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EXISTENCE OF FINANCIAL EQUILIBRIA IN A MULTIPERIOD STOCHASTIC ECONOMY

Laura Angeloni
Dipartimento di Matematica e Informatica, Universita degli Studi di Perugia
Bernard Cornet
Department of Economics, University of Kansas

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Laura Angeloni[1], Bernard Cornet[2]

Abstract

We consider the model of a stochastic financial exchange economy where time and uncertainty are represented by a finite event-tree of length T. We provide a general existence result of financial equilibria, which allows to cover several important cases of financial structures considered in the literature, such as nominal and numeraire assets, when consumers may have constraints on their portfolios.

1 Introduction

The main purpose of general equilibrium theory with incomplete markets is to study the interactions between the financial structure of the economy and the commodity structure, in a world in which time and uncertainty play a fundamental role. The first pioneering multiperiod model is due to Debreu [10], who introduced the idea of an event-tree of finite length, in order to represent time and uncertainty in a stochastic economy. Later, Magill and Schafer [24] extended the analysis of multiperiod models, describing economies in which financial equilibria coincide with contingent market equilibria. The $T-$period model was also explored, among others, by Duffie and Schafer ([12]), who proved a result of generic existence of equilibria, and we recall that a detailed presentation is provided in Magill and Quinzii ([23]).

The multiperiod model has been also extensively studied in the simple two-date model (one period $T = 1$): see, among others, [3, 26, 6, 8, 34], for the case of a finite set of states and [27, 28, 1, 30] for the case of a continuum of states. The two-date model, however, is not sufficient to capture the time evolution of realistic models. In this sense, the multiperiod model is much more flexible, and is also a necessary intermediate step before studying the infinite horizon setting (see [21, 22]). Moreover, multiperiod models may provide a framework for phenomena which do not occur in a simple two-date model: for example, in [4], Bonnisseau and Lachiri

[1]Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, Via Vanvitelli 1, 06123, Perugia, ITALIA and CERMSEM, Université de Paris 1. $106-112$, Boulevard de l'Hopital, 75647, Cedex 13, Paris, FRANCE

[2]Université Paris 1 and University of Kansas, E-mail: bern@rdcornet.com

describe a three-date economy with production in which, essentially, the second welfare theorem does not hold, while it always holds in the two-date case. As a further example, we may recall that the suitable setting to study the effect of incompleteness of markets on price volatility is a three-date model, in the way addressed in [7].

In the model we consider, time and uncertainty are represented by an event-tree with T periods and a finite set S of states of nature, namely the possible events that could occur in the future. At each node, there is a spot market where a finite set of commodities is available. Moreover, transfers of value among nodes and dates are made possible via a financial structure, namely a finite set of financial assets available at some nodes of the event-tree. The financial assets are general enough to encompass, for instance, the case of nominal assets, numeraire assets, real assets and pure spot markets; we also prove that our model encompasses the case in which retrading of financial assets is allowed at every node (see [23]), in the sense that we prove that financial equilibria, in the two settings, coincide. Finally, we consider the case of restricted participation, namely the case in which agents'portfolio sets may be constrained.

This paper considers the problem of the existence of a financial equilibrium in a stochastic economy with general financial assets and possible constraints on portfolios. The problem of the existence of an equilibrium in incomplete markets was studied, in the case of two-date models, by Cass ([5]) and Werner ([35, 36]), for nominal financial structures, Duffie ([11]) for purely financial securities under general conditions, Geanakoplos and Polemarchakis ([18]) in the case of numéraire assets. Again in the case of a two-date economy the existence of a financial equilibrium was proved by Bich and Cornet ([3]) and Florenzano ([14]) and, more recently, by Da Rocha-Triki ([25]). In the case of $T-$period economies, this problem was also faced by Duffie and Schafer ([13]) and by Florenzano and Gourdel ([15]); other existence results in the infinite horizon models can be found in [20, 29, 16].

2 The T-period financial exchange economy

2.1 Time and uncertainty in a multiperiod model

We[3] consider a multiperiod exchange economy. There are $(T+1)$ dates $t \in [0,T]:= \{0,\ldots,T\}$, a finite set of agents I and a finite set S of *states of nature* in the model, hence a finite set S^t of states that could occur at date t and $S = \bigcup_{t\in T} S^t$. The first date will be referred to as $t=0$ and T is the final date. At each date $t \neq T$, there is an a priori uncertainty about which node will prevail in the next date. The only non-stochastic event occurring at date $t=0$ will be referred to as s_0, sometimes also denoted 0. The stochastic structure of the model can be described by a finite event-tree $\mathcal{S}$ of length T, that we now formally define.

Definition 2.1 *A* finite event-tree $\mathcal{S}$ *is a quadruple* $\mathcal{S} = (S, T, (S^t)_{t\in[0,T]}, pr)$ *of a finite set of nodes* S*, a finite (horizon) time* $T \geq 1$*, a partition of nonempty subsets* $(S^t)_{t\in[0,T]}$ *of* S *such that* S^0 *contains a single element, say* $S^0 = \{0\}$*, and a* predecessor mapping $pr : S \setminus S^0 \longrightarrow S$ *satisfying* $pr(S^t) = S^{t-1}$*, for every* $t = 1,\ldots,T$.

We denote by $t(s)$ the unique $t \in [0,T]$ such that $s \in S^t$. For every $t \in [0,T]$, S^t is the set of nodes occurring at time t, $S^T = S \setminus pr(S)$ is the set of terminal nodes and $S \setminus S^T$ is the set of non-terminal nodes. For each $s \neq s_0$, $pr(s)$, also denoted s^-, is the (unique) immediate predecessor of s, and, for each $s \in S \setminus S^T$, we let $s^+ = \{\sigma \in S : \ s = pr(\sigma)\}$ be the (finite) set of immediate successors of s. Moreover, for $\tau \in [1,T]$ and every $s \in S \setminus \bigcup_{t=0}^{\tau-1} S^t$ we define, by induction, $pr^\tau(s) = pr(pr^{\tau-1}(s))$.

