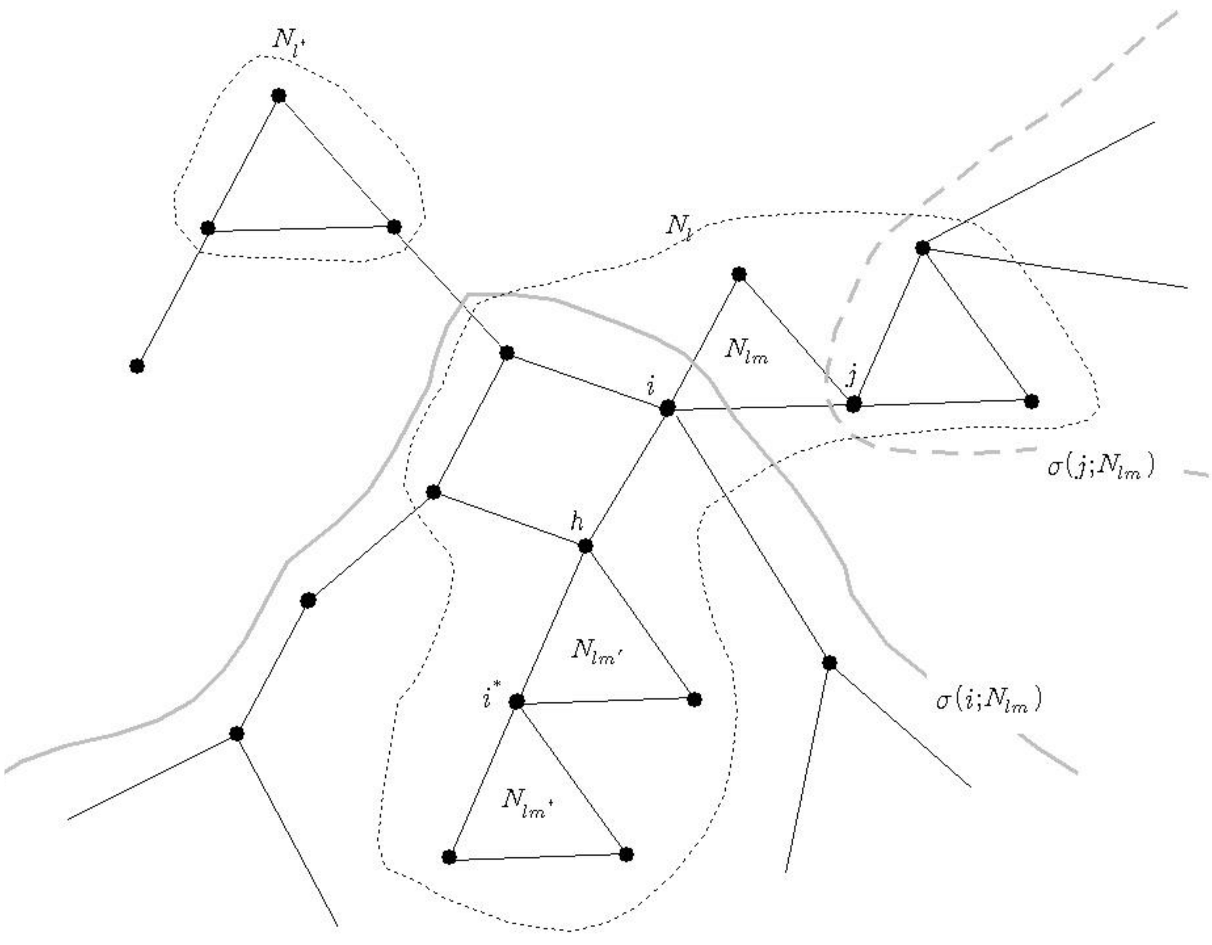

Figure 3: Proof of Theorem, Case 1 of Step 1.

Case 1. For some $l \in \{1,\ldots,L\}$, $i,j \in N_l$. See Figure 3 for an illustration of this case. Since N_l is multi-edge-connected, $\{i,j\}$ is not a bridge. Thus there is a maximal multi-node-connected subgraph of G_{N_l} containing $\{i,j\}$. That is, there is $m \in \{1,\ldots,M_l\}$ such that $i,j \in N_{lm}$. Using (10), we obtain

$$f_i(c,E)+f_j(c,E) = \left\{\begin{array}{l} A_i^{lm}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right)+A_j^{lm}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ +\sum\limits_{k\in K}\hat{W}_k^{lm}\left(\bar{c}_{\sigma(i,N_{lm})k}+\bar{c}_{\sigma(j,N_{lm})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i)\backslash\{m\}}\sum\limits_{h\in N_{lm'}\backslash\{i\}}A_h^{lm'}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(j)\backslash\{m\}}\sum\limits_{h\in N_{lm'}\backslash\{j\}}A_h^{lm'}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i)\backslash\{m\}}\sum\limits_{k\in K}\hat{W}_k^{lm'}\left(\sum\limits_{h\in N_{lm'}\backslash\{i\}}\bar{c}_{\sigma(h,N_{lm'})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(j)\backslash\{m\}}\sum\limits_{k\in K}\hat{W}_k^{lm'}\left(\sum\limits_{h\in N_{lm'}\backslash\{j\}}\bar{c}_{\sigma(h,N_{lm'})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{l':N_{l'}\in sm(N_l;i)}T_{l'}\left(\bar{c}_{\cup s(N_{l'})},\bar{c},E\right)-\sum\limits_{l':N_{l'}\in sm(N_l;j)}T_{l'}\left(\bar{c}_{\cup s(N_{l'})},\bar{c},E\right) \end{array}\right. .$$

Note that for each $m' \in \mathfrak{m}(i)\backslash\{m\}$ and each $h \in N_{lm'}\backslash\{i\}$, $\{i,j\}\cap\sigma(h,N_{lm'}) = \emptyset$ and that for each $m' \in \mathfrak{m}(j)\backslash\{m\}$ and each $h \in N_{lm'}\backslash\{j\}$, $\{i,j\}\cap\sigma(h,N_{lm'}) = \emptyset$. Thus for each $m' \in \mathfrak{m}(i)$ (resp. $m' \in \mathfrak{m}(j)$), $\sum_{h\in N_{lm'}\backslash\{i\}}\bar{c}_{\sigma(h,N_{lm'})k}$ (resp. $\sum_{h\in N_{lm'}\backslash\{j\}}\bar{c}_{\sigma(h,N_{lm'})k}$) does not depend on c_i or c_j. Neither does $\bar{c}_{\cup s(N_{l'})}$ for each l' such that $N_{l'} \in sm(N_l;i)\cup sm(N_l;j)$. For each $k \in K$, $\bar{c}_{\sigma(i,N_{lm})k}+\bar{c}_{\sigma(j,N_{lm})k}$ depends on c_i and c_j only through their sum. Therefore, it follows from the above formula that the total award of i and j cannot be changed by a reallocation of c_i and c_j.

Let $i^* \in N\backslash\{i,j\}$. If $i^* \in N_{lm^*}$ for some m^* (see Figure 3),

$$f_{i^*}(c,E) = \left\{\begin{array}{l} A_{i^*}^{lm^*}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right)+\sum\limits_{k\in K}\hat{W}_k^{lm^*}\left(\bar{c}_{\sigma(i^*,N_{lm^*})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i^*)\backslash\{m^*\}}\sum\limits_{h\in N_{lm'}\backslash\{i^*\}}A_h^{lm'}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i^*)\backslash\{m^*\}}\sum\limits_{k\in K}\hat{W}_k^{lm'}\left(\sum\limits_{h\in N_{lm'}\backslash\{i^*\}}\bar{c}_{\sigma(h,N_{lm'})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{l':N_{l'}\in sm(N_l;i^*)}T_{l'}\left(\bar{c}_{\cup s(N_{l'})},\bar{c},E\right) \end{array}\right. .$$

Note that for each $m' \in \mathfrak{m}(i^*)$ and each $h \in N_{lm'}$, either $\{i,j\}\cap\sigma(h,N_{lm'}) = \emptyset$ or $\{i,j\}\subseteq\sigma(h,N_{lm'})$ (depicted in Figure 3 is the case $\{i,j\}\subseteq\sigma(h,N_{lm'})$). Thus $\bar{c}_{\sigma(i^*,N_{lm^*})}$ and $\bar{c}_{\sigma(h,N_{lm'})}$ in the above formula depend on neither c_i nor c_j, or depend on c_i and c_j only through their sum. Also for each l' with $N_{l'} \in sm(N_l;i^*)$,

$\{i,j\} \cap (\cup s(N_{l'})) = \emptyset$ and so $\bar{c}_{\cup s(N_{l'})}$ does not depend on c_i or c_j. Thus any reallocation of c_i and c_j cannot change i^*'s award.

