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RIGHTS AND CONSENT

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Abstract

We introduce conditionally decisive rights of which the exercise depends on social consent. Decisive (or libertarian) rights by Sen (1970) is an example and much weaker forms of rights are covered by our notion, leaving more hope for escaping from the so-called paradox of Paretian liberal. Main results provide an axiomatic characterization for existence of a well-defined system of rights and its uniqueness as well as characterizations of various families of rules. Axioms considered are some combinations of *monotonicity*, *independence*, *Pareto efficiency*, and symmetry-type axioms such as *anonymity*, *neutrality*, *symmetry*, and *symmetric linkage*. In particular, on a domain of simple preference relations (trichotomous or dichotomous preferences), we show that a rule satisfies *Pareto efficiency*, *independence*, and *symmetry* if and only if it is represented by a "quasi-plurality system of rights". For the exercise of rights under a quasi-plurality system, at least either a majority (or $(n+1)/2$) consent or a 50% (or $(n-1)/2$) consent is needed.

Keywords: Rights; Consent; Monotonicity; Independence; Symmetric linkage; Symmetry; Pareto efficiency; Plurality

JEL Classification Numbers: D70, D71, D72

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1 Introduction

Suppose that there are a finite number of issues. A society needs to decide on each issue either positively (acceptance) or negatively (rejection). The social decision should reflect members' opinions that are expressed in one of the three ways, positively or negatively or neutrally (we also consider the case when opinions are either positive or negative). The systematic relationship between social decisions and members' opinions are described by a (social choice) *rule*. It is a function associating with each list of members' opinions, namely, a *problem*, a single decision. We focus on rules satisfying the two basic axioms known as *monotonicity* and *independence* (Rubinstein and Fishburn 1986, Samet and Schmeidler 2003, Kasher and Rubinstein 1997, and Ju 2003). *Monotonicity* says that the rule should respond non-negatively whenever the set of members with positive opinion expands and the set of members with negative opinion shrinks. *Independence* says that the decision on each issue should be based only on members' opinions on this issue.

Building on Samet and Schmeidler (2003),[1] we develop a weak notion of rights of which the exercise depends on social consent and a notion of systems of rights. Degree of social consent to the exercise of, say, person i's right on an issue is measured by counting the number of persons with the same opinion on the issue as i's. And i's right can be exercised (social decision on the issue equals i's opinion) when and only when i's opinion gets sufficient social consent, that is, the degree is greater than or equal to a certain level, called a *consent quota*.[2] A *system of rights* is a function mapping each issue a person who has the right on this issue and the associated consent quotas.[3] Our notion covers decisive (or libertarian) rights by Sen (1970, 1976) and Gibbard (1974). Moreover, it covers a variety of much weaker forms of rights, depending on how much consent is needed for the exercise of rights. We investigate when there exists a well-defined system of rights and whether it is unique. We also study compatibility of *Pareto efficiency* and existence of a system of rights, associating opinions with simple preference relations, called, "trichotomous" or "dichotomous" preferences. Our

[1] Samet and Schmeidler (2003) did not formally introduce their definition of rights. But their conception of rights, to be taken by us, appears in a number of places of their work, for example, in the introduction.

[2] When i's opinion is neutral, this description does not match exactly to our definition. This is because decision on each issue cannot be neutral. In this case, we require the k^{th} issue to be decided positively (acceptance) when the number of persons with the positive opinion is sufficiently larger than the number of persons with the negative opinion.

[3] There are additional restrictions on this function related with *monotonicity* and the "principle of horizontal equality of rights".

approach is axiomatic. We offer axiomatic characterization for existence of a well-defined system of rights and its uniqueness. We also characterize various families of rules imposing some combinations of *monotonicity*, *independence*, *Pareto efficiency*, and symmetry-type axioms such as *anonymity*, *neutrality*, *symmetry*, and *symmetric linkage* (to be explained later). In particular, on the domain of trichotomous (or dichotomous) preference relations, we show that a rule satisfies *Pareto efficiency*, *independence*, and *symmetry* if and only if it is represented by a "quasi-plurality system of rights". For the exercise of rights under a quasi-plurality system, at least either majority (or $100 \times (n+1)/2\%$) consent or 50% (or $100 \times (n-1)/2\%$) consent is needed. Plurality rule is one of these rules and others are close to plurality rule. For example, whenever the number of persons with positive opinion on an issue is not equal to the number of persons with the negative opinion, decision on the issue is identical to the decision by plurality rule; whenever positive opinions tie with negative opinions, a prespecified person (with the right on this issue) breaks the tie (prespecified persons may differ across issues).

In the Arrovian framework, Sen (1970, 1976, 1983) and many of his critics formulate individual rights based on (i) existence of the so-called recognized personal spheres (Gaertner, Pattanaik, and Suzumura 1992), and (ii) individuals' *decisiveness* on personal spheres (social decision on an issue in someone's sphere is decided by the person himself). Despite some fundamental differences between our model and the Arrovian framework (see Samet and Schmeidler 2003 for the details), our definition is similar to this formulation with regard to aspect (i). This is because a system of rights links issues with persons who have the rights on these issues. However, with regard to aspect (ii), our definition is substantially weaker and flexible. Our rights are rights to *influence* social decision, not necessarily decisive but conditionally decisive (decisiveness is one extreme case in our definition). They are alienable as in Blau (1975) and Gibbard (1974). But, alienation of rights in this paper relies on degree of social consent. Motivation for our weakening decisiveness component in the earlier definition comes, first of all, from realistic rights that are often conditionally decisive. For example, consider rights for smoking or for clean air. There are some places where smoking is prohibited and also other places where smoking is allowed. A person's desire is not decisive in his own smoking or his own consumption of clean air. In order for a person to exercise his right, he needs to find a place where his desire can get sufficient consent from others. Motivation comes also from the so-called paradox of Paretian liberal. As pointed out by Sen (1970, 1976, 1983), Gibbard (1974) and other subsequent works,[4] existence of decisive rights is incompatible with *Pareto efficiency*. Sen (1983, p.14) proposed studying this compatibility issue in restricted preferences domains. However, we show that the paradox prevails even on extremely restricted domains such as domains of trichotomous preferences or dichotomous preferences. Thus, unless we are not going to abandon *Pareto efficiency*, it is inevitable to think about weakening "decisiveness" component in the definition of rights and then address the following question. How much weakening is necessary for compatibility of *Pareto efficiency* and existence of a system of rights? Our answer is that only quasi-plurality system of rights can make the two requirements compatible.

In *qualification problems* studied by Samet and Schmeidler (2003), members of society decide who among themselves are qualified for a certain activity. Here there is an exogenous linkage between issues and persons, in fact, one to one correspondence. Central systems of rights conforming to this linkage give each member the right on his own qualification. Samet and Schmeidler (2003) characterize rules, called "consent rules", satisfying *monotonicity*, *independence*, and *symmetry* (social decision should not depend on names of members). This result can be interpreted as offering a necessary and sufficient condition for existence of a system of rights, namely, the combination of *monotonicity*, *independence*, and *symmetry*. *Symmetry* in Samet and Schmeidler (2003) is similar, in spirit, to the two standard axioms of social choice, called *anonymity* and *neutrality*. *Anonymity* says that names of opinion holders should not matter in the choice. *Neutrality* says that names of issues should not matter either, that is, when the names of two issues are switched, the social decision should also be switched accordingly. Since, in qualification problems, issues and persons are the same, renaming persons is naturally associated with renaming issues. So it is appealing to require that any "simultaneous renaming" of both persons and issues should not matter in the choice. This is exactly what *symmetry* requires in Samet and Schmeidler (2003). This definition cannot be extended directly in our general setting, since there is no exogenous linkage between issues and persons. We introduce a generalized notion of *symmetry*, called *symmetric linkage*. This axiom requires that there should be a linkage between issues and persons (a mapping from the set of issues to the set of persons) and each person i and issues linked to i should be treated symmetrically to person j and issues linked to j. Technically, when names of person i and all i's issues are switched simultaneously to names of person j and all j's issues, social choice should also be switched accordingly.

[4] See also Deb, Pattanaik, and Razzolini (1997) for the paradox in a framework where rights are represented as a game form.

We show that a rule satisfies *monotonicity*, *independence*, and *symmetric linkage* if and only if there is a well-defined system of rights and the system is unique. Adding *anonymity*, we establish a necessary and sufficient condition for existence of a *public system of rights* in which everyone has an equal right on every issue. Adding *neutrality* instead of *anonymity*, we characterize rules represented by either a constant public system of rights (constant consent quotas across issues) or a *monocentric system of rights* (one and only one person has rights on all issues). These results apply for both domains with trichotomous opinions and with dichotomous opinions. Finally, considering simple preference relations, dichotomous and trichotomous preference relations, we offer an axiomatic justification for plurality-like rules on the basis of *Pareto efficiency*, *independence*, and *symmetry* in a model with an exogenous linkage between issues and persons (e.g. the model in Samet and Schmeidler 2003).

The major difference between our model and the qualification problem in Samet and Schmeidler (2003) lies in the following two extensions. First, in our model, the set of issues may differ from the set of persons both in terms of elements and cardinality. There is no exogenous linkage between issues and persons. This generalization enables us to have much wider variety of applications. Second, we allow for "neutral opinion" and consider trichotomous opinions as well as dichotomous opinions considered by Samet and Schmeidler (2003). Thus, our definition of consent rules is much weaker than Samet and Schmeidler's. Consent rules are those rules represented by a well-defined system of rights. Thus, we allow for a wide spectrum of systems of rights, while Samet and Schmeidler's definition allows for systems conforming to the exogenous linkage. On the one extreme, we have monocentric systems of rights giving only a single person rights on all issues. On the other extreme, we have public systems of rights giving everyone equal right on every issue. We also find that on the trichotomous domain, consent rules may quite differ from plurality rule, while, on the dichotomous domain, they are close to plurality (or majority) rule. Much richer variety of consent rules emerge because of admissibility of neutral opinion. Capturing neutral opinion, we think, is natural because it is prevalent in realistic decision procedures (abstention can be viewed as an expression of neutral opinion).

The rest of the paper is organized as follows. In Section 2, we define our model and basic concepts. In Section 3, we introduce our notion of rights and systems of rights. In Section 4, we state our main results. We conclude with a few remarks in Section 5. Some proofs are collected in Appendix.

2 Model and Basic Concepts

Let $N \equiv \{1, \cdots, n\}$, $n \geq 2$, be the set of *persons* and $M \equiv \{1, \cdots, m\}$ the set of *issues*. Each person $i \in N$ has his *opinion* on issues in M, represented by a $1 \times m$ row vector P_i consisting of 1, 0, or -1. A *problem* is an $n \times m$ opinion matrix P consisting of n row vectors $P_1, \cdots, P_n$. An *alternative* is a list of either positive or negative decisions on all issues, formally, a vector of 1 and -1, $x \equiv (x_1, \cdots, x_k) \in \{-1, 1\}^M$, where 1 (resp. -1) in the k^{th} component means accepting the k^{th} issue (resp. rejecting the k^{th} issue). A *domain* is the set of problems which we are interested in resolving in a systematic manner. Throughout the paper, we consider the following two domains. One is the family of all opinion matrices, denoted by $\mathcal{P}$. Another is the family of those opinion matrices whose entries are either 1 or -1, denoted by $\mathcal{P}^*$. We call $\mathcal{P}$ and $\mathcal{P}^*$ the *trichotomous domain* and the *dichotomous domain*, respectively. Let $\mathcal{D}$ be either one of the two domains. The dichotomous domain is considered by Samet and Schmeidler (2003) in a special model of qualification problems.[5]

A social choice rule, or briefly, a *rule*, is a function $f \colon \mathcal{D} \to \{-1, 1\}^M$ associating with each opinion matrix a single alternative. Throughout the paper, we focus on rules satisfying the following two standard axioms.

The first axiom says that rules should not respond negatively when the opinion matrix increases.

Monotonicity. For each $P, P' \in \mathcal{D}$, if $P \geqq P'$, $f(P) \geqq f(P')$.

The second axiom says that decisions on different issues should be made independently, that is, decision on the k^{th} issue should rely only on the k^{th} column of the matrix. For each $P \in \mathcal{P}$, we denote the k^{th} column of P by P^k.

Independence. For each $P, P' \in \mathcal{D}$ and each $k \in M$, if $P^k = P'^k$, $f_k(P) = f_k(P')$.

We refer readers to Rubinstein and Fishburn (1986), Kasher and Rubinstein (1997), and Samet and Schmeidler (2003) for more discussion on the two axioms.

[5]Samet Schmeidler (2003) consider dichotomous opinions that are described by vectors of 1 and 0. Number 0 in their paper has the same meaning as -1 in this paper.

3 System of Rights and Uniqueness

To introduce our definition of rights, fix a rule f throughout this section. For each $P \in \mathcal{P}$ and each $k \in M$, let $||P_+^k|| \equiv \sum_{\{i \in N : P_{ik}=1\}} P_{ik}$ be the number of 1's in the k^{th} column vector P^k and $||P_-^k||$ the number of -1's in P^k.[6] We introduce our definitions, first, in the dichotomous domain, and then, in the trichotomous domain.

3.1 System of Rights on the Dichotomous Domain $\mathcal{P}^*$

On the dichotomous domain $\mathcal{P}^*$, person $i \in N$ has the "right to influence the social decision on the k^{th} issue", briefly, the *right on the k^{th} issue* if the decision on the k^{th} issue is made following person i's opinion whenever person i's opinion obtains sufficient consent from society: formally, there exist $q_+, q_- \in \{1, \cdots, n+1\}$ with $q_+ + q_- \leq n + 2$ such that for each $P \in \mathcal{P}^*$,

$$\begin{array}{l} \text{(i) when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_+; \\ \text{(ii) when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow ||P_-^k|| \geq q_-. \end{array} \tag{1}$$

The two numbers q_+ and q_- are called *consent-quotas.* The inequality condition $q_+ + q_- \leq n + 2$ is required because of *monotonicity*. Note that when $q_+ = n+1$ (resp. $q_- = n+1$), for any P with $P_{ik} = 1$ (resp. $P_{ik} = -1$), $f_k(P) = -1$ (resp. $f_k(P) = 1$).