[3] In this paper, we shall use the following notations. An $(S\times J)-$matrix A is an element of $\mathbb{R}^{S\times J}$, with entries $(a(s,j))_{s\in S, j\in J}$ and for every subsets $\widetilde{S} \subset S$ and $\widetilde{J} \subset J$, the $(\widetilde{S}\times\widetilde{J})-$sub-matrix of A is the $(\widetilde{S}\times\widetilde{J})-$matrix $\widetilde{A}$ with entries $\widetilde{a}(s,j) = a(s,j)$ for every $(s,j) \in \widetilde{S}\times\widetilde{J}$. We recall that the transpose of A is the unique $(J\times S)-$matrix ${}^tA$ satisfying $(Ax)\cdot y = x\cdot({}^tAy)$, for every $x\in\mathbb{R}^J$, $y \in \mathbb{R}^S$. We shall denote by $rankA$ the rank of the matrix A. Let x, y be in $\mathbb{R}^n$; we shall use the notation $x \geq y$ (resp. $x \gg y$) if $x_h \geq y_h$ (resp. $x_h \gg y_h$) for every $h = 1,\ldots,n$ and we let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : \ x \geq 0\}$, $\mathbb{R}^n_{++} = \{x\in\mathbb{R}^n : \ x \gg 0\}$. We shall also use the notation $x > y$ if $x \geq y$ and $x \neq y$.

From now on, for every node $s \in S$ we denote by $\mathcal{S}^+(s)$ and by $\mathcal{S}^-(s)$, respectively, the set of successors (not necessarily immediate) and the set of predecessors of s, defined by

$$\mathcal{S}^+(s) = \{\sigma \in S : \ \exists \tau \in [1, T] \mid s = pr^\tau(\sigma)\},$$

$$\mathcal{S}^-(s) = \{\sigma \in S : \ \exists \tau \in [1, T] \mid \sigma = pr^\tau(s)\}.$$

If $\sigma \in \mathcal{S}^+(s)$ [resp. $\sigma \in \mathcal{S}^+(s) \cup \{s\}$], we shall also use the notation $\sigma > s$ [resp. $\sigma \geq s$]. We notice that $\mathcal{S}^+(s)$ is nonempty if and only if $s \notin S^T$ and $\mathcal{S}^-(s)$ is nonempty if and only if $s \neq s_0$. Moreover, one has $\sigma \in \mathcal{S}^+(s)$ if and only if $s \in \mathcal{S}^-(\sigma)$ (and similarly $\sigma \in s^+$ if and only if $s = \sigma^-$).

2.2 The stochastic exchange economy

At each node $s \in S$, there is a spot market where a finite set H of divisible physical commodities is available. We assume that each commodity does not last for more than one period. In this model, a commodity is a couple (h, s) of a physical commodity $h \in H$ and a node $s \in S$ at which it will be available, so the commodity space is $\mathbb{R}^L$, where $L = H \times S$. An element x in $\mathbb{R}^L$ is called a *consumption*, that is $x = (x(s))_{s \in S} \in \mathbb{R}^L$, where $x(s) = (x(h, s))_{h \in H} \in \mathbb{R}^H$, for every $s \in S$.

In the following we denote by $p = (p(s))_{s \in S} \in \mathbb{R}^L$ the vector of spot prices and $p(s) = (p(h, s)_{h \in H}$ is called the spot price at node s. The spot price $p(h, s)$ is the price paid, at date $t(s)$, for the delivery of one unit of commodity h at node s. Thus the value of the consumption $x(s)$ at node $s \in S$ (evaluated in unit of account of node s) is

$$p(s) \cdot x(s) = \sum_{h \in H} p(h, s) x(h, s).$$

Without any financial instrument (or any present value factor for the different nodes), it is not possible to compare the evaluations $p(s) \cdot x(s)$ and $p(s') \cdot x(s')$ in two different nodes $s \neq s'$.

There is a finite set I of consumers. Each consumer $i \in I$ has a *consumption set* $X_i \subset \mathbb{R}^L$ which is the set of her possible consumptions; an element $x = (x_i)_{i \in I} \in \prod_{i \in I} X_i$ is called an *allocation*. The tastes of each consumer $i \in I$ are represented by a *strict preference correspondence* $P_i : \prod_{j\in I} X_j \longrightarrow X_i$, where $P_i(x)$ defines the set of consumptions that are strictly preferred by i to x_i, if $x = (x_j)_{j\in I} \in \prod_{j\in I} X_j$, that is, given the consumptions x_j for the other consumers $j \neq i$. Thus P_i represents the tastes of consumer i but also her behavior under time and uncertainty, in particular her impatience, and her attitude towards risk. If consumers'preferences are represented by utility functions $u_i : X_i \longrightarrow \mathbb{R}$, for every $i \in I$, the preferences correspondences are defined as $P_i(x) = \{x'_i \in X_i \mid u_i(x'_i) > u_i(x_i)\}$. Finally, at each node $s \in S$, every consumer $i \in I$ has a *node-endowment* $e_i(s) \in \mathbb{R}^H$ (contingent to the fact that s prevails) and we denote by $e_i = (e_i(s))_{s\in S} \in \mathbb{R}^L$ her *endowment vector* across the different nodes. The exchange economy $\mathcal{E}$ can thus be summarized by

