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FATOU'S LEMMA FOR UNBOUNDED GELFAND INTEGRABLE MAPPINGS

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Abstract. It is shown that, in the framework of Gelfand integrable mappings, the Fatou-type lemma for integrably bounded mappings, due to Cornet–Medecin [14] and the Fatou-type lemma for uniformly integrable mappings due to Balder [9], can be generalized to mean norm bounded integrable mappings.

Keywords. Fatou's lemma, Banach space, dual space, Gelfand integral, Komlós limit.

AMS Classification. 28B20, 28C20, 46G10.

1. Introduction

In Mathematical Economics, the framework to model perfect competition is to consider an atomless measure space of agents. For economies with finitely many commodities, the Fatou-type lemma of Aumann [3], Schmeidler [30] and Artstein [2] have been of most importance to prove the existence of equilibria: Aumann [4] for exchange economies and Hildenbrand [19] for production economies. For economies with infinitely many commodities, aggregation of individual consumption bundles is formalized in terms of both the Bochner and Gelfand integral. The choice of Gelfand integration is motivated by the models of spatial economies (Cornet–Medecin [13]) and models of economies with differentiated commodities (Ostroy–Zame [26] and Martins-da-Rocha [23]).

We can find in the literature many Fatou-type lemma dealing with Bochner integrals: Khan–Majundar [21], Balder [5], Yannelis [32, 33, 34], Papageorgiou [28], Balder–Hess [10]. In the framework of Gelfand integrals, Podczeck [29] and Cornet–Medecin [14] proved a Fatou-type lemma for integrably bounded mappings. Balder [9] generalized these results for uniformly integrable mappings. However in order to apply these results to economic models, we need to assume *ad-hoc* assumptions: boundedness of individual consumption sets or strict monotonicity of preferences.

We propose in this paper to generalize the Fatou-type lemma of Cornet–Medecin [14] and Balder [8] to mean norm bounded integrable mappings. Moreover we provide a simple condition under which mean norm boundedness of a sequence of mappings is implied by the boundedness of the sequence of means. This result is the crucial step of the existence results in Araujo–Martins-da-Rocha–Monteiro [1] and it should enable us to substantially weaken the monotonicity assumptions used in Mas-Colell [24], Jones [20], Ostroy–Zame [26], Podczeck [29], Cornet–Medecin [13] and Martins-da-Rocha [23].

The proof of our main result relies on an extension (Balder [6]) to vector-valued mappings of the important result by Komlós. The crucial step of the proof is to deduce a lower closure result from the Komlós-convergence of a sequence of mappings. The originality of this paper is to prove that we can deduce the lower closure result for duals of separable Banach spaces from finite dimensional (Page [27]) lower closure results.

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2. Statement of results

2.1. **Gelfand integrable mappings.** In the whole paper we assume that $(\Omega, \mathcal{A}, \mu)$ is a finite complete positive measure space, $(E, \|.\|)$ is a separable Banach space, with topological dual space E^*. We shall mainly consider on the space E^* the weak star topology $\sigma(E^*, E)$, denoted w^*, and we shall use the notation lim, cl (etc..) to specify the limit, the closure of a set (etc..) for this topology. For $x \in E$ and $f \in E^*$, we denote by $\langle x, f \rangle := f(x)$ the dual product, and by $\|.\|^*$ the dual norm on E^*, i.e. $\|f\|^* := \sup_{x \neq 0} |\langle f, x\rangle| / \|x\|$. We denote by B and B^*, the closed unit balls in $(E, \|.\|)$ and $(E^*, \|.\|^*)$, respectively. If (x_k) is a sequence in E^* we denote by $\mathrm{Ls}_k\{x_k\}$ the set of w^*-limit points of (x_k). If $C \subset E$ (resp. $C^* \subset E^*$) is a subset of E, then we note $C^\circ \subset E^*$ (resp. $[C^*]^\circ \subset E$) the negative polar cone of C (resp. C^*), i.e. $x^* \in C^\circ$ (resp. $x \in [C^*]^\circ$) if and only if for every $x \in C$ (resp. $x^* \in C^*$), $\langle x, x^* \rangle \leqslant 0$. If $F \subset E$ is a subspace of E, then the negative polar F° coincide with the orthogonal $F^\perp$ defined by $\{x^* \in E^* : \forall x \in F, \ \langle x, x^*\rangle = 0\}$. Note that if A is a subset of E^*, then

$$A \subset \bigcap_{F \in \mathcal{F}} [A + F^\perp] \subset \mathrm{cl}\, A,$$

where $\mathcal{F}$ is the collection of all finite dimensional subspaces of E. In particular if E is finite dimensional, then $A = \bigcap_{F \in \mathcal{F}} [A + F^\perp]$.

A mapping f from Ω to E^* is said to be **Gelfand measurable**,[1] if for every $x \in E$, the real valued function $a \mapsto \langle x, f(a)\rangle$ is measurable, and f is said to be **Gelfand integrable**, if for every $x \in E$, the function $a \mapsto \langle x, f(a)\rangle$ is integrable. If f is Gelfand integrable, it can be shown (Diestel–Uhl [16, pp. 52-53]) that for each $A \in \mathcal{A}$, there exists a unique $x^*_A \in E^*$ such that

$$\forall x \in E, \quad \langle x, x^*_A \rangle = \int_\Omega \langle x, f(a)\rangle d\mu(a).$$

For each $A \in \mathcal{A}$, x^*_A is noted $\int_A f d\mu$. Note that if f is a Gelfand measurable mapping, then the function $a \mapsto \|f(a)\|^*$ is measurable.[2] However if f is Gelfand integrable then $a \mapsto \|f(a)\|^*$ is not necessary integrable. A Gelfand measurable mapping f is said **norm integrable** if $a \mapsto \|f(a)\|^*$ is integrable. Obviously, a norm integrable mapping is Gelfand integrable. We recall the following notions about sequences of integrable mappings.

Definition 2.1. A sequence (f_n) of Gelfand integrable mappings from Ω to E^* is said

1. **integrably bounded** if there exists a real-valued integrable function φ such that
$$\sup_n \|f_n(a)\|^* \leqslant \varphi(a) \quad \text{a.e. },$$
2. **uniformly integrable** if the sequence of real-valued functions $(\|f_n(\cdot)\|^*)_n$ is uniformly integrable, i.e.
$$\lim_{\alpha \to \infty} \sup_n \int_{\{\|f_n\|^* \geqslant \alpha\}} \|f_n(a)\|^* \, d\mu(a) = 0 \ ,$$
3. **mean norm bounded** if
$$\sup_n \int_\Omega \|f_n(a)\|^* \, d\mu(a) < +\infty.$$

[1]We prove in Appendix that f is Gelfand measurable if and only if for each borelian $B \subset E^*$, $f^{-1}(B) := \{a \in \Omega : f(a) \in B\}$ belongs to $\mathcal{A}$.

[2]See Proposition A.1 in Appendix.