Let $R\colon M \to N \times \{(q_+, q_-) : q_+, q_- \in \{1, \cdots, n+1\} \text{ and } q_+ + q_- \leq n+2\}$ be a function mapping each issue into a list consisting of the person who has the right on this issue and the pair of consent-quotas associated with this right. Thus R has two component functions $R_1\colon M \to N$ and $R_2\colon M \to \{(q_+, q_-) : q_+, q_- \in \{1, \cdots, n+1\} \text{ and } q_+ + q_- \leq n+2\}$. We say that R satisfies the *principle of horizontal equality of rights* if persons who are "equally situated" have "equal rights", formally, if for each pair of persons i and $j \in N$ with the same number of issues under R_1, that is, $|R_1^{-1}(i)| = |R_1^{-1}(j)|$, their rights are associated with the same pair of consent-quotas, that is, for each $k \in R_1^{-1}(i)$ and each $l \in R_1^{-1}(j)$, $R_2(k) = R_2(l)$.[7]

Definition 1 (System of Rights on $\mathcal{P}^*$). A *system of rights* representing a

[6] Clearly, when $P \in \mathcal{P}^*$, for each $k \in M$, $||P_+^k|| + ||P_-^k|| = n$.

[7] One may want to strengthen the principle by requiring that all rights should be associated with the same pair of consent-quotas. However, this will exclude some interesting hierarchical rights distributions where persons may have different consent-quotas depending on their positions in a hierarchy.

rule f is a function $R\colon M \to N \times \{(q_+, q_-) : q_+, q_- = 1, \cdots, n \text{ and } q_+ + q_- \leq n+2\}$ satisfying the principle of horizontal equality of rights such that for each $k \in M$, $R_1(k)$ has the right on the k^{th} issue and this right is associated with the pair of consent-quotas $R_2(k)$: that is, denoting $i \equiv R_1(k)$ and $(q_+, q_-) \equiv R_2(k)$, for each $P \in \mathcal{P}^*$,

$$\begin{aligned}&\text{(i) when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_+;\\&\text{(ii) when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow ||P_-^k|| \geq q_-.\end{aligned}$$

Given a system of rights R, for each pair $i, j \in N$, we write $i \sim_R j$ if the same number of issues are mapped into i and j, that is, $|R_1^{-1}(i)| = |R_1^{-1}(j)|$. Let N/R be the partition of N into these equivalence classes under $\sim_R$. Then R is a system of rights of f if and only if for each $G \in N/R$, there exist consent-quotas $q_{+G}, q_{-G} \in \{1, \cdots, n\}$ such that $q_{+G} + q_{-G} \leq n+2$, and for each $i \in G$ and each $k \in R_1^{-1}(i)$, person i's right on the k^{th} issue is associated with the pair of consent-quotas (q_{+G}, q_{-G}).

We show later that each rule can have at most one system of rights. Note that one implication of the principle of horizontal equality of rights is that when a person has the rights on two issues, the two rights should be associated with the same consent-quotas.

3.2 System of Rights on the Trichotomous Domain $\mathcal{P}$

On the trichotomous domain, person $i \in N$ has the *right on the k^{th} issue* if there exist three functions $q_+\colon \{1, \ldots, n\} \to \{1, \ldots, n+1\}$, $q_0\colon \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n\}$, and $q_-\colon \{1, \ldots, n\} \to \{1, \ldots, n+1\}$ such that for each $\nu \in \{1, \ldots, n\}$, $q_+(\nu)$, $q_-(\nu) \in \{1, \ldots, \nu+1\}$, and $q_0(\nu) \in \{0, \ldots, \nu+1\}$, and for each $P \in \mathcal{P}$ with $||P_+^k|| + ||P_-^k|| = \nu$,[8]

$$\begin{aligned}&\text{(i) when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_+(\nu);\\&\text{(ii) when } P_{ik} = 0,\ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_0(\nu);\\&\text{(iii) when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow ||P_-^k|| \geq q_-(\nu).\end{aligned} \tag{2}$$

Let $q \equiv (q_+, q_0, q_-)$ be the *consent-quota function* (with a slight abuse of notation).[9] We further assume that the consent-quota function satisfies the following

[8]Note that 0 is not in the range or in the domain of q_+ and q_-. This is because the two functions are useful only when the total number of negative or positive votes is not zero.

[9]Since the three component functions q_+, q_0, q_- have different domains, q cannot be described as a function. But, including 0 in the domain of q_+ and q_- and defining the values at 0 arbitrarily will not make any difference and, this way, the problem can be avoided.

conditions: for each $\nu \in \{0, 1, \ldots, n\}$,

$$\begin{aligned}
&\text{C(i) } q_+(\nu + 1) \leq q_0(\nu) + 1; \\
&\text{C(ii) } q_0(\nu - 1) + q_-(\nu) \leq \nu + 1; \\
&\text{C(iii) } q_+(\nu) + q_-(\nu) \leq \nu + 2;
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
&\text{B(i) } q_+(\nu) \leq q_+(\nu + 1) \leq q_+(\nu) + 1; \\
&\text{B(ii) } q_0(\nu) \leq q_0(\nu + 1) \leq q_0(\nu) + 1; \\
&\text{B(iii) } q_-(\nu) \leq q_-(\nu + 1) \leq q_-(\nu) + 1.
\end{aligned} \tag{4}$$

These conditions, C(i)-C(iii) and B(i)-B(iii), are needed because of *monotonicity*. They correspond to the inequality condition $q_+ + q_- \leq n + 2$ for consent-quotas on the dichotomous domain. Let Q be the set of all consent-quota functions satisfying C(i)-C(iii) and B(i)-B(iii). Note that for each $\nu \in \{1, \ldots, n\}$, when $q_+(\nu) = \nu + 1$ (resp. $q_-(\nu) = \nu + 1$) in (2), for any P with $P_{ik} = 1$ (resp. $P_{ik} = -1$), $f_k(P) = -1$ (resp. $f_k(P) = 1$) and that for each $\nu \in \{0, \ldots, n-1\}$, when $q_0(\nu) = \nu + 1$, for any P with $P_{ik} = 0$, $f_k(P) = -1$.

Let $R\colon M \to N \times Q$ be a function mapping each issue into a list consisting of the person who has the right on this issue and the consent-quota function associated with this right. As in Section 3.1, R has two component functions $R_1\colon M \to N$ and $R_2\colon M \to Q$. We say that R satisfies the *principle of horizontal equality of rights* if for each pair of persons i and $j \in N$ with the same number of issues under R_1, that is, $|R_1^{-1}(i)| = |R_1^{-1}(j)|$, their rights are associated with the same consent-quota function, that is, for each $k \in R_1^{-1}(i)$ and each $l \in R_1^{-1}(j)$, $R_2(k) = R_2(l)$.

Definition 2 (System of Rights on $\mathcal{P}$). A *system of rights representing a rule* f is a function $R\colon M \to N \times Q$ satisfying the principle of horizontal equality of rights such that for each $k \in M$, $R_1(k)$ has the right on the k^{th} issue and this right is associated with the consent-quota function $R_2(k)$: that is, denoting $i \equiv R_1(k)$ and $(q_+(\cdot), q_0(\cdot), q_-(\cdot)) \equiv R_2(k)$, for each $\nu \in \{0, 1, \ldots, n\}$ and each $P \in \mathcal{P}$ with $||P_+^k|| + ||P_-^k|| = \nu$,

$$\begin{aligned}
&\text{(i) when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_+(\nu); \\
&\text{(ii) when } P_{ik} = 0,\ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_0(\nu); \\
&\text{(iii) when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow ||P_-^k|| \geq q_-(\nu).
\end{aligned}$$

When f is represented by a system of rights, we call f a *consent rule* (Samet and Schmeidler 2003). Evidently, the above definitions on the trichotomous domain reduces to the definitions on the dichotomous domain.

3.3 Private and Public Rights, and Uniqueness of System of Rights

We distinguish two types of rights, "private" and "public". The right on the k^{th} issue is *private* if there is an agent who has the right on the k^{th} issue and no one else has the right on the k^{th} issue. It is *public* if all agents have the "equal" right on the k^{th} issue associated with a single consent-quota function $q \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot)) \in Q$ (or, on the dichotomous domain, a list of consent-quotas (q_+, q_-)). Thus, when the right on the k^{th} issue is public, we can represent the rule using different, yet essentially the same, systems of rights just setting $R_1(k)$ differently (it can be set arbitrarily). Two rights distributions R and R' for a rule are *equivalent*, denoted by $R \equiv_r R'$, if for each k with $R_1(k) \neq R'_1(k)$, the right on the k^{th} issue is *public* (thus, $R_2(k) = R'_2(k)$). The following two extreme systems of rights are notable. Under a *public system of rights*, everyone has the public right on every issue. Under a *monocentric system of rights*, one and only one agent has the private right on every issue.

Proposition 1. *Let f be a rule represented by a system of rights R. Then*
(i) *The right on the k^{th} issue is public if and only if for each $\nu \in \{1, \ldots, n\}$, $q_+(\nu) + q_-(\nu) = \nu + 1$ and for each $\nu \in \{1, \ldots, n-1\}$, $q_+(\nu) = q_0(\nu)$.*
(ii) *For each $k \in M$, the right on the k^{th} issue is either private or public.*
(iii) *The system of rights is unique up to the equivalence relation $\equiv_r$.*

The proof is in Appendix A.

4 Results

4.1 Characterization Results Imposing Monotonicity, Independence, and Symmetric Linkage

We generalize the "symmetry" axiom considered by Samet and Schmeidler [9]. This generalization is weaker than the combination of the two standard axioms known as *anonymity* and *neutrality*. For each permutation π on N, let ${}_\pi P$ be such that for each $i \in N$ and each $k \in M$, ${}_\pi P_{ik} \equiv P_{\pi(i)k}$. For each permutation δ on M, let ${}^\delta P$ be such that for each $i \in N$ and each $k \in M$, ${}^\delta P_{ik} \equiv P_{i\delta(k)}$.

Anonymity. For each $P \in \mathcal{P}$ and each permutation $\pi \colon N \to N$, $f({}_\pi P) = f(P)$.

Neutrality. For each $P \in \mathcal{P}$, each permutation $\delta\colon M \to M$, and each $k \in M$, $f_k\left({}^{\delta}P\right) = f_{\delta(k)}(P)$.

The two axioms require symmetric treatment of agents and symmetric treatment of issues, respectively. Our next axiom is weaker than the combination of the two axioms. It requires that there be a linkage between issues and persons and that each person and the issues linked to him should be treated symmetrically to any other person and the issues linked to this person. More precisely, when names of person i and all issues linked to i are switched to names of person j and all issues linked to j, then social choice should switch accordingly. Let $\rho\colon M \to N$ be a linkage between issues and persons. Let $\pi\colon N \to N$ and $\delta\colon M \to M$ are permutations on N and M such that δ maps the set of each person i's issues *onto* the set of person $\pi(i)$'s issues. Let ${}^{\delta}_{\pi}P$ be the matrix such that for each $i \in N$ and each $k \in M$, ${}^{\delta}_{\pi}P_{ik} \equiv P_{\pi(i)\delta(k)}$. Then each person i and his issue k play the same role in ${}^{\delta}_{\pi}P$ as person $\pi(i)$ and his issue $\delta(k)$ do in P. The next axiom requires that the decision on the k^{th} issue at ${}^{\delta}_{\pi}P$ be the same as the decision on the $\delta(k)^{\text{th}}$ issue at P.

Symmetric Linkage. There exists $\rho\colon M \to N$ such that for each $\pi\colon N \to N$ and each $\delta\colon M \to M$, if for each $i \in N$, δ maps the set of i's issues $\rho^{-1}(i)$ *onto* the set of $\pi(i)$'s issues $\rho^{-1}(\pi(i))$, then for each $k \in M$, $f_k\left({}^{\delta}_{\pi}P\right) = f_{\delta(k)}(P)$.

Clearly, the combination of *anonymity* and *neutrality* implies *symmetric linkage*. Given a function $\rho\colon M \to N$, we say that a rule f satisfies *ρ-symmetry* if f satisfies *symmetric linkage* with respect to $\rho\colon M \to N$. Note that if $\pi(i) = j$ and $|\rho^{-1}(i)| \neq |\rho^{-1}(j)|$, then there is no permutation $\delta\colon M \to M$ satisfying the ontoness condition for δ stated in the definition of *symmetric linkage*. Thus, *ρ-symmetry* does not impose any restriction for such π. In particular, if $\rho^{-1}(i) = M$, *ρ-symmetry* applies to only those permutations on N not changing the name of i and all permutations on M.[10]

Now we are ready to state our results. The first result is that the combination of *monotonicity*, *independence*, and *symmetric linkage* is necessary and sufficient for existence of a system of rights.

Theorem 1. *Let $\mathcal{D} \in \{\mathcal{P}^*, \mathcal{P}\}$. A rule on $\mathcal{D}$ satisfies monotonicity, independence, and symmetric linkage if and only if there is a system of rights representing the rule. Moreover, the system is unique under the equivalence relation $\equiv_r$.*

[10] In the qualification problems considered by Samet and Schmeidler (2003), $M = N$. Thus there is an exogenous one-to-one correspondence between M and N, namely the identity function $\rho^{\text{ID}}(i) = i$, for each $i \in M$. Their symmetry axiom coincides with ρ^{ID}-*symmetry*.