$$\mathcal{E} = [\mathcal{S}; H; I; (X_i, P_i, e_i)_{i\in I}].$$

2.3 The financial structure

We consider finitely many financial assets and we denote by J the (possibly empty) set of assets[4]. An asset $j \in J$ of the economy is a contract which is issued at a given (unique) node in S, denoted by $s(j)$ and is called the *emission node* of j. Each asset j is bought (or sold) at its emission node $s(j)$ and yields payoffs only at the successor nodes σ of $s(j)$, that is, for $\sigma \in \mathcal{S}^+(s(j))$. For the sake of convenient notations, we shall in fact consider the payoff of asset j at every node $s \in S$ and assume that it is zero if s is not a successor of the emission node $s(j)$. To be able to deal with real assets, it is important to allow the payoff to depend upon the spot price vector $p \in \mathbb{R}^L$ and we denote by $v(p, s, j)$ the payoff of asset j at node $s \in S$. Formally, we assume that $v(p, s, j) = 0$ if $s \notin \mathcal{S}^+(s(j))$. With the above convention, we notice that every asset has a zero payoff at the initial node, that is $v(p, s_0, j) = 0$ for every $j \in J$; furthermore, every asset j which is emitted at the terminal date has a zero payoff, that is, if $s(j) \in S^T$, $v(p, s, j) = 0$ for every $s \in S$.

The *payoff matrix* $V(p)$ is the $(S \times J)-$matrix, whose coefficients are $v(p, s, j)$ $(s \in S, j \in J)$. We denote by $V(p, s) \in \mathbb{R}^J$ the $s-$th row of $V(p)$ and by $V(p, j) \in \mathbb{R}^S$

[4]The case of no financial assets – i.e., J is empty – is called pure spot markets.

the $j-$th column on $V(p)$.

For every consumer $i \in I$, if $z_i^j > 0$ [resp. $z_i^j < 0$], then $|z_i^j|$ will denote the quantity of asset $j \in J$ bought [resp. sold] by agent i at the emission node $s(j)$. The vector $z_i = (z_i^j)_{j \in J} \in \mathbb{R}^J$ is called the *portfolio* of agent i. We assume that each consumer $i \in I$ is endowed with a *portfolio set* $Z_i \subset \mathbb{R}^J$, which represents the set of portfolios that are (institutionally) admissible for agent i. This general framework allows us to treat, for example, the following important cases:

- $Z_i = \mathbb{R}^J$ (unconstrained portfolios);

- $Z_i \subset \underline{z}_i + \mathbb{R}^J_+$, for some $\underline{z}_i \in -\mathbb{R}^J_+$ (exogenous bounds on short sales);

- $Z_i = B_J(0,1)$[5] (bounded portfolio sets).

The price of asset j is denoted by q^j and we recall that it is paid at its emission node $s(j)$. We let $q = (q^j) \in \mathbb{R}^J$ be the asset price (vector).

Definition 2.2 *A* financial asset structure $\mathcal{F} = (J, (Z_i)_{i \in I}, (s(j))_{j \in J}, V)$, *associated to a finite event tree* $\mathcal{S}$ *and a set of agents* I, *consists of*

- *a set of assets* J,

- *a collection of portfolio sets* $Z_i \subset \mathbb{R}^J$ *for every agent* $i \in I$,

- *a node of issue* $s(j) \in S$ *for each asset* $j \in J$,

- *a payoff mapping* $V : \mathbb{R}^L \to (\mathbb{R}^S)^J$ *which associates, to every spot price* $p \in \mathbb{R}^L$ *the* $(S \times J)-$*return matrix* $V(p) = (v(p,s,j))_{s \in S, j \in J}$, *and satisfies the condition* $v(p,s,j) = 0$ *if* $s \notin \mathcal{S}^+(s(j))$.

Given the spot price $p \in \mathbb{R}^L$ *and the asset price* $q \in \mathbb{R}^J$ *we associate to* $\mathcal{F}$ *its* full matrix of returns $W_{\mathcal{F}}(p,q)$, *which is the* $(S \times J)-$*matrix of entries*

$$w_{\mathcal{F}}(p,q)(s,j) := v(p,s,j) - \delta_{s,s(j)} q^j,$$

where $\delta_{s,\sigma} = 1$ *if* $s = \sigma$ *and* $\delta_{s,\sigma} = 0$ *otherwise. We denote by* $W_{\mathcal{F}}(p,q,s) \in \mathbb{R}^J$ *the* $s-$*th row of the matrix* $W_{\mathcal{F}}(p,q)$, *and by* $W_{\mathcal{F}}(p,q,j) \in \mathbb{R}^S$ *its* $j-$*th column.*

So, for a given portfolio $z \in \mathbb{R}^J$ (and given prices (p,q)) the full flow of returns is

[5]When no confusion is possible, we shall denote by $\|\cdot\|$ the Euclidean norm in the different Euclidean spaces $\mathbb{R}^S$, $\mathbb{R}^J$, $\mathbb{R}^L$, used in this paper. By $B_L(x,r)$ we denote the closed ball centered at $x \in \mathbb{R}^L$ of radius $r > 0$, namely $B_L(x,r) = \{y \in \mathbb{R}^L : \|y - x\| \leq r\}$.