Remark 2.1. Following Neveu [25], we recall that the sequence (φ_n) of real valued functions is uniformly integrable if and only if the sequence is mean bounded, i.e.

$$\sup_n \int_\Omega |\varphi_n| d\mu < +\infty$$

and equi-continuous, i.e. for every $\varepsilon > 0$ there exists $\eta > 0$ such that for every $A \in \mathcal{A}$,

$$\mu(A) \leqslant \eta \Longrightarrow \sup_n \int_A |\varphi_n| d\mu \leqslant \varepsilon.$$

It follows that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3).

Definition 2.2. A sequence (f_n) of Gelfand integrable mappings from Ω to E^* is said C**-uniformly integrable** for a cone (of vertex 0) C of E, if for every $x \in C$, the sequence of real-valued functions $(\langle x, f_n(\cdot)\rangle^-)_n$ is uniformly integrable, where

$$\forall a \in \Omega, \quad \langle x, f_n(a)\rangle^- := \max[0, -\langle x, f_n(a)\rangle].$$

Remark 2.2. For every $x \in E$, $\langle x, f_n(\cdot)\rangle^- \leqslant |\langle x, f_n(a)\rangle| \leqslant \|x\| \, \|f_n(a)\|^*$. It follows that if a sequence of mappings is uniformly integrable then it is E-uniformly integrable. The converse is not always true, i.e. an E-uniformly integrable sequence of mappings is not always uniformly integrable.

2.2. **Fatou's lemma.** We present hereafter our main results: the Convex and Approximate Fatou's Lemma.

Theorem 2.1 (Convex Fatou's lemma)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space and $(E, \|.\|)$ be a separable Banach space. Let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^*, which is mean norm bounded and C-uniformly integrable for a cone $C \subset E$. If $\lim_n \int_\Omega f_n d\mu$ exists in E^* then there exists a Gelfand integrable mapping f such that*

$$\int_\Omega f d\mu - \lim_n \int_\Omega f_n d\mu \in C^\circ$$

and

$$f(a) \in \overline{\text{co}}\, \text{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

In fact f is norm integrable and

$$\int_\Omega \|f(a)\|^* \, d\mu(a) \leqslant \liminf_n \int_\Omega \|f_n(a)\|^* \, d\mu(a).$$

Theorem 2.1 is a direct consequence of Theorem 3.1 in Section 3.

Theorem 2.2 (Approximate Fatou's lemma)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space and $(E, \|.\|)$ be a separable Banach space. Let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^*, which is mean norm bounded and* C*-uniformly integrable for a cone $C \subset E$. If $\lim_n \int_\Omega f_n d\mu$ exists in E^* then for each finite dimensional subspace F of E, there exists a Gelfand integrable mapping f_F such that*

$$\int_\Omega f_F d\mu - \lim_n \int_\Omega f_n d\mu \in C^\circ + F^\perp$$

and

$$f_F(a) \in \text{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

In particular, if the dimension of E is finite, then there exists a Gelfand integrable mapping f_E such that

$$\int_\Omega f_E d\mu - \lim_n \int_\Omega f_n d\mu \in C^\circ$$

and

$$f_E(a) \in \mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

In fact for each finite dimensional subspace F, the mapping f_F is norm integrable and

$$\int_\Omega \|f_F(a)\|^* d\mu(a) \leqslant \liminf_n \int_\Omega \|f_n(a)\|^* d\mu(a).$$

Theorem 2.2 will be proved in Section 4 as a Corollary of Theorem 3.1. We provide hereafter a sufficient condition for a sequence of mappings to satisfy the assumptions of Theorem 2.1.

Proposition 2.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space and $(E, \|.\|)$ be a separable Banach space. Let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^* such that*

$$\forall n \in \mathbb{N}, \quad f_n(a) \in C^* + \varphi_n(a)B^* \quad \text{a.e.},$$

where C^* is a closed convex cone in E^* and (φ_n) is a sequence of uniformly integrable positive functions.

(a) *Then the sequence (f_n) is C-uniformly integrable, where $C = -(C^*)^\circ \subset E$ is the negative polar of C^*.*
(b) *Suppose that* $\lim_n \int_\Omega f_n d\mu$ *exists in E^*. If C^* has a w^*-compact sole*[3] *then the sequence (f_n) is mean norm bounded.*

Proof. Part (a) is obvious, we propose to prove part (b). Let C^* be a closed convex cone with a w^*-compact sole. There exists $e \in E$ such that for every $x^* \in C^* \setminus \{0\}$, $\langle e, x^* \rangle > 0$ and such that the following set S, defined by $S := \{x^* \in C^* : \langle e, x^* \rangle = 1\}$ is w^*-compact. It follows that S is $\|.\|^*$-bounded by $m > 0$. In particular, for every $x^* \in C^*$, $\langle e, x^* \rangle \geqslant m \|x^*\|^*$. For each $n \in \mathbb{N}$, we consider the following correspondence $F_n : a \mapsto C^* \cap [\{f_n(a)\} - \varphi_n(a)B^*]$. Applying Theorem A.1, there exists $c_n : \Omega \mapsto C^*$ and $b_n : \Omega \mapsto B^*$, two measurable mappings such that for every $n \in \mathbb{N}$,

$$\forall a \in \Omega, \quad f_n(a) = c_n(a) + \varphi_n(a)b_n(a).$$

Since the sequence $(\int_\Omega f_n d\mu)$ converges, we can then suppose (passing to a subsequence if necessary) that the sequences $(\int_\Omega c_n d\mu)$ and $(\int_\Omega \varphi_n b_n d\mu)$ converges in E^*. Now, let $v^* := \lim_n \int_\Omega c_n d\mu$, then

$$\limsup_n \int_\Omega \|c_n(a)\|^* d\mu(a) \leqslant \frac{1}{m} \langle e, v^* \rangle \mu(\Omega)$$

and the sequence (c_n) is mean norm bounded. It follows that the sequence (f_n) is mean norm bounded. □

Applying Proposition 2.1, we present hereafter a corollary of Theorems 2.1 and 2.2.

[3] That is there exists $e \in E$, such that for each $c^* \in C^* \setminus \{0\}$, $\langle e, c^* \rangle > 0$ and $S := \{c^* \in C^* : \langle e, c^* \rangle = 1\}$ is w^*-compact.

Corollary 2.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space and $(E, \|.\|)$ be a separable Banach space. Let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^* such that*

$$\forall n \in \mathbb{N}, \quad f_n(a) \in C^* + \varphi_n(a)B^* \quad \text{a.e.},$$

where C^* is closed convex cone in E^* with a w^*-compact sole, and (φ_n) is a sequence of uniformly integrable positive functions. Suppose that $\lim_n \int_\Omega f_n d\mu$ exists in E^*.