Now assume that $i^* \notin N_l$. Let l^* be such that $i^* \in N_{l^*}$. If $N_{l^*} \in s^0(N_l)$ or $N_{l^*} \in p^0(N_l)$, then we can use the same argument as above to show that i^*'s award cannot be changed by any reallocation of c_i and c_j. Otherwise, i^*'s award depends on c_i and c_j only through $\bar{c}$ and so cannot be changed by any reallocation of the two vectors.

Case 2. $\{i,j\}$ is a bridge. See Figure 4 for an illustration of this case. Let $l,p \in \{1,\ldots,L\}$ be such that $i \in N_l$ and $j \in N_p$. Let $m \in \{1,\ldots,M_l\}$ and $q \in \{1,\ldots,M_p\}$ be such that $i \in N_{lm}$ and $j \in N_{pq}$. Without loss of generality, assume $N_p \in sm(N_l)$. Then

$$f_i(c,E)+f_j(c,E) = \begin{cases} A_i^{lm}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right) + \sum\limits_{k\in K}\hat{W}_k^{lm}\left(\bar{c}_{\sigma(i,N_{lm})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i)\backslash\{m\}}\ \sum\limits_{h\in N_{lm'}\backslash\{i\}} A_h^{lm'}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i)\backslash\{m\}}\ \sum\limits_{k\in K}\hat{W}_k^{lm'}\left(\sum\limits_{h\in N_{lm'}\backslash\{i\}}\bar{c}_{\sigma(h,N_{lm'})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{l':N_{l'}\in sm(N_l;i)} T_{l'}\left(\bar{c}_{\cup s(N_{l'})},\bar{c},E\right) + f_j(c,E) \end{cases}.$$

Equivalently,

$$f_i(c,E)+f_j(c,E) = \begin{cases} A_i^{lm}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right) + \sum\limits_{k\in K}\hat{W}_k^{lm}\left(\bar{c}_{\sigma(i,N_{lm})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i)\backslash\{m\}}\ \sum\limits_{h\in N_{lm'}\backslash\{i\}} A_h^{lm'}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{m'\in\mathfrak{m}(i)\backslash\{m\}}\ \sum\limits_{k\in K}\hat{W}_k^{lm'}\left(\sum\limits_{h\in N_{lm'}\backslash\{i\}}\bar{c}_{\sigma(h,N_{lm'})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum\limits_{l'\neq p:N_{l'}\in sm(N_l;i)} T_{l'}\left(\bar{c}_{\cup s(N_{l'})},\bar{c},E\right) - T_p\left(\bar{c}_{\cup s(N_p)},\bar{c},E\right) + f_j(c,E) \end{cases}.$$

Here $T_p\left(\bar{c}_{\cup s(N_p)},\bar{c},E\right)$ is the total award of all agents in N_p or in its successors, namely agents in $\cup s(N_p)$. Note that since $j \in D^*$, $\cup s(N_p)$ is partitioned into the three sets, $\{j\}$, the union of successors of N_p originating from j, that is, $\cup_{p'\in sm(N_p;j)}[\cup s(N_{p'})]$, and the set of all agents succeeding each $j' \in N_{pq'}\backslash\{j\}$ for some $q' \in \mathfrak{m}(j)$, that is, $\cup_{q'\in\mathfrak{m}(j)}\cup_{j'\in N_{pq'}\backslash\{j\}}\sigma(j',N_{pq'})$. The total award of agents in the second set is given by $\sum\limits_{p'\in sm(N_p,j)} T_{l'}\left(\bar{c}_{\cup s(N_{l'})},\bar{c},E\right)$. For each $q' \in \mathfrak{m}(j)$ and each $j' \in N_{pq'}$, the total award of agents in $\sigma(j',N_{pq'})$ is given by

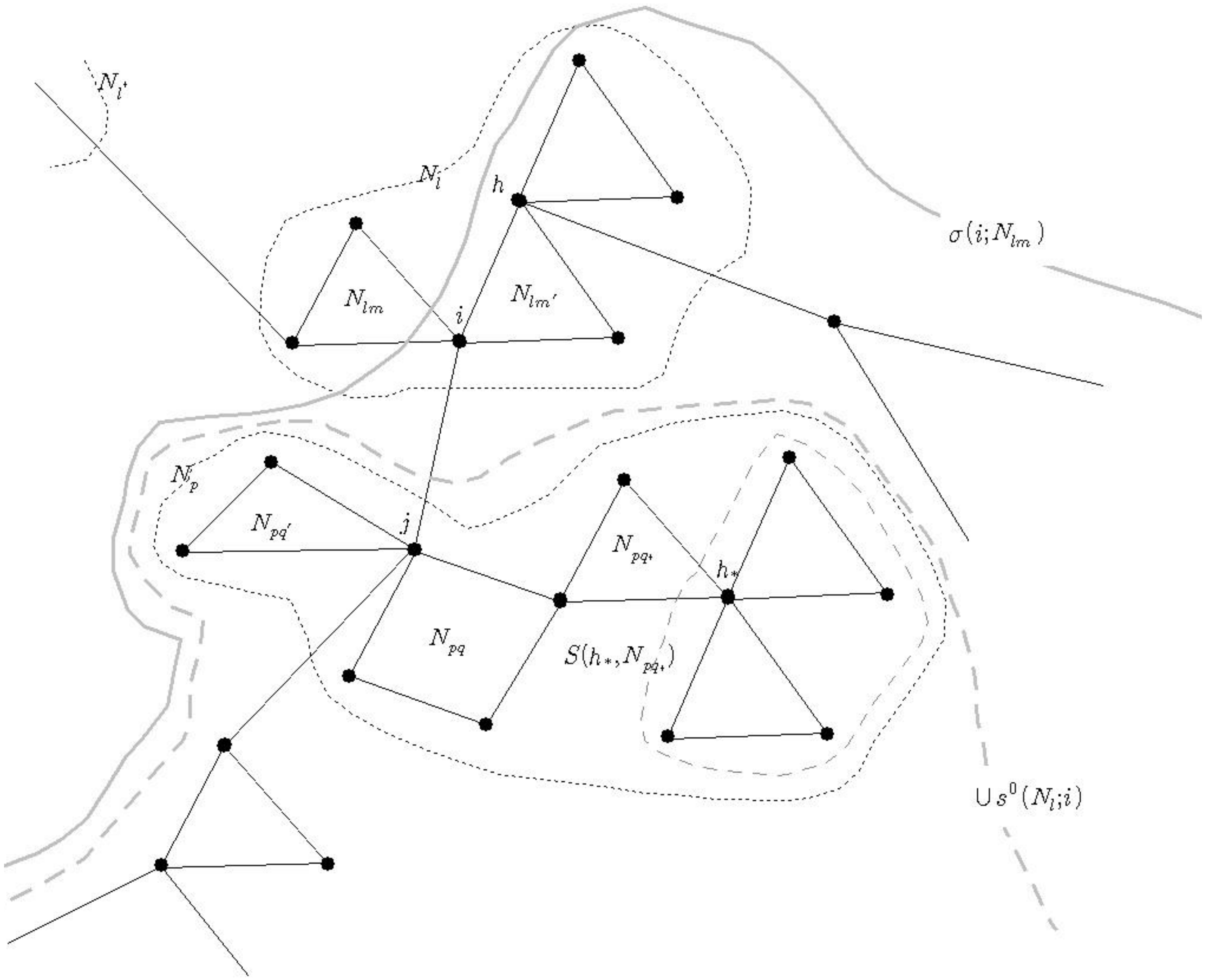

Figure 4: Proof of Theorem, Case 2 of Step 1.