$W_{\mathcal{F}}(p,q)z$ and the (full) financial return at node s is

$$[W_{\mathcal{F}}(p,q)z](s) := W_{\mathcal{F}}(p,q,s)\cdot z = \sum_{j\in J} v(p,s,j)z^j - \sum_{j\in J}\delta_{s,s(j)}q^j z^j$$
$$= \sum_{\{j\in J|\ s(j)<s\}} v(p,s,j)z^j - \sum_{\{j\in J|\ s(j)=s\}} q^j z^j,$$

and it is easy to see that, for every $\lambda\in\mathbb{R}^S$, and every $j\in J$, one has:

$$[{}^tW_{\mathcal{F}}(p,q)\lambda](j) = \sum_{s\in S}\lambda(s)v(p,s,j) - \sum_{s\in S}\lambda(s)\delta_{s,s(j)}$$
$$= \sum_{s>s(j)}\lambda(s)v(p,s,j) - \lambda(s(j))q^j. \tag{2.1}$$

In the following, when the financial structure $\mathcal{F}$ remains fixed, while only prices vary, we shall simply denote by $W(p,q)$ the full matrix of returns. In the case of unconstrained portfolios, namely $Z_i = \mathbb{R}^J$, for every $i\in I$, the financial asset structure will be simply denoted by $\mathcal{F} = (J,(s(j))_{j\in J},V)$.

2.3.1 Short-lived and long-lived assets

An asset j is said to be *short-lived*, when the payoffs are paid only at the immediate successors of its emission node, that is, formally, for every spot price $p\in\mathbb{R}^L$, $v(p,s,j)=0$ if $s\notin s(j)^+$. An asset is said to be *long-lived* if it is not short-lived. A financial structure is said to be *short-lived* if all its assets are short-lived; it is said to be *long-lived* if it is not short-lived.

The multiperiod model with short-lived assets is a natural generalization of the classical two-date model ($T=1$), which has been extensively studied in the literature due to its simple tractability. The next proposition recalls that several important properties of the two-date model are still valid in the case of short-lived financial structures. First, the list of emission nodes $(s(j))_{j\in J}$ of the (non-zero) short-lived assets is uniquely determined by the knowledge of the return matrix $V(p)$, and, secondly, the relationship between the ranks of the matrices $V(p)$ and $W_{\mathcal{F}}(p,q)$ can be simply formulated.

Proposition 2.1 *For short-lived financial structures $\mathcal{F}$, the following holds:*

(a) if, for every $j\in J$, $V(p,j)\neq 0$, then the emission node $s(j)$ is uniquely

determined by the knowledge of the payoff vector $V(p, j)$, that is, $s(j) = s^-$ for every $s \in S$ such that $v(p, s, j) \neq 0$;

(b) $rank V(p) \leq rank W_{\mathcal{F}}(p, q)$ for every $(p, q) \in \mathbb{R}^L \times \mathbb{R}^J$;

(c) $rank V(p) = rank W_{\mathcal{F}}(p, q)$ if ${}^t W_{\mathcal{F}}(p, q)\lambda = 0$ for some $\lambda \in \mathbb{R}^S_{++}$.

The proof of Proposition 2.1 is given in the Appendix.

2.4 Financial equilibria

2.4.1 Financial equilibria without retrading

We now consider a financial exchange economy, which is defined as the couple of an exchange economy $\mathcal{E}$ and a financial structure $\mathcal{F}$. It can thus be summarized by

$$(\mathcal{E}, \mathcal{F}) := [\mathcal{S}, H, I, (X_i, P_i, e_i)_{i \in I}; J, (Z_i)_{i \in I}, (s(j))_{j \in J}, V].$$

Given the spot price vector $p \in \mathbb{R}^L$ and the asset price vector $q \in \mathbb{R}^J$, the *budget set* of consumer $i \in I$ is[6]

$$\begin{aligned} B^i_{\mathcal{F}}(p, q) &= \{(x_i, z_i) \in X_i \times Z_i : \ \forall s \in S, \ \ p(s) \cdot [x_i(s) - e_i(s)] \leq [W(p, q) z_i](s)\} \\ &= \{(x_i, z_i) \in X_i \times Z_i : \ p \,\square\, (x_i - e_i) \leq W(p, q) z_i\}. \end{aligned}$$

We now introduce the notion of financial equilibrium.

Definition 2.3 *A* financial equilibrium *of the financial exchange economy $(\mathcal{E}, \mathcal{F})$ is a collection of strategies and prices $\big((x_i^*, z_i^*)_{i \in I}, p^*, q^*\big) \in (\mathbb{R}^L \times \mathbb{R}^J)^I \times \mathbb{R}^L \setminus \{0\} \times \mathbb{R}^J$ such that, if $x^* = (x_i^*)_{i \in I}$,*

(a) for every $i \in I$, (x_i^*, z_i^*) maximizes the preferences P_i, in the sense that

$$(x_i^*, z_i^*) \in B^i_{\mathcal{F}}(p^*, q^*) \ \text{and} \ [P_i(x^*) \times Z_i] \cap B^i_{\mathcal{F}}(p^*, q^*) = \emptyset;$$

(b) $\displaystyle\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$ and $\displaystyle\sum_{i \in I} z_i^* = 0$.

---

[6]For $x = (x(s))_{s \in S}, p = (p(s))_{s \in S}$ in $\mathbb{R}^L = \mathbb{R}^{H \times S}$ (with $x(s), p(s)$ in$\mathbb{R}^H$) we let $p \,\square\, x = (p(s) \cdot x(s))_{s \in S} \in \mathbb{R}^S$.

2.4.2 Retrading financial assets and equilibria

In this section we will show that, if every asset of the financial structure $\mathcal{F}$ can be retraded at each node, the previous equilibrium notion coincides with another concept widely used in the literature (see for example Magill-Quinzii [23]).

To every asset $j \in J$ and every node $\sigma > s(j)$ which is not a *maturity node*[7] of j we define the new asset $\tilde{\jmath} = (j, \sigma)$, which is issued at σ, and has the same payoffs as asset j at every node which succeeds σ. For the sake of convenient notations, we shall allow to retrade every asset j at every node $\sigma \in S$.[8]

Throughout this section we shall assume that the portfolios are unconstrained, that is, $Z_i = \mathbb{R}^J$, for every $i \in I$.