The proof is in the Appendix B. Adding *anonymity*, we obtain:

Proposition 2. *Let $\mathcal{D} \in \{\mathcal{P}^*, \mathcal{P}\}$. A rule on $\mathcal{D}$ satisfies monotonicity, independence, symmetric linkage, and anonymity if and only if it is represented by a public system of rights.*

Proof. Let $k \in M$ and $i \equiv R_1(k)$. By *anonymity*, when i has the right on the k^{th} issue, then every other agent should have the same right. Thus by Proposition 1, the right on the k^{th} issue is public. The proof for the reverse direction is straightforward. ∎

Combining this proposition and Proposition 1, we obtain:

Theorem 2. *Let $\mathcal{D} \in \{\mathcal{P}^*, \mathcal{P}\}$. The followings are equivalent.*
(i) *A rule on $\mathcal{D}$ satisfies monotonicity, independence, symmetric linkage, and anonymity.*
(ii) *A rule on $\mathcal{D}$ is represented by a system of rights R such that for each $k \in M$, letting $(q_+(\cdot), q_0(\cdot), q_-(\cdot)) \equiv R_2(k)$, for each $\nu \in \{1, \ldots, n\}$, $q_+(\nu) + q_-(\nu) = \nu + 1$ and for each $\nu \in \{1, \ldots, n-1\}$, $q_+(\nu) = q_0(\nu)$.*
(iii) *A rule on $\mathcal{D}$ is represented by a public system of rights.*

On the dichotomous domain $\mathcal{P}^*$, the condition on rights distributions, stated in part (ii) of Theorem 2, can be simplified and we obtain:

Corollary 1. *On the dichotomous domain $\mathcal{P}^*$, the followings are equivalent.*
(i) *A rule on $\mathcal{P}^*$ satisfies monotonicity, independence, symmetric linkage, and anonymity.*
(ii) *A rule on $\mathcal{P}^*$ is represented by a system of rights R such that for each $k \in M$, letting $(q_+, q_-) \equiv R_2(k)$, $q_+ + q_- = n + 1$.*
(iii) *A rule on $\mathcal{P}^*$ is represented by a public system of rights.*

Adding *neutrality* to the three axioms of Theorem 1, we characterize two extreme systems, monocentric and public systems of rights.

Theorem 3. *Let $\mathcal{D} \in \{\mathcal{P}^*, \mathcal{P}\}$. A rule on $\mathcal{D}$ satisfies monotonicity, independence, symmetric linkage, and neutrality if and only if it is represented either by a monocentric system of rights or by a constant public system of rights.*

Proof. If f is represented by a monocentric system of rights, then by the principle of horizontal equality, the consent quota functions for all issues are identical and one and only one agent has the right on each issue. Hence decisions on different issues are made neutrally. If f is represented by a constant public system of rights R, because of constancy of R, f satisfies *neutrality*.

To prove the converse, let f be a rule satisfying the stated axioms. By Theorem 1, there is a system of rights R representing f. Suppose that there is $i \in N$ who has a private right on the k^{th} issue. Then by *neutrality*, i should have the private right on every other issue and all rights are associated with the same consent quota functions. Therefore, the system of rights is monocentric. If there is no private right, then by Proposition 1, the system is public. And by *neutrality*, it is constant. ∎

We next consider *duality* (Samet and Schmeidler 2003). Each issue may be defined as representing a certain statement (a proposal) or its negation (the antiproposal): for example, qualification or disqualification. Which representation is taken does not matter for rules satisfying *duality*.

Duality. For each $P \in \mathcal{P}$, $f(-P) = -f(P)$.

On the trichotomous domain $\mathcal{P}$, *duality* is incompatible with the combination of the three axioms in Theorem 1. For example, if f is a rule satisfying the three axioms in Theorem 1, then for each $i \in N$, each $k \in \rho^{-1}(i)$, and each $P \in \mathcal{P}$ with $P_{ik} = 0$ and $||P^k_+|| = ||P^k_-||$, $f_k(-P) = f_k(P)$, violating *duality*. However, on the dichotomous domain $\mathcal{P}^*$, adding *duality*, we are able to pin down a smaller family of rules. A system of rights R has *R-duality* if for each issue $m \in M$, its consent-quotas function $R_2(m) \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ has $q_+(\cdot) = q_-(\cdot)$.

Theorem 4. *On the dichotomous domain $\mathcal{P}^*$, a rule satisfies monotonicity, independence, symmetric linkage, and duality if and only if it is represented by a system of rights with R-duality.*

Proof. Let f be a rule and R a system of rights of f such that for each $k \in M$, if we let $(q_+, q_-) \equiv R_2(k)$, $q_+ = q_-$. Let $i \in N$ and $k \in R_1^{-1}(i)$. Let $P \in \mathcal{P}^*$. Note $(-P)_{ik} = -P_{ik}$, $||(-P)^k_+|| = ||P^k_-||$, and $||(-P)^k_-|| = ||P^k_+||$. Therefore, $||(-P)^k_-|| \geq q_- \Leftrightarrow ||P^k_+|| \geq q_+$ and $||(-P)^k_+|| \geq q_+ \Leftrightarrow ||P^k_-|| \geq q_-$. Then $f(-P) = -f(P)$. Hence f satisfies *duality*.

Conversely, let f be a rule satisfying the four axioms. By Theorem 1, there exists a system of rights R representing f. Let $k \in M$, $i \equiv R_1(k)$, and $(q_+, q_-) \equiv R_2(k)$. Suppose, by contradiction, that $q_+ \neq q_-$, say, $q_+ > q_-$ (the same argument applies when $q_+ < q_-$). Let r be the number such that $q_+ > r \geq q_-$. Then there exists $P \in \mathcal{P}^*$ such that $P_{ik} = -1$ and $||P^k_-|| = r$. Then $f_k(P) = -1$. Since

$(-P)_{ik} = 1$ and $||\,(-P)^k_+\,|| = ||P^k_-|| = r < q_+$, $f_k\,(-P) = -1$, contradicting *duality*. ∎

When n is even, there is no system of rights R satisfying both conditions stated in Corollary 1 and Theorem 4. However, when n is odd, we obtain a characterization of majority rule as a corollary to Theorems 2 and 4.

Corollary 2. *Assume that the number of persons, n, is odd. On the dichotomous domain $\mathcal{P}^*$, majority rule is the only rule satisfying monotonicity, independence, symmetric linkage, anonymity, and duality.*

When we consider *neutrality* instead of *anonymity*, we obtain a characterization of the family consisting of majority rule and rules represented by monocentric systems of rights with R-duality.

Corollary 3. *Assume that the number of persons, n, is odd. On the dichotomous domain $\mathcal{P}^*$, a rule satisfies monotonicity, independence, symmetric linkage, neutrality, and duality if and only if it is majority rule or it is represented by a monocentric system of rights with R-duality.*

Proof. To prove the nontrivial direction, let f be a rule satisfying the stated axioms. Then by Theorem 3, it is represented either by a monocentric system of rights or by a constant public system of rights. In the former case, we are done. In the latter case, the rule satisfies *anonymity*. Thus it follows from Corollary 2 that f is majority rule. ∎

Dropping Symmetric Linkage

Dropping *symmetric linkage*, we characterize the following rules satisfying *monotonicity* and *independence*. These rules can be described by "power structures" between subgroups of N (Ju 2003). Let $\mathfrak{C}^* \equiv \{(C_1, C_2) \in 2^N \times 2^N : C_1 \cap C_2 = \varnothing\}$ be the set of all pairs of disjoint subgroups of N. For each $k \in M$, a *power structure for the k^{th}-issue*, denoted by $\mathfrak{C}_k \subseteq \mathfrak{C}^*$, is a subset of $\mathfrak{C}^*$. It satisfies *power monotonicity* if for each $(C_1, C_2) \in \mathfrak{C}_k$, if $(C_1', C_2') \in \mathfrak{C}^*$ is such that $C_1' \supseteq C_1$ and $C_2' \subseteq C_2$, then $(C_1', C_2') \in \mathfrak{C}_k$. A *profile of power structures* is a list $(\mathfrak{C}_k)_{k \in M}$ of power structures indexed by issues. For each $P \in \mathcal{P}$ and each $k \in M$, let

$$\begin{aligned} N\left(P^k_+\right) &\equiv \{i \in N : P_{ik} = 1\}; \\ N\left(P^k_-\right) &\equiv \{i \in N : P_{ik} = -1\}. \end{aligned}$$

A rule f is *represented by a profile* $(\mathfrak{C}_k)_{k\in M}$ if for each $P \in \mathcal{D}$ and each $k \in M$, $f_k(P) = 1$ if and only if $\left(N\left(P_+^k\right), N(P_-^k)\right) \in \mathfrak{C}_k$. Any rule represented by a profile of power structures satisfies *independence*, since it makes decisions issue by issue. Conversely, if a rule satisfies *independence*, the decision on the k^{th} issue relies only on the pair of the set of persons in favor of k and the set of persons against k. Thus, it is represented by a profile of power structures. Power monotonicity of power structures is a necessary and sufficient condition for *monotonicity*. Therefore we obtain:

Proposition 3. *Let $\mathcal{D} \in \{\mathcal{P}^*, \mathcal{P}\}$.*
(i) *A rule on $\mathcal{D}$ satisfies independence if and only if it is represented by a profile of power structures.*
(ii) *A rule on $\mathcal{D}$ satisfies independence and monotonicity if and only if it is represented by a profile of power structures satisfying power monotonicity.*

The proof is left for readers.

Let $\mathcal{I}^* \equiv \{(n_1, n_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : n_1 + n_2 \leq n\}$, where $\mathbb{Z}_+$ is the set of non-negative integers. Any subset $\mathcal{I} \subseteq \mathcal{I}^*$ is an *index set*. It is *comprehensive* if for each $(n_1, n_2) \in \mathcal{I}$ and each $(n_1', n_2') \in \mathcal{I}^*$, if $n_1' \geq n_1$ and $n_2' \leq n_2$, then $(n_1', n_2') \in \mathcal{I}$. Using Proposition 3, it is easy to characterize rules satisfying *independence* and *anonymity*. The power structures of each of these rules can be described by index sets. Formally, a *counting rule* is a rule that is represented by a profile of index sets, $(\mathcal{I}_k)_{k\in M}$, as follows: for each $P \in \mathcal{P}$ and each $k \in M$, $f_k(P) = 1 \Leftrightarrow (||P_+^k||, ||P_-^k||) \in \mathcal{I}_k$. It is easy to show that a counting rule is *monotonic* if and only if all index sets in the profile $(\mathcal{I}_k)_{k\in M}$ representing the rule are comprehensive. Thus, we obtain:

Proposition 4. *Let $\mathcal{D} \in \{\mathcal{P}^*, \mathcal{P}\}$.*
(i) *A rule on $\mathcal{D}$ satisfies independence and anonymity if and only if it is a counting rule.*
(ii) *A rule on $\mathcal{D}$ satisfies monotonicity, independence, and anonymity if and only if it is a counting rule represented by a profile of comprehensive index sets.*

The proof is left for readers.

Dropping Monotonicity

An *extended system of rights* ${}_eR$ maps each issue $k \in M$ into a person ${}_eR_1(k) \in N$ and a triple of index sets ${}_eR_2(k) = \left(\mathcal{I}_k^+, \mathcal{I}_k^0, \mathcal{I}_k^-\right)$ such that for each $i, j \in N$ with $|{}_eR_1^{-1}(i)| = |{}_eR_1^{-1}(j)|$, each $k \in {}_eR_1^{-1}(i)$, and each $l \in {}_eR_1^{-1}(j)$,

${}_eR_2\left(k\right) = {}_eR_2\left(l\right)$. A rule f is *represented by an extended system of rights* ${}_eR$ if for each $P \in \mathcal{P}$ and each $k \in M$,

$$\begin{aligned}
&\text{(i) when } P_{ik} = 1,\ f_k\left(P\right) = 1 \Leftrightarrow \left(||P_+^k||, ||P_-^k||\right) \in \mathcal{I}_k^+;\\
&\text{(ii) when } P_{ik} = 0,\ f_k\left(P\right) = 1 \Leftrightarrow \left(||P_+^k||, ||P_-^k||\right) \in \mathcal{I}_k^0;\\
&\text{(iii) when } P_{ik} = -1,\ f_k\left(P\right) = -1 \Leftrightarrow \left(||P_-^k||, ||P_+^k||\right) \in \mathcal{I}_k^-;
\end{aligned}$$

where $i \equiv {}_eR_1\left(k\right)$ and $\left(\mathcal{I}_k^+, \mathcal{I}_k^0, \mathcal{I}_k^-\right) \equiv {}_eR_2\left(k\right)$.

Proposition 5. *Let $\mathcal{D} \in \{\mathcal{P}^*, \mathcal{P}\}$. A rule over $\mathcal{D}$ satisfies independence and symmetric linkage if and only if it is represented by an extended system of rights.*

The proof is in Appendix B.

Dropping Independence

For each $P \in \mathcal{P}$, let $\chi\left(P\right) \equiv \sum_{k\in M} ||P_-^k||/|M|$. Let f be the rule represented by $\chi\left(\cdot\right)$ as follows: for each $P \in \mathcal{P}$ and each $k \in M$,

$$f_k\left(P\right) = 1 \Leftrightarrow ||P_+^k|| \geq \chi\left(P\right).$$

By definition, this rule treats agents anonymously and issues neutrally. Thus it satisfies *anonymity*, *neutrality*, and so *symmetric linkage*. If $P, P' \in \mathcal{P}$ are such that for each $k \in M$, $N\left(P_+^k\right) \subseteq N\left(P_+'^k\right)$ and $N\left(P_-^k\right) \supseteq N\left(P_-'^k\right)$, $\sum_{k\in M} ||P_-^k||/|M| \geq \sum_{k\in M} ||P_-'^k||/|M|$, that is, $\chi\left(P\right) \geq \chi\left(P'\right)$. Then for each $k \in M$, if $f_k\left(P\right) = 1$ (that is, $||P_+^k|| \geq \chi\left(P\right)$), $||P_+'^k|| \geq ||P_+^k|| \geq \chi\left(P\right) \geq \chi\left(P'\right)$ and so $f_k\left(P'\right) = 1$. Thus f satisfies *monotonicity*. The threshold level $\chi\left(P\right)$ depends on opinions on all issues. So f violates *independence*. Using different $\chi\left(\cdot\right)$, we can define other examples of rules violating *independence* but satisfying other axioms. However, we leave it for future research to characterize the family of rules satisfying *monotonicity* and *symmetric linkage*.

Anonymity and Representation by A Public System of Rights

Any rule f represented by a public system of rights R is a monotonic counting rule. This can be shown by constructing a profile of index sets as follows. By part (i) of Proposition 1, the three parts of (2) collapse into the following condition: for each $P \in \mathcal{P}$ and each $k \in M$ with $||P_+^k|| + ||P_-^k|| = \nu$,

$$f_k\left(P\right) = 1 \Leftrightarrow ||P_+^k|| \geq q_+\left(\nu\right).$$

Let $\mathcal{I}_k \equiv \{(n_1, \nu - n_1) : n_1 \geq q_+(\nu), \nu = 1, \ldots, n\}$. Then the above condition is equivalent to

$$f_k(P) = 1 \Leftrightarrow (||P_+^k||, ||P_-^k||) \in \mathcal{I}_k.$$

Thus f is the counting rule represented by $(\mathcal{I}_k)_{k \in M}$.