1. [Convex Fatou's lemma]. *There exists a Gelfand integrable mapping f such that*

$$\int_\Omega f d\mu - \lim_n \int_\Omega f_n d\mu \in -C^*$$

and

$$f(a) \in \overline{\text{co}}\, \text{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

2. [Approximate Fatou's lemma]. *For every finite dimensional subspace F of E, there exists a Gelfand integrable mapping f_F such that*

$$\int_\Omega f_F d\mu - \lim_n \int_\Omega f_n d\mu \in F^\perp - C^*$$

and

$$f_F(a) \in \text{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

3. [Finite Fatou's lemma]. *If E is finite dimensional then there exists a Gelfand integrable mapping f_* such that*

$$\int_\Omega f_* d\mu - \lim_n \int_\Omega f_n d\mu \in -C^*$$

and

$$f_*(a) \in \text{Ls}_n\{f_n(a)\} \quad \text{a.e.}$$

Remark 2.3. If E is finite dimensional, then every pointed closed convex cone has a compact sole. In particular, Corollary 2.1 generalizes the version of Fatou's lemma proved in Cornet–Topuzu–Yildiz [15].

Remark 2.4. Let T be a compact metric space and let $E = C(T)$ be the separable Banach space of continuous real-valued functions endowed with the supremum norm. The topological dual space E^* is then $M(T)$, the space of finite Radon measures on T. Let $C := C(T)_+$ and $C^* = M(T)_+$ be the natural positive cones of $C(T)$ and $M(T)$ respectively, i.e. $C(T)_+ := \{x \in C(T) : \forall t \in T,\ x(t) \geqslant 0\}$ and $M(T)_+ := \{f \in M(T) : \forall x \in C(T)_+,\ \langle x, f\rangle \geqslant 0\}$. Then $M(T)_+$ is a closed convex cone with a w^*-compact sole.[4] In particular, Corollary 2.1 can be applied in General Equilibrium Theory to prove the existence of Walras equilibria for economies with a continuum of consumers and differentiated commodities (see Martins-da-Rocha [23]).

[4]Take e in $C(T)$ defined by $e(t) = 1$ for each $t \in T$. Then for each m in $M(T)_+$, $\langle e, m\rangle = \|m\|^*$.

2.3. **The link with other results.** In the context of Cornet–Medecin [14], the sequence (f_n) is supposed to be integrably bounded. It follows that the sequence (f_n) is mean norm bounded and E-uniformly integrable. Hence Theorems 2.1 and 2.2 generalize Theorem 1 in Cornet–Medecin [14].

If a sequence (f_n) is uniformly integrable, then it is mean norm bounded and E-uniformly integrable. Hence Theorems 2.1 and 2.2 generalize Theorems 1 and 2 in Balder [9]. More precisely, in Balder [9] it is proved that if a sequence (f_n) of Gelfand integrable mappings is supposed to be uniformly integrable, then for each open neighborhood W of zero, there exists a Gelfand integrable mapping f_W such that

$$\int_\Omega f_W d\mu - \lim_n \int_\Omega f_n d\mu \in W \quad \text{and} \quad f_W(a) \in \operatorname{cl} \operatorname{Ls}_n\{f_n(a)\} \ a.e.$$

Since the sequence (f_n) is uniformly integrable, it follows that the sequence (f_n) is mean norm bounded and E-uniformly integrable. Now we can apply Theorem 2.2 to a finite dimensional subspace F of E such that $F^\perp \subset W$. Then there exists a Gelfand integrable mapping f_F such that

$$\int_\Omega f_F d\mu - \lim_n \int_\Omega f_n d\mu \in F^\perp \subset W \quad \text{and} \quad f_F(a) \in \operatorname{Ls}_n\{f_n(a)\} \ a.e.$$

Note that we succeed to prove that $f_F(a)$ belongs to $\operatorname{Ls}_n\{f_n(a)\}$ whereas Balder [9] only succeeded to prove that $f_W(a)$ belongs to the closure of $\operatorname{Ls}_n\{f_n(a)\}$.

3. A more general version of Theorem 2.1 and its proof

Our proof of Fatou's lemma relies on an extension proved by Balder [6] of the important result by Komlós (Theorem A.2 in Appendix). We first recall the following definition of Komlós convergence or simply K−convergence.

Definition 3.1. A sequence (f_m) of mappings from Ω to E^* is said to be K-convergent almost everywhere to a mapping $f : \Omega \to E^*$, denoted

$$f_m \xrightarrow{\mathrm{K}} f,$$

if for every subsequence (m_i) of (m), there exists a null set $N \in \mathcal{A}$ (i.e. $\mu(N) = 0$) such that

$$\forall a \in \Omega \setminus N, \quad (1/n)\sum_{i=1}^{n} f_{m_i}(a) \xrightarrow{w^*} f(a).$$

We propose to prove the following theorem which is slightly more general than Theorem 2.1.

Theorem 3.1. *Let $(\Omega, \mathcal{A}, \mu)$ be a finite positive complete measure space and $(E, \|.\|)$ be a separable Banach space. Let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^*, which is mean norm bounded. Then there exists a subsequence (m) of (n) and a Gelfand integrable mapping f such that*

(a) *the sequence (f_m) K-converge to f, and f is norm integrable with*

$$\int_\Omega \|f(a)\|^* \, d\mu(a) \leqslant \liminf_n \int_\Omega \|f_n(a)\|^* \, d\mu(a);$$

(b) *if the sequence (f_n) is C-uniformly integrable for a cone $C \subset E$, then*

$$\forall A \in \mathcal{A}, \quad \forall x \in C, \quad \int_A \langle x, f(a)\rangle d\mu(a) \leqslant \liminf_m \int_A \langle x, f_m(a)\rangle d\mu(a)$$

and

$$\forall A \in \mathcal{A}, \quad \int_A \|f(a)\|^* \, d\mu(a) \leqslant \liminf_m \int_A \|f_m(a)\|^* \, d\mu(a);$$

(c) *there exists* ρ *a positive integrable function such that for every finite dimensional subspace* F *of* E*,*

$$f(a) \in \operatorname{coLs}_n\{f_n(a)\} + \rho(a)B^* \cap F^{\perp} \quad \text{a.e.},$$

in particular $f(a) \in \overline{\operatorname{co}}\,\operatorname{Ls}_n\{f_n(a)\}$ *almost everywhere.*

We recall that a sequence (f_m) of Gelfand integrable mappings from Ω to E^* is said to be **weakly convergent** to a Gelfand integrable mapping f, if for every $x \in E$, the sequence of real valued functions $a \mapsto \langle x, f_n(a)\rangle$ converges to the function $a \mapsto \langle x, f(a)\rangle$ for the weak topology $\sigma(L^1_{\mathbb{R}}, L^\infty_{\mathbb{R}})$.

Remark 3.1. If a sequence (f_m) of Gelfand integrable mappings from Ω to E^* is mean norm bounded and if there exists a Gelfand integrable mapping f such that

$$\forall A \in \mathcal{A}, \quad \lim_m \int_A f_m d\mu = \int_A f d\mu$$

then the sequence (f_m) is weakly convergent to f.

A direct consequence of Theorem 3.1 is the following weak sequential compactness criteria.