Definition 2.4 *The retrading of asset $j \in J$ at node $\sigma \in S$, denoted $\tilde{\jmath} = (j, \sigma)$, is the asset issued at σ, that is, $s(j, \sigma) = \sigma$, and whose flow of payoffs is given by*

$\tilde{v}(p, s, (j, \sigma)) = v(p, s, j)$, if $\sigma < s$;

$\tilde{v}(p, s, (j, \sigma)) = 0$, otherwise.

Given the financial structure $\mathcal{F} = (J, (s(j))_{j \in J}, V)$, we associate a new financial structure $\widetilde{\mathcal{F}} = (\widetilde{J}, (s(\tilde{\jmath}))_{\tilde{\jmath} \in \widetilde{J}}, \widetilde{V})$, called the retrading extension *of $\mathcal{F}$, which consists of all the retradings (j, σ) of asset $j \in J$ at node $\sigma \in S$. Hence $\widetilde{J} = J \times S$ and the $S \times \widetilde{J}$-matrix $\widetilde{V}(p)$ has for coefficients $\tilde{v}(p, s, (j, \sigma))$, as defined above.*

We denote by $q^j(\sigma)$ the price of asset (j, σ) (i.e., the retrading of asset j at node σ), which is sometimes also called the retrading price of asset j at node *also*. So, for the financial structure $\widetilde{\mathcal{F}}$, both the asset price vector $q = (q^j(\sigma))_{j \in J, \sigma \in S}$ and the portfolio $z = (z^j(\sigma))_{j \in J, \sigma \in S}$ now belong to $\mathbb{R}^{J \times S}$. Given $p \in \mathbb{R}^L$, q and z in$\mathbb{R}^{J \times S}$,

[7]We recall that the maturity nodes of an asset j are the nodes $s > s(j)$ such that $v(p, s, j) \neq 0$ and $v(p, \sigma, j) = 0$ for every $\sigma > s$.

[8]In particular, if σ is a terminal node ($\sigma \in S^T$) the payoff of the asset (j, σ) is zero (i.e., $\tilde{v}(p, s, (j, \sigma)) = 0$ for every $s \in S$). However, these assets do not affect the equilibrium notion since, under the Non-Satiation Assumption at every Node, the corresponding equilibrium price $(q^*)^{(j,\sigma)}$ must be zero (otherwise it would lead to an arbitrage situation which does not prevail at equilibrium).

the full financial return of $\widetilde{\mathcal{F}}$ at node $s \in S$ is

$$[W_{\widetilde{\mathcal{F}}}(p,q)z](s) = \sum_{(j,\sigma)\in J\times S} \widetilde{v}(p,s,(j,\sigma))z^j(\sigma) - \sum_{(j,\sigma)\in J\times S} \delta_{s,s(j,\sigma)} q^j(\sigma)z^j(\sigma)$$
$$= \sum_{j\in J}\sum_{\sigma<s} v(p,s,j)z^j(\sigma) - \sum_{j\in J} q^j(s)z^j(s).$$

We now give the definition of equilibrium which is most often used when retrading is allowed. Given the financial structure $\mathcal{F} = (J,(s(j))_{j\in J},V)$ and given $p \in \mathbb{R}^L$, $q \in \mathbb{R}^{J\times S}$, we first define the budget set:

$$\widetilde{B}^i_{\mathcal{F}}(p,q) = \{(x_i,y_i) \in X_i \times \mathbb{R}^{J\times S} \ : \ p \,\Box\, (x_i - e_i) \le \widetilde{W}_{\mathcal{F}}(p,q)y_i\}$$

where we let, for $y = (y^j(s))_{(j,s)\in J\times S} \in \mathbb{R}^{J\times S}$:

$$[\widetilde{W}_{\mathcal{F}}(p,q)y](s) = \begin{cases} -\sum_{j\in J} q^j(s_0)y^j(s_0), \\ \displaystyle\sum_{j\in J} v(p,s,j)y^j(s^-) + \sum_{j\in J} q^j(s)y^j(s^-) - \sum_{j\in J} q^j(s)y^j(s), \ \ \forall s \ne s_0. \end{cases}$$

We recall that we have allowed the retrading of assets at terminal nodes, for the sake of convenient notations; so we don't need above to distinguish the cases $s \in S^T$ ans $s \notin S^T$.[9]

Definition 2.5 *A* financial equilibrium with retrading *of the economy $\mathcal{E}$ and the financial structure $\mathcal{F} = (J,(s(j))_{j\in J},V)$ is a collection of strategies and prices $\big((x_i^*,y_i^*)_{i\in I},p^*,q^*\big) \in (\mathbb{R}^L \times \mathbb{R}^{J\times S})^I \times \mathbb{R}^L \setminus \{0\} \times \mathbb{R}^{J\times S}$ such that, if $x^* = (x_i^*)_i$,*

(a) for every $i \in I$, $(x_i^*,y_i^*) \in \widetilde{B}^i_{\mathcal{F}}(p^*,q^*)$ and $[P_i(x^*) \times \mathbb{R}^{J\times S}] \cap \widetilde{B}^i_{\mathcal{F}}(p^*,q^*) = \emptyset$;

(b) $\displaystyle\sum_{i\in I} x_i^* = \sum_{i\in I} e_i$ and $\displaystyle\sum_{i\in I} y_i^* = 0$.