Is every (*monotonic*) counting rule represented by a public system of rights? Not necessarily. Given a public system of rights R, the set of issues M can be partitioned into subsets $K \subseteq M$ such that for any two issues $k, l \in K$ in the same element of the partition, $R_1(k)$ and $R_1(l)$ (possibly, $R_1(k) = R_1(l)$) are linked to the same number of issues under R_1, that is, $|R_1^{-1}(R_1(k))| = |R_1^{-1}(R_1(l))|$. Then any two issues k, l in the same element of this partition are associated with the same consent-quota functions. But consent-quota functions for different elements of the partition may differ. Thus any counting rule with at most n different index sets is represented by a public system of rights. To explain this, map each pair $k, l \in M$ with the same index set $\mathcal{I}$ into one person, which is possible because there are at most n index sets, and then set the three consent-quota functions $q_+(\cdot)$, $q_0(\cdot)$, and $q_-(\cdot)$ $(R_2(k) = R_2(l) = (q_+(\cdot), q_0(\cdot), q_-(\cdot)))$ as follows: for each $\nu \in \{1, \ldots, n\}$, let

$$\begin{aligned}
&q_+(\nu) \equiv \begin{cases} \min\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}\}, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}\} \neq \emptyset; \\ \nu + 1, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}\} = \emptyset. \end{cases} \\
&q_0(\nu) \equiv q_+(\nu), \\
&q_-(\nu) \equiv \nu + 1 - q_+(\nu),
\end{aligned}$$

and

$$q_0(0) = \begin{cases} 0, \text{ if } (0,0) \in \mathcal{I}, \\ 1, \text{ if } (0,0) \notin \mathcal{I}. \end{cases}$$

Then by comprehensiveness of $\mathcal{I}$, $R(\cdot)$ satisfies C(i)-C(iii) and B(i)-B(iii). If a counting rule has more than n index sets, we cannot find a function R_1 mapping each pair $k, l \in M$ associated with the same index set into one person because we are short of persons. Thus *every counting rule with at most n different index sets is represented by a public system of rights. Hence when $m \leq n$, every counting rule is represented by a public system of rights*; *so, independence and anonymity together imply symmetric linkage.*

Models with Exogenous Linkage between Issues and Persons

In this section, we apply our results to the model considered by Samet and Schmeidler (2003) and its generalization. Assume that there is an exogenous linkage between issues and persons, that is, a function $\rho\colon M \to N$. A rule is *symmetric* if it satisfies *ρ-symmetry*. When $M = N$ and ρ is the identity function, *symmetry* coincides with the definition by Samet and Schmeidler (2003). Replacing *symmetric linkage* in all our results with *symmetry*, we obtain characterizations of subfamilies of rules represented by systems of rights $R(\cdot)$ conforming to the exogenous linkage, that is, $R_1(\cdot) = \rho(\cdot)$.

Suppose that $\rho(\cdot)$ is not constant. Then no system of rights conforming to $\rho(\cdot)$ can be monocentric. Thus, it follows from Theorem 3 that a rule over $\mathcal{D} \in \{\mathcal{P}, \mathcal{P}^*\}$ satisfies *monotonicity*, *independence*, *symmetry*, and *neutrality* if and only if it is represented by a constant public system of rights conforming to the exogenous linkage ρ. Thus these four axioms together imply *anonymity*. Also it follows from Corollary 3 that when n is odd, majority rule is the only rule over $\mathcal{P}^*$ satisfying *monotonicity*, *independence*, *symmetry*, *neutrality*, and *duality*.

4.2 Pareto Efficiency and Existence of A System of Rights

Compatibility of *Pareto efficiency* and existence of a system of rights is widely studied by a number of authors followed by the celebrated work, Sen (1970). To discuss this issue in our framework, we now turn our attention to the domains of preference relations. Opinions are partial description of the following preference relations.

A *separable preference relation* R_0 orders social decisions in such a way that for each quadruple $x, x', y, y' \in \{-1,1\}^M$, if there is $k \in M$ such that $x_k = y_k$, $x'_k = y'_k$, $x_{-k} = x'_{-k}$, and $y_{-k} = y'_{-k}$,

$$\begin{aligned} x \succ_{R_0} x' \quad &\Leftrightarrow \quad y \succ_{R_0} y' \,; \\ x \sim_{R_0} x' \quad &\Leftrightarrow \quad y \sim_{R_0} y' \,, \end{aligned}$$

where $\succ_{R_0}$ and $\sim_{R_0}$ are strict and indifference relations associated with R_0. Then issues are partitioned into goods, bads, and nulls depending on whether they have positive or negative or indifferent impacts on the person's well-being. Thus, each separable preference R_0 is associated with an opinion vector P_0, each positive (resp. negative or zero) component of P_0 representing the corresponding issue as a good (resp. a bad or a null). Obviously, there are a number of separable preference relations corresponding to a single opinion vector. Let $\mathcal{R}$ be the family of profiles of separable preference relations. A *rule* over the separable preferences domain $\mathcal{R}$ associates with each profile of preference relations a single alternative in $\{-1,1\}^M$. With the stated relationship between opinions and preferences, axioms and rights defined for the opinion domain are easily extended to the corresponding notions on the separable preferences domain.

4.2.1 Sen's Paradox of Paretian Liberal

Sen (1970) shows in the Arrovian social choice model that no *Pareto efficient* preference aggregation rule gives at least two agents "libertarian rights". This is so-called Sen's paradox of Paretian liberal. Sen's reasoning does not directly apply here because of the following differences between our model and his. The alternative space, here, is a product space and, associated with this structure, preference relations have the separability restriction. In addition, while Sen (1970) considers preference aggregation rules, we consider social choice functions. Despite these differences, our notion of decisive rights is a natural counterpart to Sen's libertarian rights (in fact, our decisive rights are the same as rights formulated in Gibbard 1974; because we focus on separable preference relations, the so-called Gibbard paradox does not prevail in our model as pointed out by Sen 1983, p.14). Thus Sen's quest is still meaningful here. Does Sen's paradox prevail in our model? Not surprisingly, it does, as we show below. Furthermore, we show that the paradox prevails in a much stronger sense even after a substantial restriction on separable preference relations.

We first show that the paradox prevails on the separable preferences domain. An axiom corresponding to Sen's "minimal liberalism" postulates that there should be at least two persons who have decisive rights. Let us call the axiom, like Sen (1970), *minimal liberalism.*

Assume that persons 1 and 2 are given the decisive rights on the first and second issues respectively. Consider the following preference relations R_1 and R_2 of the two persons. The first issue is a bad for R_1 and any decision with the positive second component is preferred, under R_1, to any decision with the negative second component. The second issue is a bad for R_2 and any decision with the positive first component is preferred, under R_2, to any decision with the negative first component. Then by the decisive rights of the two persons, decisions on the first and second issues are both negative. But the two persons will be better off at any decision with positive components for both issues. This confirms that *minimal liberalism and Pareto efficiency are incompatible on the separable preferences domain.*

Preference relations in the above example are "meddlesome"' (Blau 1975); person 1 has such an extremely positive opinion on person 2's issue that positive decision on this issue is preferred to the negative decision no matter what decisions are made on the other issues. In environments without such meddlesome preference relations, the paradox of Paretian liberal may not apply.

Unfortunately, the paradox prevails even in a substantially restricted environment where only "trichotomous" or "dichotomous" preference relations are admissible. A *trichotomous preference relation* R_0 is a separable preference relation represented by a function $U_0\colon \{-1,1\}^M \to \mathbb{R}$ such that for each $x \in \{-1,0,1\}^M$, $U_0(x) = \sum_{k\in M: x_k=1} P_{0k}$, where $P_0 \in \{-1,0,1\}^M$ is the opinion vector corresponding to R_0.[11] A *dichotomous preference relation* is a trichotomous preference relation for which each issue is either a good or a bad. Let $\mathcal{R}_{\mathrm{Tri}}$ be the family of profiles of trichotomous preference relations and $\mathcal{R}_{\mathrm{Di}}$ the family of profiles of dichotomous preference relations. Note that there are one-to-one correspondences between $\mathcal{R}_{\mathrm{Tri}}$ and $\mathcal{P}$ and between $\mathcal{R}_{\mathrm{Di}}$ and $\mathcal{P}^*$.

To show the paradox, suppose that persons 1 and 2 have the decisive rights respectively on issues 1 and 2. Consider a profile of dichotomous preference relations $(R_i)_{i\in N}$ with the corresponding profile of opinion vectors $(P_i)_{i\in N}$ such that $P_1 \equiv (1,-1,-1,\ldots,-1)$, $P_2 \equiv (-1,1,-1,\ldots,-1)$, and for each $i \in N\backslash\{1,2\}$, $P_i \equiv (-1,\ldots,-1)$. Then by the decisive rights of persons 1 and 2, $f_1(R) = f_2(R) = 1$. If the rule is *Pareto efficient*, for each $k \in M\backslash\{1,2\}$, $f_k(R) = -1$. Thus $f(R) = (1,1,-1,\ldots,-1)$. Note that this alternative is indifferent to $x \equiv (-1,\ldots,-1)$ for both person 1 and person 2 and x is preferred to $f(R)$ by all others. This contradicts *Pareto efficiency*. Therefore, *when there are at least three persons, no Pareto efficient rule on the dichotomous preferences domain satisfies minimal liberalism.* Note that unlike the previous paradox on the separable preferences domain, we need the assumption on the number of persons. The case with two persons ruled out by this assumption is very limited. However, it should be noted that the paradox does not apply when there are only two persons (then decisiveness is quite close to plurality principle since one person's opinion accounts for 50%). This is an easy corollary to our results in the next section.

4.2.2 Quasi-Plurality Rules

The observations made in Section 4.2.1 show that decisiveness component in the definition of libertarian rights is extremely strong, unless we give up *Pareto efficiency*. They force us to consider non-decisive rights instead. Is it, then, possible to have non-decisive rights and at the same time to satisfy *Pareto efficiency*? It is indeed possible on the trichotomous preferences domain $\mathcal{R}_{\mathrm{Tri}}$ and also on the

[11] That is, $U_0(x) = |\{k \in M : x_k = 1 \text{ and } P_{0k} = 1\}| - |\{k \in M : x_k = 1 \text{ and } P_{0k} = -1\}|$.

dichotomous preferences domain $\mathcal{R}_{\mathrm{Di}}$ as we show in this section. Moreover, we offer an interesting characterization of plurality-like rules on the basis of *Pareto efficiency*, *independence*, and *symmetry* (or *symmetric linkage*).

Since we only consider trichotomous or dichotomous preference relations, throughout this section, we use opinion vectors to refer to the corresponding preference relations.

We will show that the following systems of rights are compatible with *Pareto efficiency*.

Definition 3 (Quasi-Plurality Systems of Rights). A *quasi-plurality system of rights* R has three functions $q_+ \colon \{1,\dots,n\} \to \{1,\dots,n+1\}$, $q_0 \colon \{0,\dots,n-1\} \to \{0,1,\dots,n+1\}$, and $q_- \colon \{1,\dots,n\} \to \{1,\dots,n+1\}$ such that for each $\nu \in \{1,\dots,n\}$,

$$q_+(\nu), q_-(\nu) \in \{\frac{\nu-1}{2}, \frac{\nu+1}{2}\}, \tag{5}$$

for each $\nu \in \{0,\dots,n-1\}$,

$$q_0(\nu) \in \{\frac{\nu-1}{2}, \frac{\nu+1}{2}\}, \tag{6}$$

and for each $k \in M$, $R_2(k) = (q_+(\cdot), q_0(\cdot), q_-(\cdot))$.

Any rule f represented by a quasi-plurality system of rights R has the following property: for each $k \in M$,

$$\begin{aligned} f_k(P) = 1 \ &\Rightarrow \ ||P_+^k|| \geq ||P_-^k||; \\ ||P_+^k|| > ||P_-^k|| \ &\Rightarrow \ f_k(P) = 1. \end{aligned} \tag{7}$$

Obviously, plurality rule is an example; it is represented by a public quasi-plurality system of rights. Quasi-plurality systems of rights are not always public. For example, for each $\nu \in \{1,\dots,n\}$, let $q_+(\nu) = q_-(\nu) \equiv (\nu-1)/2$ and for each $\nu \in \{0,\dots,n-1\}$, let $q_0(\nu) \equiv (\nu-1)/2$. Then the right on each issue is private by Proposition 1. In fact, for each $k \in M$, if $||P_+^k|| \neq ||P_-^k||$, $f_k(P)$ equals the decision by plurality rule; if $||P_+^k|| = ||P_-^k||$, $f_k(P)$ is determined by the opinion of the person, say i, who has the right on the k^{th} issue (that is, $f_k(P) = 1$ if $P_{ik} = 1$ or 0; $f_k(P) = -1$ if $P_{ik} = -1$). Thus "privateness" matters only when there is a tie between the group of persons with the positive opinion and the group of persons with the negative opinion.

Note that $\sum_{i\in N} U_i(x) = \sum_{i\in N} \sum_{\{k\in M : x_k = 1\}} P_{ik} = \sum_{\{k\in M : x_k=1\}} \left(||P_+^k|| - ||P_-^k||\right)$. Therefore, by (7), any rule represented by a quasi-plurality system of rights, maximizes the sum of utilities. Thus it satisfies *Pareto efficiency*. Note that C(i)-C(iii) in the definition of systems of rights are not used for proving *Pareto efficiency*. Dropping these properties lead us to a larger family of rules.