$A_{j'}^{pq'}\left(\bar{c}_{\cup s(N_p)},\bar{c},E\right)+\sum_{k\in K}\hat{W}_k^{pq'}\left(\bar{c}_{\sigma\left(j',N_{pq'}\right)k},\bar{c}_{\cup s(N_p)},\bar{c},E\right)$. Thus

$$T_p\left(\bar{c}_{\cup s(N_p)},\bar{c},E\right)=\begin{cases} f_j\left(c,E\right)+\sum_{p'\in sm(N_p,j)}T_{p'}\left(\bar{c}_{\cup s(N_{p'})},\bar{c},E\right)+ \\ \sum_{q'\in\mathfrak{m}(j)}\sum_{j'\in N_{pq'}\backslash\{j\}}\left(A_{j'}^{pq'}\left(\bar{c}_{\cup s(N_p)},\bar{c},E\right)+\sum_{k\in K}\hat{W}_k^{pq'}\left(\bar{c}_{\sigma\left(j',N_{pq'}\right)k},\bar{c}_{\cup s(N_p)},\bar{c},E\right)\right)\end{cases}.$$

Using this, we obtain

$$f_i(c,E)+f_j\left(c,E\right)=$$
$$\begin{cases} A_i^{lm}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right)+\sum_{k\in K}\hat{W}_k^{lm}\left(\bar{c}_{\sigma(i,N_{lm})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum_{m'\in\mathfrak{m}(i)\backslash\{m\}}\sum_{h\in N_{lm'}\backslash\{i\}}A_h^{lm'}\left(\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum_{m'\in\mathfrak{m}(i)\backslash\{m\}}\sum_{k\in K}\hat{W}_k^{lm'}\left(\sum_{h\in N_{lm'}\backslash\{i\}}\bar{c}_{\sigma(h,N_{lm'})k},\bar{c}_{\cup s(N_l)},\bar{c},E\right) \\ -\sum_{l'\neq p:N_{l'}\in sm(N_l;i)}T_{l'}\left(\bar{c}_{\cup s(N_{l'})},\bar{c},E\right)-\sum_{l':N_{l'}\in sm(N_p;j)}T_{l'}\left(\bar{c}_{\cup s(N_{l'})},\bar{c},E\right) \\ -\sum_{q'\in\mathfrak{m}(j)}\sum_{j'\in N_{pq'}\backslash\{j\}}\left(A_{j'}^{pq'}\left(\bar{c}_{\cup s(N_p)},\bar{c},E\right)+\sum_{k\in K}\hat{W}_k^{pq'}\left(\bar{c}_{\sigma\left(j',N_{pq'}\right)k},\bar{c}_{\cup s(N_p)},\bar{c},E\right)\right)\end{cases}. \quad (\star)$$

For each $q'\in\mathfrak{m}(j)$ and each $j'\in N_{pq'}\backslash\{j\}$, $j\notin S\left(j',N_{pq'}\right)$ and $j\in D^*$. So by CONS, $A_{j'}^{pq'}\left(\bar{c}_{\cup s(N_p)},\bar{c},E\right)$ and $\hat{W}_k^{pq'}(\bar{c}_{\sigma(j',N_{pq'})k},\bar{c}_{\cup s(N_p)},\bar{c},E)$, for each $k\in K$, are constant in $\bar{c}_{\cup s(N_p)}$. Thus the fifth line of $(\star)$ cannot be changed by any reallocation of c_i and c_j. Note that $\{i,j\}\subseteq\cup s\left(N_l\right)$, $\{i,j\}\subseteq\sigma(i,N_{lm})$, for each $m'\in\mathfrak{m}\left(i\right)\backslash\{m\}$ and each $h\in N_{lm'}\backslash\{i\}$, $\{i,j\}\cap\sigma(h,N_{lm'})=\emptyset$, and for each $l'\neq p$ with $N_{l'}\in sm(N_l;i)$ or $N_{l'}\in sm(N_p;j)$, $\{i,j\}\cap[\cup s(N_{l'})]=\emptyset$. Therefore, the first four lines of $(\star)$ cannot be changed by any reallocation of c_i and c_j, either.

Let $h\in N\backslash\{i,j\}$. If $i\notin\cup s\left(h\right)$ and $h\notin\cup s\left(i\right)$, h's award depends on c_i and c_j only through $\bar{c}$. So it cannot be changed by any reallocation of c_i and c_j. Similar argument applies when $i\in\cup s^0\left(N_{l'}\right)$ for some l' with $h\in N_{l'}$ or $h\in\cup s^0(N_p)$. We now consider two remaining cases, $h\in N_l$ or $h\in N_p$.

Assume $h\in N_l$. Then for each $m'\in\mathfrak{m}\left(h\right)$ and each $h'\in N_{lm'}\backslash\{h\}$, $\{i,j\}\cap\sigma(h',N_{lm'})=\emptyset$ or $\{i,j\}\subseteq\sigma(h',N_{lm'})$. Thus, $\bar{c}_{\sigma(h',N_{lm'})}$ does not depend on c_i or c_j, or depends on c_i and c_j only through c_i+c_j. In any case, h's award cannot be changed by any reallocation of c_i and c_j.

Assume $h_*\in N_p$. Let $q_*\in\{1,\ldots,M_p\}$ be such that $h_*\in N_{pq_*}$ and $j\notin$

$S(h_*, N_{pq_*})$. Then

$$f_{h_*}(c,E) = \left\{ \begin{array}{l} A_{h_*}^{pq_*}(\bar{c}_{\cup s(N_p)}, \bar{c}, E) + \sum_{k \in K} \hat{W}_k^{pq_*}\left(\bar{c}_{\sigma(h_*,N_{pq_*})k}, \bar{c}_{\cup s(N_p)}, \bar{c}, E\right) \\ - \sum_{m' \in \mathfrak{m}(h_*)\backslash\{q_*\}} \sum_{h' \in N_{pm'}\backslash\{h_*\}} A_{h'}^{pm'}\left(\bar{c}_{\cup s(N_p)}, \bar{c}, E\right) \\ - \sum_{m' \in \mathfrak{m}(h_*)\backslash\{q_*\}} \sum_{k \in K} \hat{W}_k^{pm'}\left(\sum_{h' \in N_{pm'}\backslash\{h_*\}} \bar{c}_{\sigma(h',N_{pm'})k}, \bar{c}_{\cup s(N_p)}, \bar{c}, E\right) \\ - \sum_{l' \neq p: N_{l'} \in sm(N_p;h_*)} T_{l'}\left(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right) \end{array} \right. . \tag{15}$$

Since $j \notin S(h_*, N_{pq_*})$, $\{i,j\} \cap \sigma(h_*, N_{pq_*}) = \emptyset$. For each $m' \in \mathfrak{m}(h_*)\backslash\{q_*\}$ and each $h' \in N_{pm'}\backslash\{h_*\}$, $\{i,j\} \cap \sigma(h', N_{pm'}) = \emptyset$. For each $l' \neq p$ with $N_{l'} \in sm(N_p; h_*)$, $\{i,j\} \cap [\cup s(N_{l'})] = \emptyset$. Therefore, $\bar{c}_{\sigma(h_*,N_{pq_*})}$ does not depend on c_i or c_j. And the same result holds for $\bar{c}_{\sigma(h',N_{pm'})}$ or $\bar{c}_{\cup s(N_{l'})}$ for each $m' \in \mathfrak{m}(h_*)\backslash\{q_*\}$, each $h' \in N_{pm'}\backslash\{h_*\}$, and each $l' \neq p$ with $N_{l'} \in sm(N_p; h_*)$. On the other hand $A_{h_*}^{pq_*}(\bar{c}_{\cup s(N_p)}, \bar{c}, E)$ and $\sum_{k \in K} \hat{W}_k^{pq_*}\left(\bar{c}_{\sigma(h_*,N_{pq_*})k}, \bar{c}_{\cup s(N_p)}, \bar{c}, E\right)$ are constant with respect to $\bar{c}_{\cup s(N_p)}$. Thus it follows from (15) that h_*'s award cannot be changed by any reallocation of c_i and c_j.