Corollary 3.1. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a finite positive complete measure space and* $(E, \|.\|)$ *be a separable Banach space. If* $\mathcal{H}$ *is a family of Gelfand integrable mappings from* Ω *to* E^* *which are mean norm bounded and* E*-uniformly integrable, then* $\mathcal{H}$ *is weakly sequentially compact.*

Proof. Indeed, if (f_n) is a mean norm bounded sequence of Gelfand integrable mappings, then applying part (a) of Theorem 2.1, there exists a Gelfand integrable mapping f and a subsequence (m) of (n) such that (f_m) K-converges to f. Moreover if (f_n) is E-uniformly integrable, then from part (b) of Theorem 2.1, we get that

$$\forall A \in \mathcal{A}, \quad \int_A f d\mu = \lim_m \int_A f_m d\mu.$$

In particular (f_m) weakly converges to f. □

Remark 3.2. If a sequence (f_n) of Gelfand integrable mappings is uniformly integrable, then (f_n) is mean norm bounded and E-uniformly integrable (Remark 2.1). In particular, if $\mathcal{H}$ is a family of uniformly integrable mappings, then $\mathcal{H}$ is weakly sequentially compact.

The proof of Theorem 3.1 will be given in three steps corresponding to part (a), (b) and (c).

3.1. **Proof of part (a).** The following proposition is an extension to vector-valued mappings, of the important result by Komlós (Theorem A.2 in Appendix). A very general criterion for relative sequential compactness for K-convergence of scalarly integrable mappings is given in Balder [6]. As a corollary of Balder's result we can prove the following Proposition 3.1. However in [6], extensions of Komlós results are only given for the Bochner integral. For the sake of completeness, we propose a simple and direct proof.

Proposition 3.1 (Balder [6]). *Let (f_k) be a sequence of Gelfand integrable mappings from Ω to E^*, which is mean norm bounded. Then there exists a subsequence (m) of (k) and a Gelfand integrable function $f : \Omega \to E^*$ such that the sequence (f_m) is* K*-convergent to f and there exists a real-valued integrable function $\varphi_\infty : \Omega \to \mathbb{R}_+$ such that the sequence $(\|f_m(.)\|^*)$ is* K*-convergent to φ_∞. Moreover the mapping f is in fact norm integrable and*

$$\int_\Omega \|f(a)\|^* \, d\mu(a) \leqslant \liminf_n \int_\Omega \|f_n(a)\|^* \, d\mu(a).$$

Proof. Since the sequence (f_k) is mean norm bounded, we can suppose (passing to a subsequence if necessary) that $\liminf_k \int_\Omega \|f_k(a)\|^* \, d\mu(a) = \lim_k \int_\Omega \|f_k(a)\|^* \, d\mu(a)$. We let for each $k \in \mathbb{N}$, for every $a \in \Omega$, $\psi_k(a) := \|f_k(a)\|^*$. Let (x_j) be a $\|.\|$-dense sequence in E. We define for each $j, k \in \mathbb{N}$,

$$\varphi_{j,k}(a) := \langle x_j, f_k(a) \rangle \quad \text{and} \quad \varphi_{\infty,k} := \psi_k.$$

Since the sequence (f_k) is mean norm bounded then for every $j \in \mathbb{N} \cup \{\infty\}$,

$$\sup_k \int_\Omega \mid \varphi_{j,k}(a) \mid d\mu(a) < +\infty.$$

It is now possible to apply Komlós' Theorem (Theorem A.2 in Appendix) repeatedly in a diagonal procedure. This yields a subsequence (m) of (k) and a family $(\varphi_j)_{j \in \mathbb{N} \cup \{\infty\}}$ of integrable real valued functions such that for every $j \in \mathbb{N} \cup \{\infty\}$ and every subsequence (m_i) of (m)

$$\frac{1}{n} \sum_{i=1}^n \varphi_{j,m_i}(a) \longrightarrow \varphi_j(a) \quad \text{a.e.}$$

In particular, for every $j \in \mathbb{N}$,

$$\langle x_j, \frac{1}{n} \sum_{i=1}^n f_{m_i}(a) \rangle \longrightarrow \varphi_j(a) \quad \text{a.e.} \tag{3.1}$$

and

$$\frac{1}{n} \sum_{i=1}^n \psi_{m_i}(a) \longrightarrow \varphi_\infty(a) \quad \text{a.e.} \tag{3.2}$$

Fix $a \in \Omega$ outside the exceptional null-set and for each $n \in \mathbb{N}$, define

$$g_n(a) := \frac{1}{n} \sum_{m=1}^n f_m(a).$$

Then applying (3.2), $\limsup_n \|g_n(a)\|^* \leqslant \varphi_\infty(a) < +\infty$. Now applying Banach-Alaoglu's Theorem, there exists a subsequence of $(g_n(a))$ converging for the w^*-topology to some $f(a) \in E^*$. Applying (3.1), for every $j \in \mathbb{N}$,

$$\langle x_j, f(a) \rangle = \varphi_j(a).$$

The sequence (x_j) is $\|.\|$-dense in E, it follows that for every subsequence (m_i) of (m)

$$\frac{1}{n} \sum_{i=1}^n f_{m_i}(a) \xrightarrow{w^*} f(a) \quad \text{a.e.} \tag{3.3}$$

i.e. the sequence (f_m) is K-convergent to f, in particular, the mapping f is Gelfand measurable.

Now $\|f(a)\|^* \leqslant \liminf_n \|g_n(a)\|^*$ almost everywhere in Ω. Hence applying Fatou's lemma for positive functions,

$$\int_\Omega \|f(a)\|^* \, d\mu(a) \leqslant \liminf_n \frac{1}{n} \sum_{m=1}^{n} \int_\Omega \|f_m(a)\|^* \, d\mu(a) = \liminf_k \int_\Omega \|f_k(a)\|^* \, d\mu(a)$$

and the mapping f is then Gelfand integrable. □

Remark 3.3. We refer to Balder [6] and Balder–Hess [11] for other extensions of Komlós' result, which mainly consider Bochner integration.

3.2. **Proof of part (b).** The proof of part (b) is given by the following proposition.

Proposition 3.2. *Let $\{f_k\}$ be a sequence of Gelfand integrable mappings from Ω to E^*,* K-*converging to a Gelfand integrable mapping f. Suppose that there exists a cone $C \subset E$ such that (f_k) is C-uniformly integrable. Then for every $x \in C$,*

$$\forall A \in \mathcal{A}, \qquad \int_A \langle x, f(a)\rangle d\mu(a) \leqslant \liminf_k \int_A \langle x, f_k(a)\rangle d\mu(a).$$

Proof. Let $A \in \mathcal{A}$ and $x \in C$, we pose $\alpha := \liminf_k \int_A \langle x, f_k\rangle d\mu$. Passing to a subsequence if necessary we can suppose that $\alpha = \lim_k \int_A \langle x, f_k\rangle d\mu$. For each m, we let $g_m = (1/m)\sum_{k=1}^{m} f_k$ and we define the real-valued function φ_m from Ω to $\mathbb{R}$ by $\varphi_m(a) = \langle x, g_m(a)\rangle$. We have

$$\varphi_m(a)^- \leqslant \frac{1}{m}\sum_{k=1}^{m} \langle x, f_k(a)\rangle^-.$$

Since (f_k) is C-uniformly integrable, it follows that the sequence (φ_m^-) is uniformly integrable. Since (g_m) is K-converging to f, it follows that for almost every a, $(\varphi_m(a))$ is converging to $\langle x, f(a)\rangle$. We apply Fatou's lemma (for real-valued functions) and we get

$$\int_A \langle x, f(a)\rangle d\mu(a) \leqslant \liminf_m \int_A \varphi_m(a) d\mu(a) = \alpha.$$

□

3.3. **Proof of part (c).** We propose now to prove that we can deduce a lower closure result from the Komlós convergence of a sequence of mappings. The proof of the lower closure result for an infinite dimensional separable Banach space is based on the following lower closure result proved by Page [27] for finite dimensional spaces. For the sake of completeness, we propose a simple and direct proof.