The next proposition shows that, for a given exchange economy $\mathcal{E}$, financial equilibria with retrading associated to the financial structure $\mathcal{F}$ are in a one-to-one correspondence with the financial equilibria associated to the retrading extension $\widetilde{\mathcal{F}}$ of $\mathcal{F}$. The correspondence will only change the equilibrium portfolios via the

[9]But again, at equilibrium, under a standard nonsatiation assumption (see NSS below), a no-arbitrage argument will imply that $q^j(s) = 0$ if $s \in S^T$. So allowing assets to be emitted at terminal nodes does not matter.

mapping $\varphi : \mathbb{R}^{J\times S} \longrightarrow \mathbb{R}^{J\times S}$defined by

$$\varphi(z)(j,s) = \sum_{\sigma\leq s} z(j,\sigma), \text{ for every } z \in \mathbb{R}^{J\times S},$$

and φ is easily shown to be linear and bijective[10]

Proposition 2.2 *Let $\mathcal{E}$ be an exchange economy, let $\mathcal{F} = (J, (s(j))_{j\in J}, V)$ and let $\widetilde{\mathcal{F}} = (\widetilde{J}, (s(j))_{j\in\widetilde{J}}, \widetilde{V})$ be the retrading extension of $\mathcal{F}$. Then the two following conditions are equivalent:*

(i) $((x_i^*, z_i^*)_{i\in I}, p^*, q^*)$ is a financial equilibrium of $(\mathcal{E}, \widetilde{\mathcal{F}})$.

(ii) $((x_i^*, \varphi(z_i^*))_{i\in I}, p^*, q^*)$ is a financial equilibrium with retrading of $(\mathcal{E}, \mathcal{F})$;

Proof. Since φ is linear and bijective, the equality $\sum_{i\in I} z_i^* = 0$ holds if and only if $\sum_{i\in I} z_i^{**} = 0$. Thus the end of the proof is a consequence of the following claim.

Claim 2.1 *For every $(p,q) \in \mathbb{R}^L \times \mathbb{R}^{J\times S}$ one has*

(i) for every $z \in \mathbb{R}^{J\times S}$, $W_{\widetilde{\mathcal{F}}}(p,q)z = \widetilde{W}_{\mathcal{F}}(p,q)\varphi(z)$;

(ii) $B^i_{\widetilde{\mathcal{F}}}(p,q) = \{(x_i, z_i) \| (x_i, \varphi(z_i)) \in \widetilde{B}^i_{\mathcal{F}}(p,q)\}$.

Proof. Part (i). For $s = s_0$, we have $\varphi(z)(j,s_0) = z(j,s_0)$ for every $j \in J$; from the definitions of $W_{\widetilde{\mathcal{F}}}(p,q)$ and $\widetilde{W}_{\mathcal{F}}(p,q)$, we get:

$$\begin{aligned}
[\widetilde{W}_{\mathcal{F}}(p,q)\varphi(z)](s_0) &= -\sum_{j\in J} q^j(s_0)[\varphi(z)(j,s_0)] \\
&= -\sum_{j\in J} q^j(s_0)z^j(s_0) = [W_{\widetilde{\mathcal{F}}}(p,q)z](s_0).
\end{aligned}$$

For $s \neq s_0$ we have

$$\begin{aligned}
[\widetilde{W}_{\mathcal{F}}(p,q)\varphi(z)](s) &= \sum_{j\in J} v(p,s,j)\varphi(z)(j,s^-) + \sum_{j\in J} q^j(s)\varphi(z)(j,s^-) - \sum_{j\in J} q^j(s)\varphi(z)(j,s) \\
&= \sum_{j\in J} v(p,s,j) \sum_{\sigma\leq s^-} z^j(\sigma) - \sum_{j\in J} q^j(s)[\varphi(z)(j,s) - \varphi(z)(j,s^-)] \\
&= \sum_{(j,\sigma)\in J\times S} \widetilde{v}(p,s,(j,\sigma))z^j(\sigma) - \sum_{j\in J} q^j(s)z^j(s) \\
&= [W_{\widetilde{\mathcal{F}}}(p,q)z](s).
\end{aligned}$$

[10]It is easy to see that the inverse of φ is the mapping $\psi : \mathbb{R}^{J\times S} \longrightarrow \mathbb{R}^{J\times S}$ defined by $\psi(z)(j,s) = z(j,s) - z(j,s^-)$, if $s \neq s_0$, and $\psi(z)(j,s_0) = z(j,s_0)$, if $s = s_0$.

Part (ii). It is a direct consequence of (i). $\square$

2.4.3 No-arbitrage and financial equilibria

When portfolios may be constrained, the concept of no-arbitrage has to be suitably modified. In particular, we shall make a distinction between the definitions of arbitrage-free portfolios and arbitrage-free financial structure.

Definition 2.6 *Given the financial structure $\mathcal{F} = (J, (Z_i)_{i\in I}, (s(j))_{j\in J}, V)$, the portfolio $z_i^* \in Z_i$ is said to* have no arbitrage opportunities *or to be* arbitrage-free *for agent $i \in I$ at the price $(p, q) \in\in \mathbb{R}^L \times \mathbb{R}^J$ if there is no portfolio $z_i \in Z_i$ such that $W_{\mathcal{F}}(p, q)z_i > W_{\mathcal{F}}(p, q)z_i^*$, that is, $[W_{\mathcal{F}}(p, q)z_i](s) \geq [W_{\mathcal{F}}(p, q)z_i^*](s)$, for every $s \in S$, with at least one strict inequality, or, equivalently, if*

$$W_{\mathcal{F}}(p, q)\ (Z_i - z_i^*) \cap \mathbb{R}^S_+ = \{0\}.$$

The financial structure $\mathcal{F}$ is said to be arbitrage-free *at (p, q) if there exists no portfolios $z_i \in Z_i$ $(i \in I)$ such that $W_{\mathcal{F}}(p, q)(\sum_{i\in I} z_i) > 0$, or, equivalently, if:*

$$W_{\mathcal{F}}(p, q)\Big(\sum_{i\in I} Z_i\Big) \cap \mathbb{R}^S_+\ \setminus \{0\} = \emptyset.$$

If the financial structure $\mathcal{F}$ is arbitrage-free at (p, q), and if we let $z_i^* \in Z_i$ $(i \in I)$ such that $\sum_{i\in I} z_i^* = 0$, it is easy to see that, for every $i \in I$, z_i^* is arbitrage-free at (p, q). The converse is true, for example, when some agent's portfolio set is unconstrained, that is, $Z_i = \mathbb{R}^J$ for some i.