Step 2. Every reallocation-proof rule is a member of TAW-family.

Substep 2.1. Let $G \equiv (N, D)$ be a connected graph. Let $\mathcal{N} \equiv \{N_1, \cdots, N_L\}$ and $\mathcal{R} \equiv \{G_{N_1}, \cdots, G_{N_L}\}$ be the set of maximal multi-edge-connected subgraphs of G. By Lemma 5, for each $l = 1, \cdots, L$, $|N_l| = 1$ or $|N_l| \geq 3$. By Lemma 4, for each $l = 1, \cdots, L$, G_{N_l} is composed of a finite number M_l of maximal multi-node-connected subgraphs. Let $N_{l1}, \cdots, N_{lM_l}$ be such that $\cup_{m=1}^{M_l} N_{lm} = N_l$ and for each $m = 1, \cdots, M_l$, $G_{N_{lm}}$ is a maximal multi-node-connected subgraph on G_{N_l}. Let $\mathcal{G} \equiv (\mathcal{N}, \mathcal{E})$ be the graph in Definition 3. Fix $l^* \in \{1, \cdots, L\}$ and consider the directed tree $\mathcal{G}(N_{l^*})$. Roughly speaking the following proof is the combination of the arguments used in the proofs of Propositions 6 and 4.

Let f be a rule satisfying *reallocation-proofness*. Then by Lemma 2, f satisfies *non-bossiness*. Define a function $T\colon \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^L$ such that for each $l \in L$ and each $(x, y, E) \in \mathbb{R}_+^K \times \mathbb{R}_{++}^K \times \mathbb{R}_{++}$,

$$T_l(x, y, E) \equiv \sum_{i \in \cup s(N_l)} f_i(c, E),$$

for some $(c, E) \in \mathcal{D}$ with $\bar{c}_{\cup s(N_l)} = x$ and $\bar{c} = y$. For all other (x, y, E), define $T_l(x, y, E)$ arbitrarily. Since both $\cup s(N_l)$ and $N \backslash \cup s(N_l)$ are connected in G, then by *reallocation-proofness* and *non-bossiness*, we can show that $T(\cdot)$ is well-defined as in the proof of Proposition 6.

Let $l \in \{1, \cdots, L\}$. If $|N_l|$ is a singleton and so $M_l = 1$, then let A^{l1} and $\hat{W}^{l1}$ be such that for each $(c, E) \in \mathcal{D}$, $A_i^{l1}(\bar{c}, E) + \sum_{k \in K} \hat{W}_k^{l1}(c_{ik}, \bar{c}, E) = T_l\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) - \sum_{l': N_{l'} \in sm(N_l)} T_{l'}(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E)$, where $i \in N_l$. Then (10) and (11) hold.

Now consider the case when $|N_l| \geq 3$ (recall $|N_l| = 1$ or $|N_l| \geq 3$). Fix $y \in \mathbb{R}_{++}^K$. Let $\mathcal{D}_{N_l}(y) \equiv \{(d, E) \in \mathbb{R}_{++}^{N_l \times K} \times \mathbb{R}_{++}$: for some $(c, E) \in \mathcal{D}$, $\bar{c} = y$, $c_{N_l \setminus C^*(N_l)} = d_{N_l \setminus C^*(N_l)}$, and for each $i \in C^*(N_l)$, $c_i + \bar{c}_{\cup s^0(N_l; i)} = d_i\}$. Define $g \colon \mathcal{D}_{N_l}(y) \to \mathbb{R}^{N_l}$ as follows: for each $(d, E) \in \mathcal{D}_{N_l}(y)$ and each $i \in N_l$,

$$g_i(d, E) \equiv f_i(c, E) + \sum_{j \in \cup s^o(N_l; i)} f_j(c, E), \qquad (\star)$$

for some $(c, E) \in \mathcal{D}$ such that $\bar{c} = y$, $\bar{c}_{\cup s(N_l)} = \bar{d}$, $c_{N_l \setminus C^*(N_l)} = d_{N_l \setminus C^*(N_l)}$, and for each $i \in C^*(N_l)$, $c_i + \bar{c}_{\cup s^0(N_l; i)} = d_i$. Using the same argument as in the proof of Proposition 4, we can show that $g(\cdot)$ is well-defined and that $g(\cdot)$ is a *reallocation-proof* rule on $\mathcal{D}_{N_l}(y)$ [`see Omitted Proofs, Section C.2`].

Now applying Proposition 4 and the definition of $T(\cdot)$, we conclude that there exists a list of functions $\left(A^m \colon \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^{N_{lm}}, \hat{W}^m \colon \mathbb{R}_+ \times \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^K\right)_{m=1}^{M_l}$ such that for each $(d, E) \in \mathcal{D}_{N_l}(y)$, each $m \in \{1, \cdots, M_l\}$, and each $i \in N_{lm}$,

$$g_i(d, E) = \begin{cases} A_i^m\left(\bar{d}, E\right) - \sum_{m' \in \mathfrak{m}(i) \setminus \{m\}} \sum_{j \in N_{lm'} \setminus \{i\}} A_j^{m'}(\bar{d}, E) + \sum_{k \in K} \hat{W}_k^m\left(\bar{d}_{S(i, N_{lm})k}, \bar{d}, E\right) \\ - \sum_{m' \in \mathfrak{m}(i) \setminus \{m\}} \sum_{k \in K} \hat{W}_k^{m'}\left(\sum_{j \in N_{lm'} \setminus \{i\}} \bar{d}_{S(j, N_{lm'})k}, \bar{d}, E\right) \end{cases}, \qquad (\star\star)$$

where for each $m \in \{1, \ldots, M_l\}$, $\hat{W}^m\left(\cdot, \bar{d}, E\right)$ is additive and satisfies (6). Now for each $m \in \{1, \cdots, M_l\}$ and each $(c, E) \in \mathcal{D}$, let $A^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \equiv A^m\left(\bar{c}_{\cup s(N_l)}, E\right)$ and for each $k \in K$, $\hat{W}_k^{lm}\left(\cdot, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \equiv \hat{W}_k^m\left(\cdot, \bar{c}_{\cup s(N_l)}, E\right)$. Then by definition of $T(\cdot)$, we obtain (10). Finally, we obtain (11) from the definition of $T(\cdot)$ and (6).

Substep 2.2. We now prove that the representation of f satisfies CONS. Throughout Substep 2.2, see Figure 2 for an illustration. First, note that for each $l \in \{1, \cdots, L\}$, each $m \in \{1, \cdots, M_l\}$, each$i \in N_{lm}$, $\sigma(i; N_{lm}) = \cup_{j \in S(i, N_{lm})} \cup s^0(N_l; j) \cup \{j\}$. Thus

$$\sum_{j \in \sigma(i, N_{lm})} f_j(c, E) = \sum_{j \in S(i, N_{lm})} \left[f_j(c, E) + \sum_{h \in s^0(N_l; j)} f_h(c, E)\right]. \qquad (16)$$

Let $l \in \{1, \cdots, L\}$, $m \in \{1, \cdots, M_l\}$, $i \in N_{lm}$, and $j \in D^* \cap N_l$ be such that $j \notin S(i, N_{lm})$. We need to show that for each $(c, E) \in \mathcal{D}$, $A_i^{lm}(\cdot, \bar{c}, E)$ is constant and for each $k \in K$ and each $\alpha \in \mathbb{R}_+$, $\hat{W}_k^{lm}(\alpha, \cdot, \bar{c}, E)$ is constant.