Definition 4 (Quasi-Plurality Rules). A *quasi-plurality rule* f is represented by $\rho\colon M \to N$ and $q_+\colon \{1,\dots,n\} \to \{1,\dots,n+1\}$, $q_0\colon \{0,\dots,n-1\} \to \{0,1,\dots,n+1\}$, and $q_-\colon \{1,\dots,n\} \to \{1,\dots,n+1\}$ satisfying (5) and (6) as follows: for each $P \in \mathcal{P}$ and each $k \in M$, letting $i \equiv \rho(k)$ and $\nu \equiv ||P_+^k|| + ||P_-^k||$,

$$\begin{aligned}&\text{if } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_+(\nu)\,;\\ &\text{if } P_{ik} = 0,\ f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_0(\nu)\,;\\ &\text{if } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow ||P_-^k|| \geq q_-(\nu)\,.\end{aligned} \tag{8}$$

Any rule represented by a quasi-plurality system of rights is a quasi-plurality rule. But a quasi-plurality rule is not necessarily represented by a quasi-plurality system of rights because C(i)-C(iii) are not necessarily guaranteed. Any quasi-plurality rule satisfies (7). Thus, as shown above, it maximizes the sum of utilities and satisfies *Pareto efficiency*. Moreover, our next result shows that they are the only rules satisfying *Pareto efficiency*, *independence*, and *symmetry* in the environment with a fixed linkage between issues and persons satisfying a certain property.

Theorem 5. *Assume that there is a fixed linkage ρ between issues and persons and that the number of issues linked to a person is constant across persons. Then quasi-plurality rules associated with ρ are the only rules over $\mathcal{D} \in \{\mathcal{R}_{Tri}, \mathcal{R}_{Di}\}$ satisfying Pareto efficiency, independence, and symmetry.*

The proof is in Appendix C. Note that the model considered by Samet and Schmeidler (2003) is among many examples in which the theorem applies.

Adding *neutrality* allows us to establish a similar characterization without any assumption on the exogenous linkage. An *anonymous quasi-plurality rule* is a quasi-plurality rule associated with $q_+(\cdot)$, $q_0(\cdot)$, and $q_-(\cdot)$ such that for each ν, $q_+(\nu) = q_0(\nu)$ and $q_+(\nu) + q_-(\nu) = \nu + 1$. A *monocentric quasi-plurality rule* is a quasi-plurality rule associated with a linkage $\rho\colon M \to N$ such that $|\rho(M)| = 1$ and for some ν, $q_+(\nu) \neq q_0(\nu)$ or $q_+(\nu) + q_-(\nu) \neq \nu + 1$. Note that both anonymous and monocentric quasi-plurality rules satisfy *neutrality*. We show that they are the only rules satisfying *Pareto efficiency*, *independence*, *symmetric linkage*, and *neutrality*.

Theorem 6. *A rule on $\mathcal{D} \in \{\mathcal{R}_{Tri}, \mathcal{R}_{Di}\}$ satisfies Pareto efficiency, independence, symmetric linkage, and neutrality if and only if it is either an anonymous or a* *monocentric quasi-plurality rule.*

The proof is in Appendix C. As a direct corollary, we obtain:

Corollary 4. *A rule on $\mathcal{D} \in \{\mathcal{R}_{Tri}, \mathcal{R}_{Di}\}$, represented by a system of rights, satisfies Pareto efficiency if and only if the system(s) of rights representing the rule is a quasi-plurality system.*

5 Concluding remarks

According to our definition, rights distributions are functions on the *entire* set of issues M. So all issues have persons who have the rights on these issues. A shortcoming is that our definition does not capture the case when persons have rights only on a subset of issues and no one has the right on any other issue. Our results, however, can be modified to deal with this more moderate definition. Let R be a function defined on a subset M^* of M. Let us call R a *generalized system of rights* if for each $k \in M^*$, $R_1(k)$ has the right on the k^{th} issue. Then we can show that for each rule, the maximal M^* exists uniquely.[12] Call M^* the domain of rights. Accordingly, we weaken *symmetric linkage*, replacing "there exists $R_1\colon M \to N$" with "there exists a domain of rights M^* and a function $R_1\colon M^* \to N$" in the definition of *symmetric linkage*. With these modifications, our results will continue to hold.

As we explained in front of Theorem 4, every rule represented by a system of rights on the trichotomous domain violates *duality*. Such a violation occurs for opinion matrices P with the property that for some $k \in M$, $||P^k_+|| = ||P^k_-||$. Thus it is natural to weaken *duality* to the following:

Weak Duality. For each $P \in \mathcal{D}$, if there is no $k \in M$, $||P^k_+|| = ||P^k_-||$, $f(-P) = -f(P)$.

Replacing *duality* in both Theorem 4 and Corollary 2 with *weak duality*, we can obtain similar characterization results on the trichotomous domain.

A Proof of Proposition 1

Fact 1. *Let $R \in \mathfrak{S}_f$, $k \in M$ and $R_2(k) \equiv (i, q(\cdot))$. If for each $\nu \in \{1, \dots, n\}$, $q_+(\nu) + q_-(\nu) = \nu + 1$ and for each $\nu \in \{1, \dots, n-1\}$, $q_+(\nu) = q_0(\nu)$, then the*

[12]If there are two sets M and M' over which rights distributions are defined, then, by *independence*, we can show that a rights distribution is defined over $M \cup M'$.

right on the k^{th} issue is public.

Proof. Under the stated assumption, the three parts (i)-(iii) in (2) collapse into the following condition: for each $\nu \in \{0, 1, \ldots, n\}$ and each $P \in \mathcal{P}$ with $||P_+^k|| + ||P_-^k|| = \nu$, when $\nu \in \{1, \ldots, n\}$,

$$f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq q_+(\nu), \tag{$\star$}$$

and when $\nu = 0$,

$$f_k(P) = 1 \Leftrightarrow q_0(0) = 0.$$

This condition is anonymous and so everyone has the right on the k^{th} issue associated with $q(\cdot)$. ∎

Fact 2. *If $R, R' \in \mathfrak{S}_f$, $R_1(k) \neq R_1'(k)$, $R_2(k) = q(\cdot)$, and $R_2'(k) = q'(\cdot)$, then for each $\nu \in \{1, \ldots, n\}$,*

$$q_+(\nu) + q_-'(\nu) > \nu \text{ and } q_+'(\nu) + q_-(\nu) > \nu.$$

Proof. The inequalities hold trivially for $\nu = 1$. Let $\nu \in \{2, \ldots, n\}$. Let $R_1(k) \equiv i$ and $R_1'(k) \equiv i'$. Suppose by contradiction $q_+(\nu) + q_-'(\nu) \leq \nu$. Since $\nu \geq 2$, there exists $P \in \mathcal{P}$ be such that $P_{ik} = 1$, $P_{i'k} = -1$, $||P_+^k|| + ||P_-^k|| = \nu$, and $||P_+^k|| = q_+(\nu)$. Then $||P_-^k|| = \nu - q_+(\nu) \geq q_-'(\nu)$. Since $P_{ik} = 1$, $R(k) = (i, q(\cdot))$, and $||P_+^k|| = q_+(\nu)$, then $f_k(P) = 1$. On the other hand, since $P_{i'k} = -1$, $R'(k) = (i', q'(\cdot))$, and $||P_-^k|| = \nu - q_+(\nu) \geq q_-'(\nu)$, then $f_k(P) = -1$, contradicting $f_k(P) = 1$. Similarly, we show $q_+'(\nu) + q_-(\nu) > \nu$. ∎

Fact 3. *Let $R, R' \in \mathfrak{S}_f$ be such that $R_1(k) \neq R_1'(k)$. Let $q(\cdot) \equiv R_2(k)$ and $q'(\cdot) \equiv R_2'(k)$. For each $\nu \in \{1, \ldots, n\}$, if $q_+(\nu) = q_+'(\nu)$ and $q_-(\nu) = q_-'(\nu)$, then $q_+(\nu) + q_-(\nu) = \nu + 1$. If, in addition, $q_0(\nu) = q_0'(\nu)$ and $\nu \in \{1, \ldots, n-1\}$, then $q_+(\nu) = q_0(\nu)$.*

Proof. Let $R_1(k) = i$ and $R_1'(k) = i'$. The proof is composed of two steps.

Step 1. For each $\nu \in \{1, \ldots, n\}$, if $q_+(\nu) = q_+'(\nu)$ and $q_-(\nu) = q_-'(\nu)$, then $q_+(\nu) + q_-(\nu) = \nu + 1$.

By Fact 2 and C(3) (that is, $q_+(\nu) + q_-(\nu) \leq \nu + 2$), we have $q_+(\nu) + q_-(\nu) = \nu + 1$ or $\nu + 2$. Suppose $q_+(\nu) + q_-(\nu) = \nu + 2$. Let $R_1(k) \equiv i$ and $R_1'(k) \equiv i'$. Let $P \in \mathcal{P}$ be such that $P_{ik} = 1$, $P_{i'k} = -1$, $||P_+^k|| + ||P_-^k|| = \nu$, and $||P_+^k|| = \nu - q_-(\nu) + 1$ (since $q_+(\nu), q_-(\nu) \in \{1, \ldots, \nu\}$ and $q_+(\nu) + q_-(\nu) = \nu + 2$, then $q_+(\nu) \geq 2$; thus $\nu - q_-(\nu) + 1 = q_+(\nu) - 1 \geq 1$ and such a problem P exists).

Then $||P_+^k|| = \nu - q_-(\nu) + 1 = q_+(\nu) - 1 < q_+(\nu)$ and $||P_-^k|| = q_-(\nu) - 1 < q_-(\nu)$. Since $P_{ik} = 1$, $R(k) = (i, q(\cdot))$, and $||P_+^k|| < q_+(\nu)$, then $f_k(P) = -1$. Since $P_{i'k} = -1$, $R'(k) = (i', q(\cdot))$, and $||P_-^k|| = \nu - ||P_+^k|| = q_-(\nu) - 1 < q_-(\nu)$, then $f_k(P) = 1$, contradicting $f_k(P) = -1$.

Step 2. If, in addition, $q_0(\nu) = q_0'(\nu)$ and $\nu \in \{1, \ldots, n-1\}$, then $q_+(\nu) = q_0(\nu)$.

Suppose $q_+(\nu) < q_0(\nu)$ (then $1 \leq q_+(\nu) \leq \nu - 1$ and $1 \leq q_0(\nu) \leq \nu$). Then there is $P \in \mathcal{P}$ such that $P_{ik} = 1$, $P_{i'k} = 0$, and $||P_+^k|| = q_+(\nu)$. Using i's right on the k^{th} issue associated with $q(\cdot)$, we obtain $f_k(P) = 1$. On the other hand, using i''s right on the k^{th} issue, we obtain $f_k(P) = -1$, contradicting $f_k(P) = 1$.

Now suppose $q_+(\nu) > q_0(\nu)$. We consider two cases one by one.

Case 1. $q_0(\nu) = 0$. Then whenever $q_+(\nu) \geq 2$, there is $P \in \mathcal{P}$ such that $P_{ik} = 0$, $P_{i'k} = 1$, $||P_+^k|| = 1$, and $||P_-^k|| = \nu - 1$. Since $P_{ik} = 0$ and $||P_+^k|| = 1 \geq q_0(\nu) = 0$, then by i's right on the k^{th} issue, $f_k(P) = 1$. Since $P_{i'k} = 1$ and $||P_+^k|| = 1 < q_+(\nu)$, then by i''s right on the k^{th} issue, $f_k(P) = -1$, contradicting $f_k(P) = 1$.

Therefore, $q_+(\nu) = 1$ and by Step 1, $q_-(\nu) = \nu$. Since $\nu \leq n-1$, there is $\bar{P} \in \mathcal{P}$ such that $\bar{P}_{ik} = 0$, $\bar{P}_{i'k} = -1$, $||\bar{P}_-^k|| = \nu$, and $||\bar{P}_+^k|| = 0$. Since $\bar{P}_{ik} = 0$ and $q_0(\nu) = 0$, then by i's right on the k^{th} issue, $f_k\left(\bar{P}\right) = 1$. Since $\bar{P}_{i'k} = -1$ and $||\bar{P}_-^k|| = \nu = q_-(\nu)$, then by i''s right on the k^{th} issue, $f_k\left(\bar{P}\right) = -1$, contradicting $f_k\left(\bar{P}\right) = 1$.

Case 2. $q_0(\nu) \geq 1$. Then there is $P \in \mathcal{P}$ such that $P_{ik} = 1$, $P_{i'k} = 0$, $||P_+^k|| = q_0(\nu)$, and $||P_+^k|| + ||P_-^k|| = \nu$. Then by i's right on the k^{th} issue, $f_k(P) = -1$. On the other hand, by i''s right on the k^{th} issue, $f_k(P) = 1$, contradicting $f_k(P) = -1$. ∎

Fact 4. *Let $R, R' \in \mathfrak{S}_f$, $R_1(k) \neq R_1'(k)$, $R_2(k) = q(\cdot)$, and $R_2'(k) = q'(\cdot)$. For each $\nu \in \{1, \ldots, n\}$, if $(q_+(\nu), q_-(\nu)) \neq \left(q_+'(\nu), q_-'(\nu)\right)$, then*
(i) *$q_+(\nu) + q_-(\nu) = \nu + 2$ or $q_+'(\nu) + q_-'(\nu) = \nu + 2$;*
(ii) *one of the following four conditions holds*

$$\begin{aligned}
\text{(ii.1) } q_+'(\nu) &= q_+(\nu) + 1 \textit{ and } q_-'(\nu) = q_-(\nu) - 1;\\
\text{(ii.2) } q_+'(\nu) &= q_+(\nu) - 1 \textit{ and } q_-'(\nu) = q_-(\nu) + 1;\\
\text{(ii.3) } q_+'(\nu) &= q_+(\nu) - 1 \textit{ and } q_-'(\nu) = q_-(\nu) - 1;\\
\text{(ii.4) } q_+'(\nu) &= q_+(\nu) + 1 \textit{ and } q_-'(\nu) = q_-(\nu) + 1.
\end{aligned}$$

Proof. *Part* (i). If $q_+(\nu) + q_-(\nu) \leq \nu$ and $q_+'(\nu) + q_-'(\nu) \leq \nu$, then either $q_+(\nu) + q_-'(\nu) \leq \nu$ or $q_+'(\nu) + q_-(\nu) \leq \nu$, contradicting Fact 2. Thus $q_+(\nu) +$

$q_-(\nu) \geq \nu+1$ or $q'_+(\nu)+q'_-(\nu) \geq \nu+1$. Since $(q_+(\nu), q_-(\nu)) \neq \left(q'_+(\nu), q'_-(\nu)\right)$, then by Fact 1, $q_+(\nu)+q_-(\nu) \neq \nu+1$ and $q'_+(\nu)+q'_-(\nu) \neq \nu+1$. Thus, $q_+(\nu)+q_-(\nu) = \nu+2$ or $q'_+(\nu)+q'_-(\nu) = \nu+2$.