Proposition 3.3 (Page [27])**.** *Let E be a finite dimensional vector space and let (f_n) be a sequence of integrable mappings from Ω to E^*. Suppose that the sequence (f_n) is mean norm bounded and* K-*convergent to an integrable mapping f. Then*

$$f(a) \in \mathrm{coLs}_n\{f_n(a)\} \quad \text{a.e.}$$

Proof. Let (f_n) be a sequence of mean norm bounded mappings from Ω to E^*, K-converging to f. Following Gaposhkin's lemma A.1, there exists a subsequence (n_k) of (n) such that for each k, $f_{n_k} = g_k + h_k$, where the sequence (g_k) is uniformly integrable and the sequence (h_k) converges almost everywhere to 0. Since (f_{n_k}) K-converges to f, it follows that (g_k) K-converges to f. ¿From Proposition 3.2, we have that the sequence (g_k) weakly converges to f. Now applying Proposition C in Artstein [2], we get that $f(a)$ belongs to $\mathrm{coLs}_k\{g_k(a)\}$ almost every where. Since $\mathrm{Ls}_k\{g_k(a)\} \subset \mathrm{Ls}_n\{f_n(a)\}$, it follows that $f(a) \in \mathrm{coLs}_n\{f_n(a)\}$ almost everywhere. □

Remark 3.4. The proof of Proposition 3.3 (actually in Page [27] a more general result is proved) is based on Proposition C in Artstein [2]. Note that Proposition C in Artstein [2] is a corollary of Propositions 3.1, 3.2 and 3.3. Indeed, let (f_n) be sequence of integrable mappings from Ω to E^* (E is finite dimensional) such that (f_n) weakly converges to an integrable mapping f. The sequence (f_n) is then mean norm bounded. Applying Propositions 3.1 and 3.3, there exists a subsequence (m) of (n) and an integrable mapping g such that (f_m) K-converges to g and that $g(a) \in \mathrm{coLs}_m\{f_m(a)\}$ almost everywhere. But since (f_n) weakly converges, it follows from Proposition IV.2.3 in Neveu [25] that (f_n) is uniformly integrable. Applying Proposition 3.2, the sequence (f_m) weakly converges to g. Hence $g = f$ almost everywhere and $f(a) \in \mathrm{coLs}_n\{f_n(a)\}$ almost everywhere.

Applying Proposition 3.3, we now provide a proof of the lower closure result in the general setting.

Proposition 3.4. *Let (f_n) be a sequence of Gelfand integrable mappings from Ω to E^*. Suppose that the sequence (f_n) is mean norm bounded and is* K*-convergent to a Gelfand integrable mapping f. Then there exists ρ a positive integrable function such that for every finite dimensional subspace F of E,*

$$f(a) \in \mathrm{coLs}_n\{f_n(a)\} + \rho(a)B^* \cap F^{\perp} \quad \text{a.e.}$$

Proof. The sequence $(\|f_n(.)\|^*)$ is mean norm bounded. Applying Komlós' Theorem (Theorem A.2 in Appendix) and passing to a subsequence if necessary, we can suppose that the sequence $(\|f_n(.)\|^*)$ is K-convergent to an integrable function ψ from Ω to $\mathbb{R}$. Let F be a finite dimensional subspace of E. We consider π the following projection from E^* to F^*, defined by

$$\forall x^* \in E^*, \quad \pi(x^*) = [x \in F \mapsto \langle x, x^* \rangle].$$

It follows that the sequence $([\|f_n(.)\|^*, \pi(f_n)])$ is K-convergent to $[\psi, \pi(f)]$. Applying Proposition 3.3,

$$[\psi(a), \pi(f(a))] \in \mathrm{coLs}_n \left\{ [\|f_n(a)\|^*, \pi(f_n(a))] \right\} \quad \text{a.e.}$$

Let $a \in \Omega$ outside the exceptional null set. There exists a finite set I, a finite family $(\lambda_i)_{i \in I} \in [0,1]^I$ such that $\sum_{i \in I} \lambda_i = 1$, and there exists a finite family $(\varphi_i)_{i \in I}$ of strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$, such that

$$[\psi(a), \pi(f(a))] = \sum_{i \in I} \lambda_i \lim_n \left[\left\| f_{\varphi_i(n)}(a) \right\|^*, \pi(f_{\varphi_i(n)}(a)) \right].$$

Let $i \in I$, the sequence $(\left\| f_{\varphi_i(n)}(a) \right\|^*)$ converges, passing to a subsequence if necessary, we can suppose that the sequence $(f_{\varphi_i(n)}(a))$ w^*-converges to some $h_i(a) \in \mathrm{Ls}_n\{f_n(a)\} \subset E^*$. It follows that

$$\pi[f(a)] = \sum_{i \in I} \lambda_i \pi[h_i(a)] \in \pi \left[\mathrm{coLs}_n\{f_n(a)\} \right].$$

Note that $\left\| \sum_{i \in I} \lambda_i h_i(a) \right\|^* \leqslant \sum_{i \in I} \lambda_i \|h_i(a)\|^* \leqslant \sum_{i \in I} \lambda_i \lim_n \left\| f_{\varphi_i(n)}(a) \right\|^* = \psi(a)$, hence

$$f(a) \in \mathrm{coLs}_n\{f_n(a)\} + \rho(a)B^* \cap F^{\perp},$$

where $\rho(a) := \psi(a) + \|f(a)\|^*$. □

The proof of part (c) of Theorem 3.1 will follow from Proposition 3.4 and the following proposition.