Since $j \in D^*$, there is $j' \in C^*$ such that $\{j, j'\}$ is a bridge. Let l', m' be such that $j' \in N_{l'm'}$. Consider the coalition $S \equiv \sigma(j', N_{l'm'}) \backslash \sigma(i, N_{lm})$. Since $j \notin S(i, N_{lm})$, $j \notin \sigma(i, N_{lm})$ and S is connected. Let $(c, E) \in \mathcal{D}$ be such that for each $h \in \sigma(i, N_{lm}) \backslash \{i\}$, $c_h = 0$. Thus $\bar{c}_{\sigma(i,N_{lm})} = c_i$. Using (16), ($\star$), and ($\star\star$), we obtain

$$\begin{aligned}
\sum_{h \in \sigma(j', N_{l'm'})} f_h(c, E) &= A_h^{l'm'}\left(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{l'm'}\left(\bar{c}_{\sigma(j', N_{l'm'}),k}, \bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right); \\
\sum_{h \in \sigma(i, N_{lm})} f_h(c, E) &= A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(\bar{c}_{\sigma(i,N_{lm})k}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \\
&= A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{lm}\left(c_{ik}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right).
\end{aligned}$$

Thus

$$\sum_{h \in S} f_h(c, E) = \left\{ \begin{array}{c} A_h^{l'm'}\left(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right) + \sum\limits_{k \in K} \hat{W}_k^{l'm'}\left(\bar{c}_{\sigma(j', N_{l'm'})k}, \bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right) \\ -A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) - \sum\limits_{k \in K} \hat{W}_k^{lm}\left(c_{ik}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) \end{array} \right. . \tag{17}$$

If $c_i = 0$, then $\hat{W}_k^{lm}\left(c_{ik}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) = 0$ for each $k \in K$. Thus,

$$\sum_{h \in S} f_h(c, E) = A_h^{l'm'}\left(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{l'm'}\left(\bar{c}_{\sigma(j', N_{l'm'}),k}, \bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right) - A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right).$$

Let $c' \in \mathbb{R}_+^{N \times K}$ be such that for some $t \in \mathbb{R}^K$, $c'_{j'} = c_{j'} - t$, $c'_j = c_j + t$, and $c'_{N \backslash \{j, j'\}} = c_{N \backslash \{j, j'\}}$. Then

$$\sum_{h \in S} f_h(c', E) = A_h^{l'm'}\left(\bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right) + \sum_{k \in K} \hat{W}_k^{l'm'}\left(\bar{c}_{\sigma(j', N_{l'm'}),k}, \bar{c}_{\cup s(N_{l'})}, \bar{c}, E\right) - A_i^{lm}\left(\bar{c}_{\cup s(N_l)} + t, \bar{c}, E\right).$$

By *reallocation-proofness*, the total award of agents in S should be the same at (c', E) and (c, E). Thus, $A_i^{lm}\left(\bar{c}_{\cup s(N_l)}, \bar{c}, E\right) = A_i^{lm}\left(\bar{c}_{\cup s(N_l)} + t, \bar{c}, E\right)$. Now using this result and (17), and considering $(c, E) \in \mathcal{D}$ for which $c_{ik} \neq 0$ and $c_{ik'} = 0$ for each $k' \neq k$, we can show that $\hat{W}_k^{lm}\left(c_{ik}, \bar{c}_{\cup s(N_l)}, \bar{c}, E\right) = \hat{W}_k^{lm}\left(c_{ik}, \bar{c}_{\cup s(N_l)} + t, \bar{c}, E\right)$. ∎

Proof of Proposition 8. Parts (i), (iii), and (iv) are easily obtained from Propositions 7 and 5. The proof of $T_1(\cdot) = \cdots = T_L(\cdot)$ in part (ii) is the same as in Proposition 7. Let $T_0(\cdot) \equiv T_1(\cdot) = \cdots = T_L(\cdot)$.

Claim 1. For each $(c, E) \in \mathcal{D}$, $T_0(\cdot, \bar{c}, E)$ is additive.

Proof. Assume that $L \geq 2$ and there is $l \in \{1, \ldots, L\}$ such that $|N_l| \geq 3$. Without loss of generality, let $|N_1| \geq 3$ and $sm(N_1) \neq \emptyset$ (if $sm(N_1) = \emptyset$, then since $L \geq 2$, we can change the root N_{l^*} so that $sm(N_1) \neq \emptyset$). Assume that $N_2 \in sm(N_2)$, $1 \in N_1$, $2 \in N_2$, and $\{1,2\} \in D$. Since $|N_1| \geq 3$ and N_1 is multi-edge-connected, there are $i, j \in N_1 \backslash \{1\}$ such that $\{i,1\}, \{j,1\} \in D$. Let $y, z, d \in \mathbb{R}_+^K$ be such that $y + z \leq d$. Let $(c, E) \in \mathcal{D}$ be such that $\bar{c} = d$ and for each $h \in \cup s(N_1) \backslash \{i,j\}$, $c_h = 0$ (so $c_1 = c_2 = 0$), $c_i = y$, and $c_j = z$. Then by *no award for nulls*, for each $h \in \cup s(N_1) \backslash \{i,j\}$, $f_h(c,E) = 0$, and $\bar{c}_{\cup s(N_1)} = y + z$. Thus

$$f_i(c,E) + f_j(c,E) = T_0(y+z, \bar{c}, E). \tag{\dagger}$$

Let c' be such that $c'_i = c_2 (= 0)$, $c'_2 = c_i (= y)$ and $c'_{N \backslash \{i,2\}} = c_{N \backslash \{i,2\}}$. Since $\{i,1,2\}$ are connected, then by *reallocation-proofness*,

$$f_i(c,E) + f_1(c,E) + f_2(c,E) = f_i(c',E) + f_1(c',E) + f_2(c',E).$$

Thus by *no award for nulls*

$$f_i(c,E) = f_2(c',E). \tag{\ddagger}$$

By *no award for nulls*, for each $h \in \cup s(N_2) \backslash \{2\}$, $f_h(c',E) = 0$, and $\bar{c}_{\cup s(N_2)} = y$. Thus, $f_2(c',E) = T_0(y, \bar{c}, E)$. Using (‡), we obtain

$$f_i(c,E) = T_0(y, \bar{c}, E).$$

Similarly, we can show

$$f_j(c,E) = T_0(z, \bar{c}, E).$$

Thus, by (†),

$$T_0(y, \bar{c}, E) + T_0(z, \bar{c}, E) = T_0(y+z, \bar{c}, E).$$

Therefore, $T_0(\cdot, \bar{c}, E)$ is additive. ♦

Claim 2. For each $(c,E) \in \mathcal{D}$ and each $i \in N$, $f_i(c,E) = T_0(c_i, \bar{c}, E)$.

Proof. Let $l \in \{1, \ldots, L\}$ be such that N_l is an end node of $\mathcal{G}(N_{l^*})$. Let $m \in \{1, \ldots, L\}$ be such that $N_m = pm(N_l)$. Let $i_l \in N_l$ and $i_m \in N_m$ be such that $\{i_l, i_m\} \in D$ (thus $\{i_l, i_m\}$ is a bridge). Let $(c,E) \in \mathcal{D}$. Let c' be such that $c'_{i_l} = c_{i_l} + c_{i_m}$, $c'_{i_m} = 0$, and $c'_{N \backslash \{i_l, i_m\}} = c_{N \backslash \{i_l, i_m\}}$. Since $N_l \cup \{i_m\}$ is connected and N_l is an end node on $\mathcal{G}(N_{l^*})$, then by *reallocation-proofness* and the definition of $T_0(\cdot)$,

$$\begin{aligned} f_{i_m}(c,E) + \sum_{i \in N_l} f_i(c,E) &= f_{i_m}(c,E) + T_0(\bar{c}_{N_l}, \bar{c}, E) \\ &= f_{i_m}(c',E) + \sum_{i \in N_l} f_i(c',E) = f_{i_m}(c',E) + T_0\left(\bar{c}'_{N_l}, \bar{c}, E\right) \\ &= f_{i_m}(c',E) + T_0(c_{i_m} + \bar{c}_{N_l}, \bar{c}, E). \end{aligned}$$