Part (ii). Suppose $q_+(\nu)+q_-(\nu) = \nu+2$ (the same argument applies when $q'_+(\nu)+q'_-(\nu) = \nu+2$). If $q'_+(\nu)+q'_-(\nu) \leq \nu-1$, then $q_+(\nu)+q_-(\nu)+q'_+(\nu)+q'_-(\nu) \leq 2\nu+1$. Then $q_+(\nu)+q'_-(\nu) \leq \nu+1/2$ or $q'_+(\nu)+q_-(\nu) \leq \nu+1/2$; that is, $q_+(\nu)+q'_-(\nu) \leq \nu$ or $q'_+(\nu)+q_-(\nu) \leq \nu$, contradicting Fact 2. Thus $q'_+(\nu)+q'_-(\nu) = \nu$ or $\nu+2$. In the former case, by Fact 2, we obtain (ii.3). In the latter case, we obtain (ii.1) or (ii.2). ∎

Now we are ready to prove Proposition 1.

Proof of Proposition 1. Part (i) follows directly from Fact 1 and Fact 3. Part (iii) follows directly from part (ii). To prove part (ii), we only have to show that for each $R, R' \in \mathfrak{S}_f$ and each $k \in M$, if $R_1(k) \neq R'_1(k)$, then $R_2(k) = R'_2(k) : \equiv q(\cdot)$ and for each $\nu \in \{0, 1, \ldots, n\}$, $q_+(\nu)+q_-(\nu) = \nu+1$ and $q_+(\nu) = q_0(\nu)$.

Let $R, R' \in \mathfrak{S}_f$. Let $k \in M$. Let $q(\cdot) \equiv R_2(k)$, $q'(\cdot) \equiv R'_2(k)$, $i \equiv R_1(\cdot)$ and $i' \equiv R'_1(k)$. Assume $i \neq i'$.

Claim 1. For each $\nu \in \{1, \ldots, n\}$, $q_+(\nu)+q_-(\nu) = \nu+1$, $q'_+(\nu)+q'_-(\nu) = \nu+1$, $q_+(\cdot) = q'_+(\cdot)$, and $q_-(\cdot) = q'_-(\cdot)$.

Proof. We only have to show that for each $\nu \in \{1, \ldots, n\}$, $q_+(\nu)+q_-(\nu) = \nu+1$ and $q'_+(\nu)+q'_-(\nu) = \nu+1$ (then by part (i) of Fact 4, $q_+(\cdot) = q'_+(\cdot)$ and $q_-(\cdot) = q_-(\cdot)$). Suppose by contradiction that for some $\nu \in \{1, \ldots, n\}$, $q_+(\nu)+q_-(\nu) \neq \nu+1$. Then by Fact 3, $(q_+(\nu), q_-(\nu)) \neq \left(q'_+(\nu), q'_-(\nu)\right)$. Thus by Fact 4, part (i), $q_+(\nu)+q_-(\nu) = \nu+2$ or $q'_+(\nu)+q'_-(\nu) = \nu+2$.

Suppose $q_+(\nu)+q_-(\nu) = \nu+2$ (we use the same argument for the other case). Then one of the three cases (ii.1), (ii.2), and (ii.3) in part (ii) of Fact 4 applies (this is shown in the proof of part (ii) of Fact 4).

Consider case (ii.1). That is, $q'_+(\nu) = q_+(\nu)-1$ and $q'_-(\nu) = q_-(\nu)+1$ (the same argument applies for case (ii.2)). Since $q_+(\nu)+q_-(\nu) = \nu+2$, then $q_+(\nu) \geq 2$ and $q_-(\nu) \geq 2$, (if not, either $q_+(\nu) > \nu$ or $q_-(\nu) > \nu$, which is not possible). Then $q'_+(\nu) \geq 1$ and $q'_-(\nu) \geq 3$. Let $P \in \mathcal{P}^*$ be such that $P_{ik} = P_{i'k} = -1$ and $||P^k_-|| = q_-(\nu) (= q'_-(\nu)-1)$ (since $q_-(\nu) \geq 2$, such P exists). Since $R(k) = (i, (q_+(\nu), q_-(\nu)))$, $P_{ik} = -1$, and $||P^k_-|| = q_-(\nu)$, then $f_k(P) = -1$. Since $R'(k) = \left(i', \left(q'_+(\nu), q'_-(\nu)\right)\right)$, $P_{i'k} = -1$, and $||P^k_-|| = q_-(\nu) = q'_-(\nu)-1 < q'_-(\nu)$, then $f_k(P) = 1$, contradicting $f_k(P) = -1$.

Now, consider case (ii.3). That is, $q'_+(\nu) = q_+(\nu) - 1$ and $q'_-(\nu) = q_-(\nu) - 1$. If $q'_+(\nu) \geq 2$, there is P such that $P_{ik} = P_{i'k} = 1$ and $||P^k_+|| = q'_+(\nu)$. Since $R(k) = (i, (q_+(\nu), q_-(\nu)))$, $P_{ik} = 1$, and $||P^k_+|| = q'_+(\nu) = q_+(\nu) - 1 < q_+(\nu)$, then $f_k(P) = -1$. Since $R'(k) = \left(i', \left(q'_+(\nu), q'_-(\nu)\right)\right)$, $P_{i'k} = 1$, and $||P^k_+|| = q'_+(\nu)$, then $f_k(P) = 1$, contradicting $f_k(P) = -1$. If $q'_+(\nu) = 1$, then $q_+(\nu) = 2$, $q_-(\nu) = \nu$, and $q'_-(\nu) = \nu - 1$. Then there is P such that $P_{ik} = P_{i'k} = -1$ and $||P^k_-|| = \nu - 1$. Since $R(k) = (i, (q_+(\nu), q_-(\nu)))$, $P_{ik} = -1$, and $||P^k_-|| = \nu - 1 < q_-(\nu)\,(= \nu)$, then $f_k(P) = 1$. Since $R'(k) = \left(i', \left(q'_+(\nu), q'_-(\nu)\right)\right)$, $P_{i'k} = -1$, and $||P^k_-|| = \nu - 1 = q'_-(\nu)$, then $f_k(P) = -1$, contradicting $f_k(P) = 1$. ♦

Claim 2. For each $\nu \in \{1, \ldots, n-1\}$, $q_+(\nu) = q_0(\nu)$ and $q'_+(\nu) = q'_0(\nu)$.

Proof. Suppose by contradiction that for some $\nu \in \{1, \ldots, n-1\}$, $q_+(\nu) \neq q_0(\nu)$. If $q_0(\nu) = 0$, then by C(i) and Claim 1, $q_+(\nu) = 1$ and $q_-(\nu) = \nu$. Let P be such that $P_{ik} = 0$, $P_{i'k} = -1$, $||P^k_+|| = 0$, and $||P^k_-|| = \nu$. Then by i's right on the k^{th} issue, $f_k(P) = 1$. By i''s right, $f_k(P) = -1$, contradicting $f_k(P) = 1$. Hence $q_0(\nu) \geq 1$.

Suppose that $q_+(\nu) > q_0(\nu)$. Then by C(i), $q_+(\nu) = q_0(\nu) + 1$. Let P be such that $P_{ik} = 0$, $P_{i'k} = 1$, and $||P^k_+|| = q_0(\nu)$. Then by i's right, $f_k(P) = 1$. By i''s right, $f_k(P) = -1$, a contradiction. Suppose that $q_+(\nu) < q_0(\nu)$. Let P be such that $P_{ik} = 0$, $P_{i'k} = 1$, and $||P^k_+|| = q_+(\nu)$. Then by i''s right, $f_k(P) = 1$. By i's right, $f_k(P) = -1$, contradicting $f_k(P) = 1$. ♦

The two claims complete the proof of part (ii). ∎

B Proof of Theorem 1

Let $\mathcal{I}^* \equiv \{(n_1, n_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : n_1 + n_2 \leq n\}$, where $\mathbb{Z}_+$ is the set of non-negative integers. A subset $\mathcal{J} \subseteq \mathcal{I}^*$ is *comprehensive* if for each $(n_1, n_2) \in \mathcal{J}$ and each $(n'_1, n'_2) \in \mathcal{I}^*$, if $n'_1 \geq n_1$ and $n'_2 \leq n_2$, then $(n'_1, n'_2) \in \mathcal{J}$.

Lemma 1. *A person $i \in N$ has the right on the k^{th} issue if and only if there exist three comprehensive subsets of $\mathcal{I}^*$, denoted by $\mathcal{I}^+$, $\mathcal{I}^0$, and $\mathcal{I}^-$, such that for each $P \in \mathcal{P}$,*

$$\begin{array}{ll}\text{(i) } \textit{when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow (||P^k_+||, ||P^k_-||) \in \mathcal{I}^+; & \\ \text{(ii) } \textit{when } P_{ik} = 0,\ f_k(P) = 1 \Leftrightarrow (||P^k_+||, ||P^k_-||) \in \mathcal{I}^0; & (\star) \\ \text{(iii) } \textit{when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow (||P^k_-||, ||P^k_+||) \in \mathcal{I}^-, & \end{array}$$

and

$$\text{C*(i)}\ \ (n_1,n_2)\in\mathcal{I}^0\Rightarrow(n_1+1,n_2)\in\mathcal{I}^+;$$
$$\text{C*(ii)}\ \ (n_1,n_2)\notin\mathcal{I}^-\Rightarrow(n_2,n_1-1)\in\mathcal{I}^0;$$
$$\text{C*(iii)}\ \ (n_1,n_2)\notin\mathcal{I}^-\Rightarrow(n_2+1,n_1-1)\in\mathcal{I}^+.$$

Proof. The proof is composed of two steps.

Step 1. *Suppose that person $i\in N$ has the right on the k^{th} issue. Let $q(\cdot)$ be the consent-quota function. For each $s\in\{+,0,-\}$, let $\mathcal{I}^s\equiv\{(n_1,n_2)\in\mathcal{I}^*:n_1\geq q_s(n_1+n_2)\}$. Then* (2)*,* C(i)-C(iii)*, and* D(i)-D(iii) *imply* ($\star$)*,* C*(i)-C*(iii)*, and comprehensiveness of the three sets $\mathcal{I}^+$, $\mathcal{I}^0$, and $\mathcal{I}^-$.*

If $P_{ik}=1$ and $f_k(P)=1$, then by (2), $||P_+^k||\geq q_+\left(||P_+^k||+||P_-^k||\right)$. Hence $(||P_+^k||,||P_-^k||)\in\mathcal{I}^+$. Conversely, if $(||P_+^k||,||P_-^k||)\in\mathcal{I}^+$, $||P_+^k||\geq q_+\left(||P_+^k||+||P_-^k||\right)$. Hence by (†), $f_k(P)=1$. Similarly, we can show the remaining two parts of ($\star$).

Suppose $(n_1,n_2)\in\mathcal{I}^0$. Then $n_1\geq q_0(n_1+n_2)$. By C(i), $q_+(n_1+n_2+1)\leq q_0(n_1+n_2)+1\leq n_1+1$. Hence $(n_1+1,n_2)\in\mathcal{I}^+$. So C*(i).

Suppose $(n_1,n_2)\notin\mathcal{I}^-$. Then $n_1<q_-(n_1+n_2)$. By C(ii), $q_0(n_1+n_2-1)+q_-(n_1+n_2)\leq n_1+n_2+1$. Thus $q_0(n_1+n_2-1)+n_1<n_1+n_2+1$, that is, $q_0(n_1+n_2-1)\leq n_2$. This implies $(n_2,n_1-1)\in\mathcal{I}^0$. So C*(ii).

Suppose $(n_1,n_2)\notin\mathcal{I}^-$. Then $n_1<q_-(n_1+n_2)$. By C(iii), $q_+(n_1+n_2)+q_-(n_1+n_2)\leq n_1+n_2+2$. Thus $q_+(n_1+n_2)+n_1<n_1+n_2+2$, that is, $q_+(n_1+n_2)\leq n_2+1$. This implies $(n_2+1,n_1-1)\in\mathcal{I}^+$. So C*(iii).

Let $(n_1,n_2)\in\mathcal{I}^+$. To prove comprehensiveness of $\mathcal{I}^+$, we only have to show that if $(n_1+1,n_2)\in\mathcal{I}^*$, $(n_1+1,n_2)\in\mathcal{I}^+$ and if $(n_1,n_2-1)\in\mathcal{I}^*$, $(n_1,n_2-1)\in\mathcal{I}^+$. By definition of $\mathcal{I}^+$, $n_1\geq q_+(n_1+n_2)$. By the second inequality of B(i), $q_+(n_1+n_2+1)\leq q_+(n_1+n_2)+1\leq n_1+1$. Hence $(n_1+1,n_2)\in\mathcal{I}^+$. On the other hand by the first inequality of B(i), $q_+(n_1+n_2-1)\leq q_+(n_1+n_2)\leq n_1$. Hence $(n_1,n_2-1)\in\mathcal{I}^+$.

Similarly we prove comprehensiveness of the two remaining sets $\mathcal{I}^0$ and $\mathcal{I}^-$.