Proposition 3.5. *Let L be a multifunction from Ω to E^*, let f be a mapping from Ω to E^* and let ρ be a positive function such that for every finite dimensional subspace F of E,*

$$f(a) \in L(a) + \rho(a)B^* \cap F^\perp \quad \text{a.e.}$$

Then

$$f(a) \in \operatorname{cl} L(a) \quad \text{a.e.}$$

Proof. Let (e_i) be a dense sequence in E, and for each $n \in \mathbb{N}$, we let F_n be the vector subspace of E generated by $\{e_0, e_1, \dots, e_n\}$. It follows that there exists $\Omega' \subset \Omega$ with $\mu(\Omega \setminus \Omega') = 0$ and such that

$$\forall a \in \Omega', \quad f(a) \in \bigcap_{n\in\mathbb{N}} \left(L(a) + \rho(a)B^* \cap F_n^\perp\right).$$

Let $a \in \Omega'$, there exists a sequence $(z_n(a))_n$ in E^* satisfying $f(a) - z_n(a)$ belongs to $L(a)$ and $z_n(a)$ belongs to $\rho(a)B^* \cap F_n^\perp$. Passing to a subsequence if necessary, we can suppose that $(z_n(a))_n$ is w^*-convergent to $z(a)$. It follows that $f(a) - z(a)$ belongs to $\operatorname{cl} L(a)$. Moreover, since $z_n(a)$ belongs to $F_n^\perp$, we have that for every i, $\langle e_i, z(a)\rangle = 0$. In particular $z(a) = 0$ and $f(a)$ belongs to $\operatorname{cl} L(a)$. □

4. Proof of Theorem 2.2

We propose to first prove Theorem 2.2 when $(\Omega, \mathcal{A}, \mu)$ is non atomic and then we provide the proof in the general case.

4.1. **The case $(\Omega, \mathcal{A}, \mu)$ is non atomic.** Let (f_n) be a sequence of Gelfand integrable mappings, which is mean norm bounded, and let F be a finite dimensional subspace of E. Applying Theorem 3.1, we can suppose, passing to a subsequence if necessary that there exists f a Gelfand integrable mapping from Ω to E^* and ψ an integrable function from Ω to $[0, +\infty)$ such that

$$(\|f_n(.)\|^*, f_n) \xrightarrow{\mathrm{K}} (\psi, f) \quad \text{a.e.}$$

and

$$(\psi(a), f(a)) \in \operatorname{coLs}_n\{(\|f_n(a)\|^*, f_n(a))\} + (\mathbb{R} \times F)^\perp \quad \text{a.e.}$$

Let π be the following projection from E^* to F^*, the algebraic dual of F, defined by

$$\forall x^* \in E^*, \quad \pi(x^*) = [x \in F \mapsto \langle x, x^*\rangle].$$

Then

$$(\psi(a), \pi[f(a)]) \in \operatorname{coLs}_n\{(\|f_n(a)\|^*, \pi[f_n(a)])\} \quad \text{a.e.}$$

Following Carathéodory's theorem, we let $I := \{1, \dots, \ell + 2\}$, where ℓ is the dimension of F. Then, for almost every $a \in \Omega$, there exists $(\lambda_i(a))_{i\in I} \in [0,1]^I$ such that $\sum_{i\in I} \lambda_i(a) = 1$ and $(\varphi_i)_{i\in I}$ strictly increasing functions from $\mathbb{N}$ to $\mathbb{N}$, such that

$$(\psi(a), \pi[f(a)]) = \sum_{i\in I} \lambda_i(a) \lim_n (\left\|f_{\varphi_i(n)}(a)\right\|^*, \pi[f_{\varphi_i(n)}(a)]).$$

In particular, for each $i \in I$, $\psi_i(a) := \lim_n \left\|f_{\varphi_i(n)}(a)\right\|^* < +\infty$. It follows that there exists $s_i(a) \in \operatorname{Ls}_n\{f_n(a)\}$ such that $\lim_n f_{\varphi_i(n)}(a) = s_i(a)$, and

$$(\psi(a), \pi[f(a)]) = \sum_{i\in I} \lambda_i(a)(\psi_i(a), \pi[s_i(a)]) \quad \text{a.e.}$$

Applying Theorem A.1, Proposition A.4 and Corollary A.1, we can suppose that for each $i \in I$, the functions λ_i and ψ_i are measurable and the mappings s_i are Gelfand measurable selections of $\text{Ls}_n\{f_n(.)\}$. Note that for each $i \in I$, for almost every $a \in \Omega$, $\|s_i(a)\|^* \leqslant \psi_i(a)$. It follows that

$$\int_\Omega \sum_{i\in I} \lambda_i(a)\, \|s_i(a)\|^* \leqslant \int_\Omega \psi(a) d\mu(a) < +\infty$$

and hence that

$$\int_\Omega \sum_{i\in I} \lambda_i(a) \left[\|s_i(a)\|^* + \|\pi[s_i(a)]\|^*\right] d\mu(a) \leqslant 2\int_\Omega \psi d\mu < \infty.$$

Applying the Extended Lyapunov Theorem A.3, there exists a measurable partition $(B_i)_{i\in I}$ of Ω such that $(\|s_i(.)\|^*, \pi[s_i(.)])$ is integrable over B_i and such that

$$\int_\Omega \sum_{i\in I} \lambda_i(a)(\|s_i(a)\|^*, \pi[s_i(a)])d\mu(a) = \sum_{i\in I}\int_{B_i} (\|s_i(a)\|^*, \pi[s_i(a)])d\mu(a).$$

Let $f_F := \sum_{i\in I} \chi_{B_i} s_i$,[5] then f_F is a Gelfand measurable selection of $\text{Ls}_n\{f_n(.)\}$, and moreover

$$\int_\Omega \|f_F(a)\|^* d\mu(a) = \sum_{i\in I}\int_{B_i} \|s_i(a)\|^* d\mu(a) \leqslant \int_\Omega \psi d\mu < \infty.$$

It follows that f_F is Gelfand integrable. Now

$$\pi[\int_\Omega f_F d\mu] = \sum_{i\in I}\int_{B_i} \pi[s_i(a)]d\mu(a) = \int_\Omega \sum_i \lambda_i(a)\pi[s_i(a)]d\mu(a) = \int_\Omega \pi[f(a)]d\mu(a).$$

Hence

$$\int_\Omega f d\mu - \int_\Omega f_F d\mu \in F^\perp.$$

4.2. **The general case.** We now provide the proof of Theorem 2.2 in the general case, i.e. without assuming anymore that $(\Omega, \mathcal{A}, \mu)$ is non-atomic. This is a classical result that the set Ω can be partitioned into a non atomic part $\Omega^{na} \in \mathcal{A}$ and a purely atomic part $\Omega^{pa} \in \mathcal{A}$, and that the set Ω^{pa} can be written as the disjoint union of at most countably many measurable atoms $(A_i)_{i\in I}$ ($I \subset N$). Furthermore, for every $i \in I$ and every $n \in N$, the measurable mapping $f_n : \Omega \to E^*$ takes a constant value $f_n^i \in E^*$ for a.e. $a \in A_i$. Since the sequence (f_n) is mean norm bounded, for each $i \in I$, the sequence (f_n^i) is norm bounded, and thus remains in a w^*-compact subset of E^* by Alaoglu's theorem. Consequently, by a diagonal extraction argument, there exists a subsequence (n_k) of (n) such that for every $i \in I$, $(f_{n_k}^i)_k$ w^*-converges to some element $\bar{f}^i \in E^*$. We let $f^{pa} : \Omega^{pa} \to E^*$ be defined by $f^{pa}(a) = \bar{f}^i$ if $a \in A^i$. We have shown that

$$f^{pa}(a) \in \text{Ls}_n\{f_n(a)\} \quad \text{a.e. in } \Omega^{pa}.$$

We now show that

$$\lim_k \int_{\Omega^{pa}} f_{n_k}(a)d\mu(a) = \int_{\Omega^{pa}} f^{pa}(a)d\mu(a).$$

[5] For each measurable set $A \in \mathcal{A}$, χ_A is the characteristic function associated to A, i.e for every $a \in \Omega$, $\chi_A(a) = 1$ if a belongs to A and $\chi_A(a) = 0$ elsewhere.