By *no award for nulls*, $f_{i_m}(c', E) = 0$. Thus by additivity of $T_0(\cdot, \bar{c}, E)$,

$$f_{i_m}(c, E) = T_0(c_{i_m}, \bar{c}, E). \qquad (\star)$$

Let $i \in N_m$ be such that $\{i, i_m\} \in D$. Let c'' be such that $c''_i = 0$, $c''_{i_m} = c_i + c_{i_m}$, and $c''_{N\setminus\{i,i_m\}} = c_{N\setminus\{i,i_m\}}$. Then by $(\star)$, $f_{i_m}(c'', E) = T_0(c''_{i_m}, \bar{c}, E)$. By *reallocation-proofness* and *no award for nulls*,

$$f_i(c, E) + f_{i_m}(c, E) = f_{i_m}(c'', E) = T_0\left(c''_{i_m}, \bar{c}, E\right).$$

By $(\star)$ and additivity of $T_0(\cdot, \bar{c}, E)$, $f_i(c, E) = T_0(c_i, \bar{c}, E)$. The same argument can be used to show: for each $i \in N_m$,

$$f_i(c, E) = T_0(c_i, \bar{c}, E). \qquad (\star\star)$$

Now let c^* be such that $c^*_{i_m} = c_{i_l} + c_{i_m}$, $c^*_{i_l} = 0$, and $c^*_{N\setminus\{i_l,i_m\}} = c_{N\setminus\{i_l,i_m\}}$. Then by *reallocation-proofness*, *no award for nulls*, and $(\star)$,

$$f_{i_l}(c, E) + T_0(c_{i_m}, \bar{c}, E) = f_{i_m}(c^*, E) = T_0(c^*_{i_m}, \bar{c}, E).$$

Thus by additivity of $T_0(\cdot, \bar{c}, E)$,

$$f_{i_l}(c, E) = T_0(c_{i_l}, \bar{c}, E).$$

Using this and the same argument that is used for $(\star\star)$, we can show: for each $i \in N_l$,

$$f_i(c, E) = T_0(c_i, \bar{c}, E).$$

Now moving backward on the three $\mathcal{G}(N_{l^*})$, we can show this equation for each $i \in N$. ♦ ∎

References

[1] Aczél, J. and J.G. Dhombres (1989) *Functional equations in several variables,* Cambridge University Press.

[2] Chambers, C.P. and W. Thomson (2002) "Group order preservation and the proportional rule for the adjudication of conflicting claims," forthcoming in *Mathematical Social Sciences.*

[3] Chun, Y. (1988) "The proportional solution for rights problems," *Mathematical Social Sciences* 15: 231–246.

[4] de Frutos, M.A. (1999) "Coalitional manipulations in a bankruptcy problem," *Review of Economic Design* 4: 255–272.

[5] Diestel, R. (2000) *Graph Theory*, 2nd Ed., Springer.

[6] Ju, B.-G. (2003): "Manipulation via merging and splitting in claims problems," *Review of Economic Design*, 8, 205–215.

[7] Ju, B.-G., E. Miyagawa, and T. Sakai (2003) "Non-manipulable division rules in claim problems and generalizations," University of Kansas, Working Paper.

[8] Moreno-Ternero, J.D. (2004) "Bankruptcy rules and coalitional manipulation," Yale University, mimeo.

[9] Moulin, H. (1985a) "Egalitarianism and utilitarianism in quasi-linear bargaining," *Econometrica* 53: 49–67.

[10] Moulin, H. (1985b) "Binary choices with compensations," Virginia Polytechnic Institute and State University, mimeo.

[11] Moulin, H. (1987) "Equal or proportional division of a surplus, and other methods," *International Journal of Game Theory* 16: 161–186.

[12] Moulin, H. and S. Shenker (1992) "Serial Cost Sharing," *Econometrica*, 60, 1009–1038.

[13] O'Neill, B. (1982) "A problem of rights arbitration from the Talmud," *Mathematical Social Sciences* 2: 345–371.

[14] Satterthwaite, M. and H. Sonnenschein (1981) "Strategy-proof allocation mechanisms at differentiable points", *Review of Economic Studies*

[15] Thomson, W. (2002a) "Axiomatic and strategic analysis of bankruptcy and taxation problems: a survey", forthcoming in *Mathematical Social Science.*

[16] Thomson, W. (2002b) *How to Divide When There Isn't Enough: From the Talmud to Modern Game Theory.*

[17] Wilson, R. (1979) *Introduction to Graph Theory*, Longman, London.

C Omitted Proofs

C.1 Structure of Connected Graph

In this section, we prove Lemmas 4 and 5. We begin with some useful facts on multi-edge-connected graphs and multi-node-connected graphs.

Fact 1. *When there are at least three nodes, multi-node-connectivity implies multi-edge-connectivity.*

Proof. Let $G \equiv (N, D)$ be multi-node-connected. Assume $|N| \geq 3$. Suppose by contradiction that G is not multi-edge-connected. Let $ij \in D$ be a bridge. Then $G' \equiv (N, D \backslash \{ij\})$ is disconnected. Then since $|N| \geq 3$, i or j has an adjacent node in $N \backslash \{i, j\}$ on G'. Suppose that i has an adjacent node $h \in N \backslash \{i, j\}$ on G'. Then there is no path from h to j on G'. Since the set of edges of $G_{N \backslash \{i\}}$ is a subset of the set of edges of G', that is, $D \backslash \{ij\}$, then there is no path from h to j on $G_{N \backslash \{i\}}$ either. Thus $G_{N \backslash \{i\}}$ is disconnected. This shows that i is a cutnode, contradicting multi-node-connectivity of G. ∎

Fact 2. *When $N \equiv \{i, j\}$ and $D \equiv \{ij\}$, $G \equiv (N, D)$ is multi-node-connected but not multi-edge-connected.*

Fact 3. *If G is multi-edge-connected, $M \subseteq N$, and G_M is a maximal multi-node-connected subgraph, then $|M| \geq 3$.*

Proof. Suppose $|M| = 1$, say $M = \{i\}$. Then because G is connected, there is $j \neq i$ such that $ij \in D$. Then $G_{\{i,j\}}$ is multi-node-connected, contradicting the maximal multi-edge-connectivity of G_M. Suppose that $|M| = 2$, say, $M = \{i, j\}$. Let $G' \equiv (N, D \backslash \{ij\})$. Let M_i be the set of nodes connected with i on G' and M_j the set of nodes connected with j on G'. Since G is multi-edge-connected, then ij is not a bridge. So $M_i \cap M_j \neq \emptyset$. Let $h \in M_i \cap M_j$. Let $p(i, h)$ be a path in G'_{M_i} from i to h and $p(h, j)$ a path in G'_{M_j} from h to j. Let M' be the set of nodes in the two paths. Clearly, $M \subseteq M'$. Then $G_{M'}$ has a spanning cycle and so it is a multi-node-connected graph, contradicting the maximal multi-edge-connectivity of G_M. ∎

Fact 4. *Let G be multi-edge-connected. Let $M, M' \subseteq N$ be such that G_M and $G_{M'}$ are maximal multi-node-connected subgraphs and $M \neq M'$. Then*
(i) *Either $|M \cap M'| = 0$ or 1.*
(ii) *If $i \in M \cap M'$, i is a cutnode on G.*

(iii) *If* $i \in M \cap M'$, $h \in M$, *and* $h' \in M'$, *every path from* h *to* h' *contains* i, *that is,* i *is between* h *and* h'.