Step 2. *Let $\mathcal{I}^+$, $\mathcal{I}^0$, and $\mathcal{I}^-$ be the three comprehensive sets satisfying* ($\star$) *and* C*(i)-C*(iii)*. For each $\nu\in\{1,\ldots,n\}$ and each $s\in\{+,0,-\}$, let*

$$q_s(\nu)\equiv\begin{cases}\min\{n_1:(n_1,\nu-n_1)\in\mathcal{I}^s\},\text{ if }\{n_1:(n_1,\nu-n_1)\in\mathcal{I}^s\}\neq\emptyset;\\ \nu+1,\text{ if }\{n_1:(n_1,\nu-n_1)\in\mathcal{I}^s\}=\emptyset.\end{cases}$$

Then ($\star$)*,* C*(i)-C*(iii)*, and comprehensiveness of the three sets $\mathcal{I}^+$, $\mathcal{I}^0$, and $\mathcal{I}^-$ imply* (2)*,* C(i)-C(iii)*, and* D(i)-D(iii)*.*

Let $\nu\in\{1,\ldots,n\}$. When $\{n_1:(n_1,\nu-n_1)\in\mathcal{I}^+\}=\emptyset$, there is no P with $P_{ik}=1$, $||P_+^k||+||P_-^k||=\nu$, and $f(P)=1$. Thus if we set $q_+(\nu)=\nu+1$, then for each P with $P_{ik} = 1$ and $||P_+^k|| + ||P_-^k|| = \nu$, $||P_+^k|| < q_+(\nu) (= \nu + 1)$ and we obtain part (i) of (2). Now consider the case when $\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^+\} \neq \emptyset$. If $P_{ik} = 1$ and $f_k(P) = 1$, then by ($\star$), $(||P_+^k||, ||P_-^k||) \in \mathcal{I}^+$. Hence $q_+ \left(||P_+^k|| + ||P_-^k||\right) \leq ||P_+^k||$. Conversely, if $q_+ \left(||P_+^k|| + ||P_-^k||\right) \leq ||P_+^k||$, then since $(q_+ \left(||P_+^k|| + ||P_-^k||\right), ||P_+^k|| + ||P_-^k|| - q_+ \left(||P_+^k|| + ||P_-^k||\right)) \in \mathcal{I}^+$, by comprehensiveness of $\mathcal{I}^+$, $(||P_+^k||, ||P_-^k||) \in \mathcal{I}^+$. Hence $f(P) = 1$. Similarly, we can show the remaining two parts of (2).

Below we show C(i)-C(iii) and D(i)-D(iii) for the case when for each $s \in \{+, 0, -\}$, $\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^s\} \neq \emptyset$.

By C*(i), $(q_0(\nu) + 1, \nu - q_0(\nu)) \in \mathcal{I}^+$. Hence $q_+(\nu + 1) \leq q_0(\nu) + 1$. So C(i). Since $q_-(\nu) = \min\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^-\}$, $(q_-(\nu) - 1, \nu - q_-(\nu) + 1) \notin \mathcal{I}^-$. By C*(ii), $(\nu - q_-(\nu) + 1, q_-(\nu) - 1 - 1) \in \mathcal{I}^0$. Hence $q_0(\nu - 1) \leq \nu - q_-(\nu) + 1$. So C(ii). Since $(q_-(\nu) - 1, \nu - q_-(\nu) + 1) \notin \mathcal{I}_-$, then by C*(iii), $(\nu - q_-(\nu) + 1 + 1, q_-(\nu) - 1 - 1) \in \mathcal{I}^+$. Hence $q_+(\nu) \leq \nu - q_-(\nu) + 1 + 1$. So C(iii).

For each $s \in \{+, 0, -\}$, $(q_s(\nu), \nu - q_s(\nu)) \in \mathcal{I}^s$. By comprehensiveness of $\mathcal{I}^s$, $(q_s(\nu)+1, \nu - q_s(\nu)) \in \mathcal{I}^s$. Hence $q_s(\nu + 1) \leq q_s(\nu)+1$. Since $(q_s(\nu + 1), \nu + 1 - q_s(\nu + 1)) \in \mathcal{I}^s$, by comprehensiveness, $(q_s(\nu + 1), \nu - q_s(\nu + 1)) \in \mathcal{I}^s$. Hence $q_s(\nu) \leq q_s(\nu + 1)$.

Finally, consider the case when for some $s \in \{+, 0, -\}$, $\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^s\} = \emptyset$. Assume $\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^+\} = \emptyset$. Then $q_+(\nu) = \nu + 1$. Then the second inequality of B(i) holds trivially. By comprehensiveness of $\mathcal{I}^+$, there is no $n_1 \leq \nu$ such that $(n_1, \nu + 1 - n_1) \in \mathcal{I}^+$. Thus, $q_+(\nu + 1) \geq \nu + 1$. So the first inequality of B(i) also holds. To show $q_+(\nu) \leq q_0(\nu - 1) + 1$, note that by C*(i), $(\nu - 1, 0) \notin \mathcal{I}^0$ (since $(\nu, 0) \notin \mathcal{I}^+$). Thus $q_0(\nu - 1) \geq \nu$ and so we obtain the inequality we wanted to show. To show $q_+(\nu) + q_-(\nu) \leq \nu + 2$, note that by C*(iii), $(1, \nu - 1) \in \mathcal{I}^-$ (since $(\nu, 0) \notin \mathcal{I}^+$). Thus $q_-(\nu) \leq 1$ which implies the inequality we wanted to show. There for we obtain all conditions associated $q_+(\nu)$ in C(i)-C(iii) and B(i)-B(iii). The same argument can be used to show the remaining parts of C(i)-C(iii) and B(i)-B(iii). ∎

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The proof is composed of two steps corresponding to the "if" part and the "only if" part.

Step 1. *If a rule f is represented by a system of rights R, it satisfies monotonicity, independence, and symmetric linkage.*

Let f be represented by a rights distribution R. Let $\rho \equiv R_1$. Let N/ρ be the partition of N into subsets of persons with the same number of linked issues under ρ (that is for each $G \in N/\rho$ and each $i, j \in G$, $|\rho^{-1}(i)| = |\rho^{-1}(j)|$. By Lemma 1, for each $G \in N/R$, there exists a triple of comprehensive subsets of $\mathcal{I}^*$, $(\mathcal{I}_G^+, \mathcal{I}_G^0, \mathcal{I}_G^-)$ satisfying $(\star)$ and C*(i)-C*(iii).

To show that f satisfies *ρ-symmetry* (so *symmetric personal spheres*), let $\pi\colon N \to N$ be a permutation on N. Let $\delta\colon M \to M$ be a permutation on M such that for each $i \in N$, δ maps $\rho^{-1}(i)$ onto $\rho^{-1}(\pi(i))$. Then because of the ontoness of δ, i and $\pi(i)$ are associated with the same number of issues under ρ, that is, $i \sim_R \pi(i)$. Let $G \in N/\rho$ contain i (and $\pi(i)$). Let $k \in \rho^{-1}(i)$. Then the decision on the k^{th} issue at ${}^{\delta}_{\pi}P$ depends on i's opinion on the k^{th} issue, $||{}^{\delta}_{\pi}P^k_+||$, $||{}^{\delta}_{\pi}P^k_-||$, and the triple $(\mathcal{I}_G^+, \mathcal{I}_G^0, \mathcal{I}_G^-)$. Similarly, the decision on the $\delta(k)^{\text{th}}$ issue at P depends on $\pi(i)$'s opinion on the $\delta(k)^{\text{th}}$ issue, $||P^{\delta(k)}_+||$, $||P^{\delta(k)}_-||$, and the triple $(\mathcal{I}_G^+, \mathcal{I}_G^0, \mathcal{I}_G^-)$. Since ${}^{\delta}_{\pi}P_{ik} = P_{\pi(i)\delta(k)}$, $||P^{\delta(k)}_+|| = ||{}^{\delta}_{\pi}P^k_+||$, and $||P^{\delta(k)}_-|| = ||{}^{\delta}_{\pi}P^k_-||$, then $f_k\left({}^{\delta}_{\pi}P\right) = f_{\delta(k)}(P)$.

To show *monotonicity*, let $P' \geq P$ and $k \in M$ be such that $f_k(P) = 1$. Let $i \equiv \rho(k)$. Let $G \in N/R$ be such that $i \in G$. We only have to show that $f_k(P') = 1$. When $P'_{ik} = P_{ik}$, it follows directly from the comprehensiveness condition of the three sets that $f_k(P') = 1$. There are two cases remaining.

Case 1. $P_{ik} = 0 \neq P'_{ik}$ and $(||P^k_+||, ||P^k_-||) \in \mathcal{I}_G^0$. Then $P'_{ik} = 1$. Hence $||P'^k_+|| \geq ||P^k_+|| + 1$ and $||P'^k_-|| \leq ||P^k_-||$. By comprehensiveness and condition C*(i), $(||P'^k_+||, ||P'^k_-||) \in \mathcal{I}_G^+$. Therefore, $f_k(P') = 1$.

Case 2. $P_{ik} = -1 \neq P'_{ik}$ and $(||P^k_-||, ||P^k_+||) \notin \mathcal{I}_G^-$. Then either $P'_{ik} = 0$ or $P'_{ik} = 1$. If $P'_{ik} = 0$, $||P'^k_+|| \geq ||P^k_+||$ and $||P'^k_-|| \leq ||P^k_-|| - 1$. Then by comprehensiveness and condition C*(ii), $(||P'^k_+||, ||P'^k_-||) \in \mathcal{I}_G^0$. Thus, $f_k(P') = 1$. If $P'_{ik} = 1$, $||P'^k_+|| \geq ||P^k_+|| + 1$ and $||P'^k_-|| \leq ||P^k_-|| - 1$. Then by comprehensiveness and condition C*(iii), $(||P'^k_+||, ||P'^k_-||) \in \mathcal{I}_G^+$. Thus $f_k(P') = 1$.

Step 2. *If a rule satisfies monotonicity, independence, and symmetric linkage, it has a non-empty system of rights.*

Let f be a rule satisfying the three axioms. Then by Proposition 3, f is represented by a profile $(\mathfrak{C}_k)_{k \in M}$. By *symmetric linkage*, there exists $\rho\colon M \to N$ such that f satisfies *ρ-symmetry*. We now identify a system of rights of f in the following three substeps.

Step 2.1. *For each pair $i, j \in N$ with $|\rho^{-1}(i)| = |\rho^{-1}(j)|$, each $k \in \rho^{-1}(i)$, each $l \in \rho^{-1}(j)$, and each $(C_1, C_2), (C'_1, C'_2) \in \mathfrak{C}^*$ with $|C_1 \cap \{i\}| = |C'_1 \cap \{j\}|$ and $|C_2 \cap \{i\}| = |C'_2 \cap \{j\}|$ (or equivalently, $[i \in C_1 \Leftrightarrow j \in C'_1]$ and $[i \in C_2 \Leftrightarrow j \in C'_2]$), if $|C_1| = |C'_1|$ and $|C_2| = |C'_2|$, then $(C_1, C_2) \in \mathfrak{C}_k \Leftrightarrow (C'_1, C'_2) \in \mathfrak{C}_l$.*

Let $i, j \in N$, $k \in \rho^{-1}(i)$, $l \in \rho^{-1}(j)$, and $(C_1, C_2), (C_1', C_2') \in \mathfrak{C}^*$ be given as above. Consider the case $i \in C_1$ and $j \in C_1'$ (the proofs for the other cases are similar). Suppose $(C_1, C_2) \in \mathfrak{C}_k$. Let P be such that $N\left(P_+^k\right) \equiv C_1$ and $N\left(P_-^k\right) \equiv C_2$. So $f_k(P) = 1$. Let π be a permutation on N such that $\pi(i) = j$, $\pi(C_1) = C_1'$, and $\pi(C_2) = C_2'$ (since $|C_1| = |C_1'|$ and $|C_2| = |C_2'|$, such permutation π exists). Let δ be a permutation on M such that $\delta(k) = l$, $\delta(l) = k$, and for all other $k' \in M\backslash\{k, l\}$, $\delta(k') = k'$. Then $N\left({}^{\delta}_{\pi}P_+^l\right) = \pi^{-1}(N(P_+^{\delta(l)})) = \pi^{-1}(C_1) = C_1'$. Similarly, $N\left({}^{\delta}_{\pi}P_-^l\right) = C_2'$. By *$\rho$-symmetry*, $f_l\left({}^{\delta}_{\pi}P\right) = f_{\delta(l)}(P) = f_k(P) = 1$. Therefore, $(C_1', C_2') \in \mathfrak{C}_l$. The proof of the opposite direction is similar.

One notable implication of Step 2.1 is that for each $i \in N$ and each pair $k, l \in \rho^{-1}(i)$, $\mathfrak{C}_k = \mathfrak{C}_l$.

Step 2.2. *There is a system of rights representing f.*

Let N/ρ be the partition of N such that for each pair $i, j \in N$, i and j are in the same set $G \in N/\rho$ if and only if $|\rho^{-1}(i)| = |\rho^{-1}(j)|$. Let $G \in N/\rho$. Pick $i \in G$ and $k \in \rho^{-1}(i)$. Let $\mathcal{I}_G^+ \equiv \{(|C_1|, |C_2|) \colon (C_1, C_2) \in \mathfrak{C}_k \text{ and } i \in C_1\}$, $\mathcal{I}_G^0 \equiv \{(|C_1|, |C_2|) \colon (C_1, C_2) \in \mathfrak{C}_k \text{ and } i \notin C_1 \cup C_2\}$, and $\mathcal{I}_G^- \equiv \{(|C_2|, |C_1|) \colon (C_1, C_2) \notin \mathfrak{C}_k \text{ and } i \in C_2\}$. For each $j \in G$ and each $l \in \rho^{-1}(j)$, let $R_2(l) \equiv (\mathcal{I}_G^+, \mathcal{I}_G^0, \mathcal{I}_G^-)$. Let $R \equiv (R_1, R_2)$. Then by definition, R satisfies the principle of horizontal equality of rights. We next show that for each $P \in \mathcal{P}$, each $j \in G$, and each $l \in \rho^{-1}(j)$,

$$\begin{aligned}
\text{if } P_{jl} &= 1,\ f_l(P) = 1 \Leftrightarrow (||P_+^l||, ||P_-^l||) \in \mathcal{I}_G^+; && (9)\\
\text{if } P_{jl} &= 0,\ f_l(P) = 1 \Leftrightarrow (||P_+^l||, ||P_-^l||) \in \mathcal{I}_G^0; && (10)\\
\text{if } P_{jl} &= -1,\ f_l(P) = -1 \Leftrightarrow (||P_-^l||, ||P_+^l||) \in \mathcal{I}_G^-. && (11)
\end{aligned}$$

When $j = i$, Step 2.1 says that the decision on the k^{th} issue relies on person i's opinion, the number of agreeing persons, and the number of disagreeing persons. Therefore, since for each $l \in \rho^{-1}(i)$, $\mathfrak{C}_l = \mathfrak{C}_k$, then (9)-(11) hold when $j = i$. When $j \in G\backslash\{i\}$, Step 2.1 says that for each $l \in \rho^{-1}(j)$, the decision on the l^{th} issue is made symmetrically to the decision on the k^{th} issue. Therefore, (9)-(11) hold also for j and l.