We recall that equilibrium portfolios are necessarily arbitrage-free under the following Non-Satiation Assumption:

Assumption NS (i) For every $x^* \in \prod_{i\in I} X_i$ such that $\sum_{i\in I} x_i^* = \sum_{i\in I} e_i$,

(Non-Satiation at Every Node) for every $s_i \in S$, there exists $x = (x_i)_i \in \prod_{i\in I} X_i$ such that, for each $s \neq s_i$, $x_i(s) = x_i^*(s)$ and $x_i \in P_i(x^*)$;

(ii) if $x_i \in P_i(x^*)$, then $[x_i, x_i^*[\subset P_i(x^*)$.

Proposition 2.3 *Under (NS), if $((x_i^*, z_i^*)_{i\in I}, p^*, q^*)$ is a financial equilibrium of the economy $(\mathcal{E}, \mathcal{F})$, then z_i^* is arbitrage-free at (p^*, q^*) for every $i \in I$.*

Proof. By contradiction. If, for some $i \in I$, the portfolio z_i^* is not arbitrage-free at (p^*, q^*), then there exists $z_i \in Z_i$ such that $W_{\mathcal{F}}(p^*, q^*)z_i > W_{\mathcal{F}}(p^*, q^*)z_i^*$, namely $[W_{\mathcal{F}}(p^*, q^*)z_i](s) \geq [W_{\mathcal{F}}(p^*, q^*)z_i^*](s)$, for every $s \in S$, with at least one strict inequality, say for $\bar{s} \in S$.

Since $\sum_{i \in I}(x_i^* - e_i) = 0$, from Assumption (NS.i), there exists $x = (x_i)_i \in \prod_{i \in I} X_i$ such that, for each $s \neq \bar{s}$, $x_i(s) = x_i^*(s)$ and $x_i \in P_i(x^*)$. Let us consider $\lambda \in]0, 1[$ and define $x_i^\lambda := \lambda x_i + (1 - \lambda)x_i^*$; then, by Assumption (NS.ii), $x_i^\lambda \in]x_i, x_i^*[\subset P_i(x^*)$.

In the following, we prove that, for $\lambda > 0$ small enough, $(x_i^\lambda, z_i) \in B_{\mathcal{F}}^i(p^*, q^*)$, which will contradict the fact that $[P_i(x^*) \times Z_i] \cap B_{\mathcal{F}}^i(p^*, q^*) = \emptyset$ (since $((x_i^*, z_i^*)_{i \in I}, p^*, q^*)$ is a financial equilibrium). Indeed, since $(x_i^*, z_i^*) \in B_{\mathcal{F}}^i(p^*, q^*)$, for every $s \neq \bar{s}$ we have:

$$p^*(s) \cdot [x_i^\lambda(s) - e_i(s)] = p^*(s) \cdot [x_i^*(s) - e_i(s)] \leq [W_{\mathcal{F}}(p^*, q^*)z_i^*](s) \leq [W_{\mathcal{F}}(p^*, q^*)z_i](s).$$

Now, for $s = \bar{s}$, we have

$$p^*(\bar{s}) \cdot [x_i^*(\bar{s}) - e_i(\bar{s})] \leq [W_{\mathcal{F}}(p^*, q^*)z_i^*](\bar{s}) < [W_{\mathcal{F}}(p^*, q^*)z_i](\bar{s}).$$

But, when $\lambda \to 0$, $x_i^\lambda \to x_i^*$, hence for $\lambda > 0$ small enough we have

$$p^*(\bar{s}) \cdot [x_i^\lambda(\bar{s}) - e_i(\bar{s})] < [W_{\mathcal{F}}(p^*, q^*)z_i](\bar{s}).$$

Consequently, $(x_i^\lambda, z_i) \in B_{\mathcal{F}}^i(p^*, q^*)$. □

2.4.4 A characterization of no-arbitrage

The following result provides an important property of no-arbitrage portfolios.

Theorem 2.1 *Let $\mathcal{F} = (J, (Z_i)_{i \in I}, (s(j))_{j \in J}, V)$, let $(p, q) \in \mathbb{R}^L \times \mathbb{R}^J$, for $i \in I$, let $z_i \in Z_i$, assume that Z_i is convex and consider the following statements:*

(i) there exists $\lambda_i = (\lambda_i(s))_{s \in S} \in \mathbb{R}^S_{++}$ such that ${}^tW_{\mathcal{F}}(p, q)\lambda_i \in N_{Z_i}(z_i)$,[11]

or, equivalently, there exists $\eta \in N_{Z_i}(z_i)$ such that:

$$-\lambda_i(s(j))q^j + \sum_{s > s(j)} \lambda_i(s)v(p, s, j) = \eta_j \text{ for every } j \in J.$$

(ii) the portfolio z_i is arbitrage-free for agent $i \in I$ at (p, q).

[11]We recall that $N_{Z_i}(z_i)$ is the normal cone to Z_i at z_i, which is defined as $N_{Z_i}(z_i) := \{\eta \in \mathbb{R}^J : \eta \cdot z_i \geq \eta \cdot z_i', \ \forall z_i' \in Z_i\}$.

The implication [(i) $\Rightarrow$ (ii)] always holds and the converse is true under the additional assumption that Z_i is a polyhedral set[12].