Proof. Proof of (i). Suppose by contradiction that $M \cap M'$ contains at least two nodes. For each $i \in M \backslash M'$, since i is not a cutnode in G_M, $G_{M \backslash \{i\}}$ is connected. Since $i \notin M \cap M' \neq \emptyset$, every $j \in M \backslash \{i\}$ has a path to a node in $M \cap M'$, which has a path to any node in M'. Thus, $G_{(M \cup M') \backslash \{i\}}$ is connected. So i is not a cutnode in $G_{M \cup M'}$. Similarly, we show that each $i \in M' \backslash M$ is not a cutnode in $G_{M \cup M'}$. Now let $i \in M \cap M'$. Since $|M \cap M'| \geq 2$, there is $j \in (M \cap M') \backslash \{i\}$. Since both G_M and $G_{M'}$ are multi-node-connected, both $G_{M \backslash \{i\}}$ and $G_{M' \backslash \{i\}}$ are connected. Because $j \in (M \cap M') \backslash \{i\}$, any node in $M \backslash \{i\}$ has a path, by way of j, to any node in $M' \backslash \{i\}$ on $G_{\{M \cup M'\} \backslash \{i\}}$. Hence $G_{\{M \cup M'\} \backslash \{i\}}$ is connected and i is not a cutnode. This holds for each $i \in M \cap M'$. Therefore, $G_{M \cup M'}$ does not have any cutnode and $G_{M \cup M'}$ is multi-node-connected. This contradicts the maximal multi-node-connectivity of G_M.

Proof of (ii). Now let $i \in M \cap M'$. If i is not a cutnode, $G_{N \backslash \{i\}}$ is connected. Pick $h \in M \backslash \{i\}$ and $h' \in M' \backslash \{i\}$. Then there is a path from h to h' on $G_{N \backslash \{i\}}$. Now combining this path with $M \cup M'$, we obtain a multi-node-connected subgraph, contradicting the maximal multi-node-connectivity of G_M.

Proof of (iii). This follows easily from (ii). ∎

Now we are ready to prove Lemma 4.

Proof of Lemma 4. Let $G \equiv (N, D)$ be a multi-edge-connected graph.

Proof of part (i): We first show that N is divided into a finite number of subsets $N_1, \cdots, N_L$ with $\cup_{l=1}^{L} N_l = N$ such that for each $l = 1, \cdots, L$, $|N_l| \geq 3$ and G_{N_l} is a maximal multi-node-connected subgraph on G. Pick a node $i \in N$. Find all maximal multi-node-connected subgraphs containing i. Let $N_1, \cdots, N_m$ be the sets of nodes of these subgraphs. Then because of multi-edge-connectivity of G and Fact 3, $|N_1|, \cdots, |N_m| \geq 3$. If $\cup_{k=1}^{m} N_k = N$, we are done. Otherwise, since G is connected, pick $j \in N \backslash \cup_{k=1}^{m} N_k$ and find all maximal multi-node-connected subgraphs containing j. Denote the sets of nodes of these subgraphs by $N_{m+1}, \cdots, N_{m+n}$. Then $|N_{m+1}|, \cdots, |N_{m+n}| \geq 3$. If $\cup_{k=1}^{m+n} N_k = N$, we are done. Otherwise, iterate the same procedure. Since N is finite, the iteration will end after a finite number of steps and, at the end, we get a list of subsets of N, $N_1, \cdots, N_L$, with the desired properties.

To prove the uniqueness, let $\{N_1, \cdots, N_L\}$ and $\{N'_1, \cdots, N_{L'}\}$ be two families of subsets of N satisfying the stated properties. Pick a node $i \in N$. Let

$\{N_1, \cdots, N_m\}$ be the subfamily of elements in $\{N_1, \cdots, N_L\}$, which include i. Let $\{N'_1, \cdots, N'_{m'}\}$ be the subfamily of elements in $\{N'_1, \cdots, N_{L'}\}$, which include i. For each element N_k in the former subfamily, find $j \in N_k$ that is adjacent to i. Then there exists an element $N'_{k'}$ in the latter family which include both i and j (that is, ij is an edge of $G_{N'_{k'}}$). Therefore, by Fact 4, $N_k = N'_{k'}$. This shows $\{N_1, \cdots, N_m\} \subseteq \{N'_1, \cdots, N'_{m'}\}$. Similarly, we can show the reverse inclusion. Therefore, $\{N_1, \cdots, N_L\} = \{N'_1, \cdots, N_{L'}\}$.

Proof of part (ii): Suppose by contradiction that there exist $N_{l_1}, \cdots, N_{l_r} \in \{N_1, \cdots, N_L\}$ with $r \geq 3$ such that $N_{l_1} \cap N_{l_2} \neq \emptyset, \cdots, N_{l_{r-1}} \cap N_{l_r} \neq \emptyset$, and $N_{l_1} = N_{l_r}$. Then if we let $M \equiv N_{l_1} \cup \cdots \cup N_{l_r}$, G_M is multi-node-connected. This contradicts the maximal multi-node-connectivity of G_{N_k} for each $k = 1, \cdots, r$. ∎

We use the next fact to prove Lemma 5.

Fact 5. *If G_M and $G_{M'}$ are maximal multi-edge-connected subgraphs on G, then either $M = M'$ or $M \cap M' = \emptyset$.*

Proof. Let $M, M' \subseteq N$ be given as above. Assume $M \neq M'$. Suppose to the contrary $M \cap M' \neq \emptyset$. Since G_M has no bridge disconnecting G_M and $M \cap M' \neq \emptyset$, there is no bridge in G_M disconnecting $G_{M \cup M'}$. Similarly, there is no bridge in $G_{M'}$ disconnecting $G_{M \cup M'}$. Therefore, $G_{M \cup M'}$ has no bridge and so it is multi-edge-connected. This contradicts maximal multi-edge-connectivity of G_M and $G_{M'}$. ∎

Fact 6. *Assume that $G \equiv (N, D)$ is a connected graph and that N is partitioned into a finite number of subsets $N_1, \cdots, N_L$ such that for each $l = 1, \cdots, L$, $|N_l| = 1$ or $|N_l| \geq 3$ and G_{N_l} is a maximal multi-edge-connected subgraph on G. Then*
(i) *For each $l, l' = 1, \cdots, L$ with $l \neq l'$, there can be at most one edge $ii' \in D$ such that $i \in N_l$ and $i' \in N_{l'}$. If there is such an edge $ii' \in D$, it is a bridge.*
(ii) *For each $l, l' = 1, \cdots, L$ with $l \neq l'$, if $i \in N_l$, $i' \in N_{l'}$, and $ii' \in D$, then for each $j \in N_l$ and each $j' \in N_{l'}$, every path from j to j' contains ii', that is, both i and i' are between j and j'.*

Proof. Proof of part (i): Let $l, l' \in \{1, \cdots, L\}$ be such that $l \neq l'$. Suppose to the contrary that at least two edges $ii', jj' \in D$ such that $i, j \in N_l$ and $i', j' \in N_{l'}$. Then any of these edges connecting N_l and $N_{l'}$ is not a bridge on $G_{N_l \cup N_{l'}}$. Since neither G_{N_l} nor $G_{N_{l'}}$ has a bridge, then no edge in G_{N_l} or $G_{N_{l'}}$ is a bridge on $G_{N_l \cup N_{l'}}$. Therefore, $G_{N_l \cup N_{l'}}$ has no bridge and so it is multi-edge-connected. This contradicts to maximal multi-edge-connectivity of G_{N_l} and $G_{N_{l'}}$.

Now assume that $ii' \in D$ is such that $i \in N_l$ and $i' \in N_{l'}$. If ii' is not a bridge, then we can find a path from a node in N_l to another node in $N_{l'}$, which does not include ii'. Now combining this path, N_l, and $N_{l'}$, we can construct a larger multi-edge-connected subgraph than G_{N_l} and $G_{N_{l'}}$, contradicting maximal multi-edge-connectivity of G_{N_l} and $G_{N_{l'}}$.

Proof of part (ii): The proof follows directly from the definition of bridge. ∎

Now we are ready to prove Lemma 5.

Proof of Lemma 5. Let $G \equiv (N, D)$ be a connected graph.