We now prove that for each $G \in N/\rho$, each of the three sets $\mathcal{I}_G^+, \mathcal{I}_G^0, \mathcal{I}_G^-$ is comprehensive and they satisfy C*(i)-C*(iii). Comprehensiveness is a direct consequence of power monotonicity of power structures. To show property C*(i), let $(n_1, n_2) \in \mathcal{I}_G^0$. Suppose to the contrary $(n_1 + 1, n_2) \notin \mathcal{I}_G^+$. Let $i \in G$, $k \in \rho^{-1}(i)$, and $P \in \mathcal{P}$ be such that $P_{ik} = 0$, $||P_+^k|| = n_1$, and $||P_-^k|| = n_2$. Then $f_k(P) = 1$. Let P' have the same components as P except $P_{ik}' \equiv 1$. Then

$P' \geq P$, $||P'^k_+|| = n_1 + 1$, and $||P^k_-|| = n_2$. Since $(n_1 + 1, n_2) \notin \mathcal{I}^+_G$, $f_k(P') = -1$, contradicting *monotonicity*.

To show C*(ii), suppose to the contrary that $(n_1, n_2) \notin \mathcal{I}^-_G$ and $(n_2, n_1 - 1) \notin \mathcal{I}^0_G$. Let $i \in G$, $k \in \rho^{-1}(i)$, and $P \in \mathcal{P}$ be such that $P_{ik} = -1$, $||P^k_-|| = n_1$, and $||P^k_+|| = n_2$. Then $f_k(P) = 1$. Let P' have the same components as P except $P'_{ik} \equiv 0$. Then $P' \geq P$, $||P'^k_+|| = n_2$, and $||P^k_-|| = n_1 - 1$. Since $(n_2, n_1 - 1) \notin \mathcal{I}^0_G$, $f_k(P') = -1$, contradicting *monotonicity*.

To show C*(iii), suppose to the contrary that $(n_1, n_2) \notin \mathcal{I}^-_G$ and $(n_2 + 1, n_1 - 1) \notin \mathcal{I}^+_G$. Let $i \in G$, $k \in \rho^{-1}(i)$, and $P \in \mathcal{P}$ be such that $P_{ik} = -1$, $||P^k_-|| = n_1$, and $||P^k_+|| = n_2$. Then $f_k(P) = 1$. Let P' have the same components as P except $P'_{ik} \equiv 1$. Then $P' \geq P$, $||P'^k_+|| = n_2 + 1$, and $||P^k_-|| = n_1 - 1$. Since $(n_2 + 1, n_1 - 1) \notin \mathcal{I}^0_G$, $f_k(P') = -1$, contradicting *monotonicity*.

Finally, we construct a system of rights representing f. Let $R_1 \equiv \rho$. Let $G \in N/\rho$. For each $k \in M$ with $R_1(k) \in G$, define three functions $q_+ \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, $q_0 \colon \{0, 1, \ldots, n-1\} \to \{0, 1, \ldots, n-1\}$, and $q_- \colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ as follows. For each $\nu \in \{1, \ldots, n\}$, let

$$q_+(\nu) \equiv \begin{cases} \min\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^+_G\}, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^+_G\} \neq \emptyset; \\ \nu + 1, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^+_G\} = \emptyset, \end{cases}$$

and

$$q_-(\nu) \equiv \begin{cases} \min\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^-_G\}, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^-_G\} \neq \emptyset; \\ \nu + 1, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^-_G\} = \emptyset. \end{cases}$$

For each $\nu \in \{0, 1, \ldots, n - 1\}$, let

$$q_0(\nu) \equiv \begin{cases} \min\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^0_G\}, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^0_G\} \neq \emptyset; \\ \nu + 1, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}^0_G\} = \emptyset. \end{cases}$$

Let $R_2(k) \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot))$. Since each agents in the same partition of N/ρ have the same consent-quota functions, R satisfies the principle of horizontal equality of rights. Finally, it follows from Lemma 1 that f is represented by R. ∎

Proof of Proposition 5. Note that in Step 2 of the proof of Theorem 1, *monotonicity* plays a role only to show that index sets $\mathcal{I}^+_G$, $\mathcal{I}^0_G$, and $\mathcal{I}^-_G$ are comprehensive and satisfy C*(i)-C*(iii). Without these extra properties, R defined there, is an extended system of rights. ∎

C Proofs of Theorems 5 and 6

Proof of Theorem 5. Let $\rho\colon M \to N$ be the exogenous linkage. Let f be a rule over $\mathcal{P}$ (or $\mathcal{R}_{\text{Tri}}$, recall that we will treat each opinion matrix as a profile of trichotomous preference relations) satisfying the three axioms (the proof for $\mathcal{P}^*$ or $\mathcal{R}_{\text{Di}}$ is essentially the same). Without loss of generality, we assume $N \subseteq M$ (since the number of objects linked to a person is constant across persons, we may label at least one object by the label of the person linked to it) and for each $i \in \{1, \ldots, n\}$, $\rho(i) = i$. By Proposition 5 and the assumption on ρ, there exist three index sets $\mathcal{I}^+$, $\mathcal{I}^0$, and $\mathcal{I}^-$ such that for each $P \in \mathcal{P}$ and each $k \in M$, if $i \equiv \rho(k)$,

$$
\begin{aligned}
&\text{(i) when } P_{ik} = 1,\ f_k(R) = 1 \Leftrightarrow \left(||P^k_+||, ||P^k_-||\right) \in \mathcal{I}^+;\\
&\text{(ii) when } P_{ik} = 0,\ f_k(R) = 1 \Leftrightarrow \left(||P^k_+||, ||P^k_-||\right) \in \mathcal{I}^0;\\
&\text{(iii) when } P_{ik} = -1,\ f_k(R) = -1 \Leftrightarrow \left(||P^k_-||, ||P^k_+||\right) \in \mathcal{I}^-.
\end{aligned} \tag{12}
$$

Claim 1. For each $s \in \{+, 0, -\}$,

$$
\begin{aligned}
&\{(t_1, t_2) \in \mathcal{I}^* : t_1 > t_2\} \subseteq \mathcal{I}^s;\\
&\{(t_1, t_2) \in \mathcal{I}^* : t_1 < t_2\} \cap \mathcal{I}^s = \emptyset.
\end{aligned} \tag{13}
$$

Proof. Let $(t_1, t_2) \in \mathcal{I}^*$ be such that $t_1 > t_2$. Suppose by contradiction $(t_1, t_2) \notin \mathcal{I}^+$. Let $[0] \equiv n$. For each $l \in \{1, \ldots, n\}$, let $[l] \equiv l$, $[n + l] \equiv l$, and $[-l] \equiv [n - l]$. Let P be the opinion matrix such that for each $i \in \{1, \ldots, n\}$,

$$
\begin{aligned}
\text{if } l &= 0, 1, \ldots, t_1 - 1,\ P_{[i+l]i} = 1;\\
\text{if } l &= t_1, \ldots, t_1 + t_2 - 1,\ P_{[i+l]i} = -1;\\
\text{if } l &= t_1 + t_2, \ldots, n,\ P_{[i+l]i} = 0;
\end{aligned}
$$

for each $k \in M \backslash \{1, \ldots, n\}$ and each $i \in N$,

$$P_{ik} = -1.$$

See Figure 1 for an example of such P. Then for each $i \in \{1, \ldots, n\}$, there are t_1 persons, $\{[i], [i+1], \ldots, [i + t_1 - 1]\}$, who have the positive opinion on the i^{th} issue, t_2 persons, $\{[i+t_1], \ldots, [i+t_1+t_2-1]\}$, who have the negative opinion, and $n-t_1-t_2$ remaining persons with the null opinion. Hence for each $i \in \{1, \ldots, n\}$, $||P^i_+|| = t_1$ and $||P^i_-|| = t_2$. Let $i, j \in \{1, \ldots, n\}$. Let $\pi\colon N \to N$ and $\delta\colon M \to M$

$$P \equiv \begin{pmatrix} 1 & 0 & -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 1 & 1 \end{pmatrix} ; \; {}^{\delta}_{\pi}P = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 1: Construction of P in the proof of Theorem 5. An example with $|N| = |M| = 6$, $t_1 = 3$, $t_2 = 2$, $i = 1$, and $j = 2$. Let $\pi\colon N \to N$ be the transposition of 1 and 2 and $\delta\colon M \to M$ the same transposition.

be two permutations on N and on M transposing i and j. Then the i^{th} and the j^{th} columns in ${}^{\delta}_{\pi}P$ are obtained by making an one-to-one and onto switch between the i^{th} and the j^{th} columns in P, not necessarily preserving the row positions of entries.[13] Thus, $||{}^{\delta}_{\pi}P^i_+|| = ||P^j_+||$, $||{}^{\delta}_{\pi}P^i_-|| = ||P^j_+||$, $||{}^{\delta}_{\pi}P^j_+|| = ||P^i_+||$, and $||{}^{\delta}_{\pi}P^j_-|| = ||P^i_+||$. By *symmetry*, $f_i\left({}^{\delta}_{\pi}P\right) = f_j(P)$ and $f_j\left({}^{\delta}_{\pi}P\right) = f_i(P)$. Since $||P^i_+|| = ||P^j_+||$ and $||P^i_-|| = ||P^j_-||$, then $||P^i_+|| = ||{}^{\delta}_{\pi}P^j_+||$, $||P^i_-|| = ||{}^{\delta}_{\pi}P^i_-||$, $||P^j_+|| = ||{}^{\delta}_{\pi}P^j_+||$, and $||P^j_-|| = ||{}^{\delta}_{\pi}P^j_-||$. So $f_i(P) = f_i\left({}^{\delta}_{\pi}P\right)$ and $f_j(P) = f_j\left({}^{\delta}_{\pi}P\right)$. Hence $f_i(P) = f_j(P)$. Since $(t_1, t_2) \notin \mathcal{I}$, $f_N(P) = (-1, \ldots, -1)$. On the other hand, by *Pareto efficiency*, $f_{M \setminus N} = (-1, \ldots, -1)$. For each $i \in N$, let $U_i(\cdot)$ be the representation of the trichotomous preference relation P_i. Then for each $i \in N$, $U_i(f(P)) = 0$. Let x be such that $x_N \equiv (1, \ldots, 1)$ and $x_{M\setminus N} \equiv (-1, \ldots, -1)$. Then for each $i \in N$, $U_i(x) = t_1 - t_2 > 0$, contradicting *Pareto efficiency*.

Let $(t_1, t_2) \in \mathcal{I}^*$ be such that $t_1 < t_2$. Suppose by contradiction $(t_1, t_2) \in \mathcal{I}^+$. Then using the same argument as above, we show $f_N(P) = (1, \ldots, 1)$ and $f_{M\setminus N}(P) = (-1, \ldots, -1)$. Let $x \equiv (-1, \ldots, -1)$. Then for each $i \in N$, $U_i(f(P)) = t_1 - t_2 < 0 = U_i(x)$, contradicting *Pareto efficiency*.

Similar arguments can be used to prove the same properties for $\mathcal{I}^0$ and $\mathcal{I}^-$. ♦

Note that the properties stated in (13) imply comprehensiveness of the three index sets and (7). Finally, for each $s \in \{+, 0, -\}$, let $q_s(\nu) \equiv \min\{t_1 : (t_1, \nu - t_1) \in \mathcal{I}^s\}$ for each ν. Then by (13), for each ν, $q_+(\nu), q_0(\nu), q_-(\nu) \in \{(\nu+1)/2, (\nu-1)/2\}$. Because of comprehensiveness of the three index sets, (12) implies (8). ∎

Proof of Theorem 6. Let f be a rule over $\mathcal{P}$ satisfying the four axioms (the

[13] Note that P_{ii} and P_{ji} in the i^{th} column are switched into P_{jj} and P_{ij} in the j^{th} column respectively. Other entries in the i^{th} column are switched into the entries in the j^{th} column in the same rows.

proof for $\mathcal{P}^*$ or $\mathcal{R}_{\mathrm{Di}}$ is essentially the same). By Proposition 5, f is represented by an extended system of rights ${}_eR(\cdot)$. Then by *neutrality*, for each pair $l, k \in M$, ${}_eR_2(l) = {}_eR_2(k)$. Thus there exist three index sets $\mathcal{I}^+$, $\mathcal{I}^0$, and $\mathcal{I}^-$ such that for each $P \in \mathcal{P}$ and each $k \in M$, if $i \equiv \rho(k)$,

$$\begin{array}{l} \text{(i) when } P_{ik} = 1,\ f_k(R) = 1 \Leftrightarrow \left(||P_+^k||, ||P_-^k||\right) \in \mathcal{I}^+; \\ \text{(ii) when } P_{ik} = 0,\ f_k(R) = 1 \Leftrightarrow \left(||P_+^k||, ||P_-^k||\right) \in \mathcal{I}^0; \\ \text{(iii) when } P_{ik} = -1,\ f_k(R) = -1 \Leftrightarrow \left(||P_-^k||, ||P_+^k||\right) \in \mathcal{I}^-. \end{array} \tag{14}$$

Using essentially the same argument as in the proof of Theorem 5, we can show that the three index sets satisfy (13). For each $s \in \{+, 0, -\}$, let $q_s(\nu) \equiv \min\{t_1 : (t_1, \nu - t_1) \in \mathcal{I}^s\}$ for each ν. Then by (13), for each ν, $q_+(\nu), q_0(\nu), q_-(\nu) \in \{(\nu+1)/2, (\nu-1)/2\}$. Because of comprehensiveness of the three index sets, (12) implies (8).

Assume that there is ν such that $q_+(\nu) \neq q_0(\nu)$ or $q_+(\nu) + q_-(\nu) \neq \nu + 1$. Then by *neutrality*, $\rho(\cdot)$ should be constant. Thus f is a monocentric quasi-plurality rule. ∎

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