This is clearly a consequence of Lebesgue dominated convergence theorem applied for every fixed $x \in E$, to the sequence $(\langle x, f_{n_k}\rangle)_k$ which is integrably bounded over Ω^{pa} (since the sequence $(f_{n_k})_k$ is also integrably bounded over Ω^{pa}).

We now consider the non atomic part Ω^{na} and we first remark that $\lim_k \int_{\Omega^{na}} f_{n_k} d\mu$ exists since

$$\lim_k \int_{\Omega^{na}} f_{n_k} d\mu = \lim_k \int_{\Omega} f_{n_k} d\mu - \lim_k \int_{\Omega^{pa}} f_{n_k} d\mu.$$

We can now apply to the non atomic part Ω^{na} the version of Fatou's lemma proved previously. Thus, for every finite dimensional subspace F of E, there exists $f_F^{na} : \Omega^{na} \to E^*$ such that

$$f_F^{na}(a) \in \mathrm{Ls}_n\{f_n(a)\} \quad \text{a.e. in } \Omega^{na}$$

and

$$\int_{\Omega^{na}} f_F^{na} d\mu - \lim_k \int_{\Omega^{na}} f_{n_k} d\mu \in F^{\perp} + C^{\circ}.$$

We now define the mapping $f_F : \Omega \to E^*$ by $f_F(a) := f^{pa}(a)$ if $a \in \Omega^{pa}$ and $f_F(a) := f_F^{na}(a)$ if $a \in \Omega^{na}$. One checks that the mapping f_F satisfies the conditions of Theorem 2.2.

Appendix A. Appendix

A.1. **Measurable mappings.** Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite measure space and $(E, \|.\|)$ be a separable Banach space. We note $\mathcal{B}$ the Borel σ-algebra of (E^*, w^*). We recall that a mapping f from Ω to E^* is said Gelfand measurable if for every $x \in E$, the function $a \mapsto \langle x, f(a)\rangle$ is measurable. The mapping f is said measurable, if for every $B \in \mathcal{B}$, $f^{-1}(B)$ belongs to $\mathcal{A}$.

Proposition A.1. *Let f be a mapping from Ω to E^*. Then f is Gelfand measurable if and only if it is measurable. Moreover, if f is measurable then the function $a \mapsto \|f(a)\|^*$ is measurable.*

Proof. Let (x_i) a norm dense sequence in B the unit ball of E. For each $i \in \mathbb{N}$ and each $\alpha > 0$, we let $V_{i,\alpha} := \{x^* \in E^* \ : \ |\langle x_i, x^*\rangle| < \alpha\}$. We note $\mathcal{D}$ the σ-algebra generated by the family of all $V_{i,\alpha}$. Since $V_{i,\alpha}$ is open in (E^*, w^*), we have $\mathcal{D} \subset \mathcal{B}$. It follows that if f is measurable then f is Gelfand measurable. Note that

$$\bigcup_{i\in\mathbb{N}} \bigcap_{n>0} V_{i,\alpha+1/n} = \alpha B^* = \{x^* \in E^* \ : \ \|x^*\|^* \leqslant \alpha\} \in \mathcal{D}.$$

It follows that if f is Gelfand measurable then the mapping $a \mapsto \|f(a)\|^*$ is measurable.

Let d be the following distance defined on E^*,

$$\forall (x^*, y^*) \in E^* \times E^*, \quad d(x^*, y^*) = \sum_{i\geqslant 0} \frac{|\langle x_i, x^* - y^*\rangle|}{2^i}.$$

Let $\mathcal{B}_d$ be the Borel σ-algebra defined by d. Note that $\mathcal{B}_d \subset \mathcal{D}$. The topology defined by the distance d coincide with the w^*-topology on closed bounded subsets of E^*. It follows that if W is a w^*-open subset of E^*, then for each $k \in \mathbb{N}$, $W \cap kB^*$ is d-open, in particular, $W \cap kB^* \in \mathcal{D}$. Since $W = \bigcup_k W \cap kB^*$, it follows that $W \in \mathcal{D}$, and then $\mathcal{B} \subset \mathcal{D}$. Hence $\mathcal{B} = \mathcal{D}$ and the result follows. □

A.2. **Measurable selections.** Let $(\Omega, \mathcal{A}, \mu)$ be a complete finite measure space and E be a separable Banach space. A multifunction F from Ω into E^* is said **graph measurable** if the graph G_F of F belongs to the σ-algebra $\mathcal{A} \otimes \mathcal{B}$, where

$$G_F := \{(a, x^*) \in \Omega \times E^* \ : \ x^* \in F(a)\}.$$

A mapping f from Ω to E^* is a selection of F if $f(a) \in F(a)$ for almost every $a \in \Omega$. We provide hereafter a classical result of existence of measurable selections.

Theorem A.1 (Aumann Selection Theorem)**.** *We consider E a separable Banach space and $(\Omega, \mathcal{A}, \mu)$ a complete finite measure space. Let F be a graph measurable multifunction from Ω to E^* with non empty values. Then there exists a measurable mapping f from Ω to E^* such that*

$$\forall a \in \Omega, \quad f(a) \in F(a).$$

In particular f is a measurable selection of F.

The proof of this theorem is given in Castaing–Valadier [12, Theorem III.22, p.74]. We provide hereafter a direct application of this theorem.

Corollary A.1. *We consider E a separable Banach space and $(\Omega, \mathcal{A}, \mu)$ a complete finite measure space. Let F be a graph measurable multifunction from Ω to E^* with non empty values, let I be a finite set and let f be a measurable selection of F. Suppose that for almost every $a \in \Omega$, for each $i \in I$, there exist $\lambda_i(a) \in [0, 1]$ and $f_i(a) \in F(a)$ such that*

$$f(a) = \sum_{i \in I} \lambda_i(a) f_i(a) \quad \text{and} \quad \sum_{i \in I} \lambda_i(a) = 1.$$

Then for each $i \in I$, λ_i may be chosen as a measurable function from Ω to $[0, 1]$ and f_i may be chosen as a measurable selection of F.