Proof of part (i): Since any edge is not a multi-edge-connected subgraph, then if $M \subseteq N$ and G_M is multi-edge-connected, either $|M| = 1$ or $|M| \geq 3$. The proof of the existence of a partition of N satisfying the property stated in part (i) is similar to the proof of part (i) in Lemma 4. The only difference is in showing that for any two subsets of N, $M \neq M'$, if G_M and $G_{M'}$ are maximal multi-edge-connected subgraphs on G, then $M \cap M' = \emptyset$. This is shown in Fact 5.

To prove the uniqueness, let $\{N_1, \cdots, N_L\}$ and $\{N'_1, \cdots, N_{L'}\}$ be two partitions of N satisfying the stated properties. Pick a node $i \in N$. Without loss of generality, let N_l and $N'_{l'}$ be the members of the two partitions, which include i. Since $N_l \cap N'_{l'} \neq \emptyset$, then by Fact 5, $N_l = N'_{l'}$. Since this holds for every $i \in N$, the two partitions must be identical.

Proof of part (ii): Suppose by contradiction that there exist $r \geq 3$, $N_{l_1}, \cdots, N_{l_r} \in \{N_1, \cdots, N_L\}$, $i_1 \in N_{l_1}, \cdots, i_{r-1} \in N_{r-1}$, and $j_2 \in N_{l_2}, \cdots, j_r \in N_r$ such that $N_{l_1} = N_{l_r}$ and $i_1 j_2, i_2 j_3, \cdots, i_{r-1} j_r \in D$. Note that for each $s \in \{2, \cdots, r-2\}$, $i_s j_{s+1}$ connects N_{l_s} and $N_{l_{s+1}}$, and $i_{r-1} j_r$ connects N_{l_r} and N_{l_1}. Therefore, since each member of $\{N_{l_1}, \cdots, N_{l_r}\}$ is connected, then there is a path from i_1 to j_2 not containing $i_1 j_2 \in D$. This means that deleting $i_1 j_2$ does not disconnect G. So $i_1 j_2$ is not a bridge, contradicting part (i) of Fact 6. ∎

C.2 Omitted Part in Substep 2.1 of the Proof of Theorem

Here we prove that $g(\cdot)$, defined in $(\star)$, is well-defined and that it is a *reallocation-proof* rule on $\mathcal{D}_{N_l}(y)$.

To show its well-definedness, let $d \in \mathcal{D}_{N_l}(y)$, $(c, E) \in \mathcal{D}$, and $c' \in \mathbb{R}_+^{N \times K}$ be such that $\bar{c} = \bar{c}' = y$, $\bar{c}_{\cup s(N_l)} = \bar{c}'_{\cup s(N_l)} = \bar{d}$, $c_{N_l \setminus C^*(N_l)} = c'_{N_l \setminus C^*(N_l)} = d_{N_l \setminus C^*(N_l)}$, and for each $i \in C^*(N_l)$, $c_i + \bar{c}_{\cup s^0(N_l;i)} = c'_i + \bar{c}'_{\cup s^0(N_l;i)} = d_i$. Since $N \setminus \cup s(N_l)$ is connected, then by *reallocation-proofness* and *non-bossiness*, $c_{N \setminus \cup s(N_l)}$ is irrelevant in this definition. So without loss of generality, we may assume that

$c_{N\backslash\cup s(N_l)} = c'_{N\backslash\cup s(N_l)}$. For each $i \in C^*(N_l)$, let $S_i \equiv \{i\} \cup [\cup s^0(N_l;i)]$. Then S_i is connected. So by *reallocation-proofness* and *non-bossiness*, if coalition S_i changes c_{S_i} to c'_{S_i}, then the total award of S_i and the awards of all others in $N\backslash S_i$ do not change. After making these changes for each $i \in C^*(N_l)$, we end up with c' and, throughout this process, the total award of coalition $S_i = \{i\} \cup [\cup s^0(N_l;i)]$ for each $i \in C^*(N_l)$, and the awards for all $j \in N_l\backslash C^*(N_l)$ do not change. Therefore, for each $i \in N_l\backslash C^*(N_l)$, $f_i(c,E) = f_i(c',E)$, and for each $i \in C^*(N_l)$, $f_i(c,E) + \sum_{j\in\cup s^o(N_l;i)} f_j(c,E) = f_i(c',E) + \sum_{j\in\cup s^o(N_l;i)} f_j(c',E)$.

We now show that g is a rule over $\mathcal{D}_{N_l}(y)$ satisfying *pairwise reallocation-proofness* and *pairwise non-bossiness* under $\mathcal{C}(G_{N_l})$ and, therefore, *reallocation-proofness* under $\mathcal{C}(G_{N_l})$. Let $i^*, j^* \in N_l$ be such that $i^*j^* \in D_{N_l}$. Consider first the case when $i^*, j^* \in N_l\backslash C^*(N_l)$. Then it follows from *pairwise reallocation-proofness* and *pairwise non-bossiness* of f and the definition of g that this pair $\{i^*, j^*\}$ cannot change their total award or awards of others by any reallocation of their characteristic vectors. Now consider the case when $i^* \in C^*(N_l)$ or $j^* \in C^*(N_l)$. Suppose $i^* \in C^*(N_l)$ and $j^* \notin C^*(N_l)$ (the same argument applies for other cases). Let $(d,E), (d',E) \in \mathcal{D}_{N_l}(y)$ be such that $d_{N_l\backslash\{i^*,j^*\}} = d'_{N_l\backslash\{i^*,j^*\}}$ and $d_{i^*} + d_{j^*} = d'_{i^*} + d'_{j^*}$. Let $c \in \mathbb{R}_+^{N\times K}$ be such that $\bar{c} = y$, $\bar{c}_{\cup s(N_l)} = \bar{d}$, $c_{N_l\backslash C^*(N_l)} = d_{N_l\backslash C^*(N_l)}$, and for each $i \in C^*(N_l)$, $c_i + \bar{c}_{\cup s^0(N_l;i)} = d_i$. Let $c' \in \mathbb{R}_+^{N\times K}$ be such that $c'_{N\backslash[\{i^*,j^*\}\cup[\cup s^0(N_l;i^*)]]} = c_{N\backslash[\{i^*,j^*\}\cup[\cup s^0(N_l;i^*)]]}$, $c'_{i^*} + \bar{c}'_{\cup s^0(N_l;i^*)} = d'_{i^*}$, and $c'_{j^*} = d'_{j^*}$. Since $d_{i^*} + d_{j^*} = d'_{i^*} + d'_{j^*}$, $c_{i^*} + \bar{c}_{\cup s^0(N_l;i^*)} + c_{j^*} = c'_{i^*} + \bar{c}'_{\cup s^0(N_l;i^*)} + c'_{j^*}$. Since i^*j^* is an edge and $\{i^*\} \cup [\cup s^0(N_l;i^*)]$ is connected, $\{i^*,j^*\} \cup [\cup s^0(N_l;i^*)]$ is connected. Thus by *reallocation-proofness* and *non-bossiness* of f,

$$\begin{aligned} \sum_{i\in\{i^*\}\cup[\cup s^0(N_l;i^*)]} f_i(c',E) + f_{j^*}(c',E) &= \sum_{i\in\{i^*\}\cup[\cup s^0(N_l;i^*)]} f_i(c,E) + f_{j^*}(c,E); \\ f_{N\backslash(\{i^*,j^*\}\cup[\cup s^0(N_l;i^*)])}(c',E) &= f_{N\backslash(\{i^*,j^*\}\cup[\cup s^0(N_l;i^*)])}(c,E). \end{aligned}$$

Therefore,

$$\begin{aligned} g_{i^*}(d',E) + g_{j^*}(d',E) &= g_{i^*}(d,E) + g_{j^*}(d,E); \\ g_{N\backslash\{i^*,j^*\}}(c',E) &= g_{N\backslash\{i^*,j^*\}}(c,E). \end{aligned}$$

This shows that g satisfies *pairwise reallocation-proofness* and *pairwise non-bossiness* under $\mathcal{C}(G_{N_l})$.