Proof. We let $\Sigma(I)$ be the set of all $(\alpha_i) \in [0, 1]^I$ such that $\sum_i \alpha_i = 1$. Let π be the linear mapping from $\Sigma(I) \times (E^*)^I$ to E^* defined by

$$\forall [(\alpha_i), (x_i^*)] \in \Sigma(I) \times (E^*)^I, \quad \pi[(\alpha_i), (x_i^*)] := \sum_{i \in I} \alpha_i x_i^*.$$

For each $a \in \Omega$, we let

$$H(a) := \pi^{-1}(\{f(a)\}) \cap \left(\Sigma(I) \times F(a)^I\right).$$

The multifunction H is graph measurable with non empty values. The proof of the corollary follows from the application of Theorem A.1 to the multifunction H. □

A.3. **Measurability of limes superior.** We consider E a separable Banach space and $(\Omega, \mathcal{A}, \mu)$ a (possibly not complete) finite measure space. A multifunction F from Ω into E^* is said measurable if for each w^*-open subset V of E^*, $F^-(V) := \{a \in \Omega \ : \ F(a) \cap V \neq \emptyset\}$ belongs to $\mathcal{A}$.

Proposition A.2. *Let F be a multifunction from Ω to E^*.*

1. *Suppose that $(\Omega, \mathcal{A}, \mu)$ is complete. If F is graph measurable then F is measurable.*
2. *Suppose that F is closed valued. If F is measurable then F is graph measurable.*

Proof. The part (1) follows from the Projection Theorem in Castaing–Valadier [12, Theorem III.23]. Now we prove part (2) of the proposition. Since E is a separable Banach space, E^* is the countable union of w^*-compact metrizable subsets. It follows from Schwartz [31] that E^* is a Lusin space, in particular, there exists a separable and completely metrizable topology τ, stronger than the w^*

topology, but generating the same Borel sets. Since F is w^*-closed valued, it is τ-closed valued. Applying Proposition III.13 in Castaing–Valadier [12], the graph of F is measurable. □

Proposition A.3. *Let F and F_n, $n \in \mathbb{N}$ be graph measurable multifunctions from Ω into E^*.*

1. *The multifunction $\operatorname{cl} F$ defined by $a \mapsto \operatorname{cl} F(a)$ is graph measurable.*
2. *The multifunction $\bigcup_n F_n$ and $\bigcap_n F_n$ are graph measurable.*

Proof. Proof of (1). The multifunction F is graph measurable, and then following Proposition A.2, F is measurable. Let V be a w^*-open subset of E^*. For each $a \in A$,

$$F(a) \cap V \neq \emptyset \Longleftrightarrow [\operatorname{cl} F(a)] \cap V \neq \emptyset.$$

It follows that if F is measurable then cl$F$ is measurable. Once again applying Proposition A.2, the multifunction cl$F$ is graph measurable.
Proof of (2). This an immediate consequence of

$$\text{Graph}\,(\cup_n F_n) = \cup_n \text{Graph}\,(F_n) \quad \text{and} \quad \text{Graph}\,(\cap_n F_n) = \cap_n \text{Graph}\,(F_n)$$

□

If (C_n) is a sequence of subsets of E^*, we denote by $\text{Ls}_n C_n$ the sequential limes superior of (C_n) relative to w^*, i.e.

$$\text{Ls}_n C_n := \{x \in E^* \ : \ x = \lim_k x_k \ , \ x_k \in C_{n_k}\}.$$

Proposition A.4. *Let (F_n) be a sequence of graph measurable multifunctions from Ω into E^*. The multifunction $a \mapsto \text{Ls}_n F_n(a)$ is graph measurable. In particular, if (f_n) is a sequence of measurable mappings from Ω to E^*, then the multifunction $a \mapsto \text{Ls}_n\{f_n(a)\}$ is graph measurable.*

Proof. Note that if (C_n) is a sequence of non-empty subsets of E^*, then

$$\text{Ls}_n C_n = \bigcup_{p \in \mathbb{N}} \text{Ls}_n (C_n \cap pB^*).$$

Indeed, let $x \in \text{Ls}_n C_n$. There exists a sequence (x_k) and a subsequence (n_k) of (n) such that $x_k \in C_{n_k}$ for each $k \in \mathbb{N}$ and

$$x_k \xrightarrow{w^*} x.$$

It follows that the sequence (x_k) is $\|.\|^*$-bounded. Hence following Proposition A.3, we can suppose without any loss of generality that there exists a w^*-compact convex and metrizable subset K of E^*, such that

$$\forall a \in \Omega, \quad \bigcup_n F_n(a) \subset K.$$

Hence

$$\text{Ls}_n F_n(a) = \bigcap_n \operatorname{cl} \bigcup_{p \geqslant n} F_p(a).$$

Following Proposition A.3, the multifunction

$$a \mapsto \text{Ls}_n F_n(a)$$

is graph measurable. This ends the proof of Claim A.4. □

Remark A.1. We refer to Hess [18] for related results of measurability of limes superior.

A.4. **Komlós limits.** Let E be a separable Banach space and $(\Omega, \mathcal{A}, \mu)$ a finite measure space. A sequence (f_n) of mappings from Ω to E^* is said K-convergent to a mapping f, if for every subsequence (n_i) of (n)

$$\frac{1}{n}\sum_{i=1}^{n} f_{n_i}(a) \xrightarrow{w^*} f(a) \quad \text{a.e.}$$

Theorem A.2 (Komlós)**.** *Suppose that (φ_k) is a sequence of integrable real valued functions such that*

$$\sup_k \int_\Omega \mid \varphi_k \mid d\mu < +\infty.$$

Then there exists a subsequence (m) of (k) and an integrable real valued function φ such that (φ_m) is K*-convergent to φ.*

This theorem is due to Komlós [22].

A.5. **Gaposhkin.**

Lemma A.1 (Gaposhkin's lemma)**.** *Let E be a finite dimensional vector space and $(\Omega, \mathcal{A}, \mu)$ a finite measure space. If (f_n) is a mean norm bounded sequence of integrable mappings from Ω to E^*, then there exists a subsequence (n_k) of (n) such that for each $k \in \mathbb{N}$, $f_{n_k} = g_k + h_k$, where the sequence (g_k) is uniformly integrable and where the sequence (h_k) converges almost everywhere to* 0*.*

This lemma is due to Gaposhkin, Lemma C.I in [17].

A.6. **Lyapunov.**

Theorem A.3 (Extended Lyapunov)**.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space, let I be a finite set, let $\ell \in \mathbb{N}$, let $(f_i)_{i\in I}$ be measurable functions from $(\Omega, \mathcal{A}, \mu)$ to $\mathbb{R}^\ell$ and let $(\lambda_i)_{i\in I}$ measurable functions from Ω to $[0,1]$ with $\sum_{i\in I} \lambda_i(a) = 1$. Suppose that*

$$\int_\Omega \sum_{i\in I} \lambda_i(a)|f_i(a)|d\mu(a) < +\infty.$$

If $(\Omega, \mathcal{A}, \mu)$ is non atomic then there exists a measurable partition $(B_i)_{i\in I}$ of Ω such that for each $i \in I$, the function f_i is integrable over B_i and

$$\int_\Omega \sum_{i\in I} \lambda_i(a) f_i(a) d\mu(a) = \sum_{i\in I} \int_{B_i} f_i d\mu.$$

This theorem proved in Balder [7] is a corollary of the classical Lyapunov theorem.

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