The above Theorem 2.1 is a consequence of Proposition 5.1, stated and proved in the Appendix, the main part (i.e., the existence of positive node prices $\lambda_i(s)$) being due to Koopmans [19].

3 Existence of financial equilibria

3.1 The main existence result

We will prove the existence of a financial equilibrium when agents may have constrained portfolios, that is, without assuming that $Z_i = \mathbb{R}^J$. We shall allow the financial structure to be general enough to cover important cases such as bounded assets (as in Radner [32]), nominal assets, and numéraire assets; our approach however does not cover the general case of real assets which needs a different treatment. Let us consider, the financial economy

$$(\mathcal{E}, \mathcal{F}) = [\mathcal{S}, H, I, (X_i, P_i, e_i)_{i \in I}, (J, (Z_i)_{i \in I}), (s(j))_{j \in J}, V].$$

We introduce the following assumptions.

Assumption (C) (**Consumption Side**) *For all $i \in I$ and all $x^* = (x_i^*) \in \prod_{i \in I} X_i$,*

(i) X_i is a closed and convex subset of $\mathbb{R}^L$;

(ii) the preference correspondence P_i, from $\prod_{i \in I} X_i$ to X_i, is lower semicontinuous[13] *and $P_i(x^*)$ is convex;*

[12] A subset $C \subset \mathbb{R}^n$ is said to be *polyhedral* if it is the intersection of finitely many closed half-spaces, namely $C = \{x \in \mathbb{R}^n : \ Ax \leq b\}$, where A is a real $(m \times n)-$matrix, and $b \in \mathbb{R}^m$. Note that polyhedral sets are always closed and convex and that the empty set and the whole space $\mathbb{R}^n$ are both polyhedral.

[13] A correspondence $\varphi : X \longrightarrow Y$ is said to be lower semicontinuous at $x_0 \in X$ if, for every open set $V \subset Y$ such that $V \cap \varphi(x_0)$ is not empty, there exists a neighborhood U of x_0 in X such that, for all $x \in U$, $V \cap \varphi(x)$ is nonempty. The correspondence φ is said to be lower semicontinuous if it is lower semicontinuous at each point of X.

(iii) *for every* $x_i \in P_i(x^*)$ *for every* $x'_i \in X_i, x'_i \neq x_i$, $[x'_i, x_i[\cap P_i(x^*) \neq \emptyset$; [14]

(iv) $x_i \notin P_i(x)$;

(v) *(Non-Satiation of Preferences at Every Node) if* $\sum_{i \in I} x^*_i = \sum_{i \in I} e_i$, *for every* $s \in S$ *there exists* $x \in \prod_{i \in I} X_i$ *such that, for each* $s' \neq s$, $x_i(s') = x^*_i(s')$ *and* $x_i \in P_i(x^*)$;

(vi) *(Strong Survival Assumption)* $e_i \in \mathrm{int} X_i$.

Assumption (F) (**Financial Side**)

(i) The application $p \mapsto V(p)$ *is continuous, or, equivalently, the application* $p \mapsto v(p, s, j)$ *is continuous, for every* $s \in S$, $j \in J$;

(ii) for every $i \in I$, Z_i *is a closed, convex subset of* $\mathbb{R}^J$ *containing* 0;

(iii) there exists $i_0 \in I$ *such that* $0 \in intZ_{i_0}$.

We now state the last assumption for which we need to define the set of admissible consumptions and portfolios for a fixed $\lambda \in \mathbb{R}^S_{++}$, that is,

$$B(\lambda) := \{(x_i, z_i)_i \in \prod_{i \in I} X_i \times Z_i : \ \exists (p, q) \in B_L(0,1) \times \mathbb{R}^J, \ {}^tW_{\mathcal{F}}(p,q)\lambda \in B_J(0,1),$$
$$(x_i, z_i) \in B^i_{\mathcal{F}}(p,q) \text{ for every } i \in I, \ \sum_{i \in I} x_i = \sum_{i \in I} e_i, \ \sum_{i \in I} z_i = 0\}.$$

Boundedness Assumption (B_λ) *The set* $B(\lambda)$ *is bounded.*

Assumption (B_λ) will be discussed after the statement of the main result and we will provide different cases under which it is satisfied.

Theorem 3.1 *(a) Let* $(\mathcal{E}, \mathcal{F})$ *be a financial economy satisfying Assumptions* (C), (F) *and let* $\lambda \in \mathbb{R}^S_{++}$ *satisfying* (B_λ). *Let* $i_0 \in I$ *be some agent such that* $0 \in$ *int* Z_{i_0},

[14] This is satisfied, in particular, when $P_i(x^*)$ is open in X_i (for its relative topology).

then there exists a financial equilibrium $\big((x_i^*, z_i^*)_{i\in I}, p^*, q^*\big)$ *of* $(\mathcal{E}, \mathcal{F})$ *such that, for every* $s \in S$, $p^*(s) \neq 0$ *and*

$$^tW_{\mathcal{F}}(p^*, q^*)\lambda \in N_{Z_{i_0}}(z_{i_0}^*),$$

or, equivalently, there exists $\eta^* \in N_{Z_{i_0}}(z_{i_0}^*)$ *such that*

$$\lambda(s(j))q^{*j} = \sum_{s>s(j)} \lambda(s)v(p^*, s, j) - \eta^{*j} \text{ for every } j \in J.$$

(b) If moreover $z_{i_0}^* \in \mathrm{int} Z_{i_0}$, *then* $^tW_{\mathcal{F}}(p^*, q^*)\lambda = 0$, *or, equivalently,*

$$\lambda(s(j))q^{*j} = \sum_{s>s(j)} \lambda(s)v(p^*, s, j) \text{ for every } j \in J,$$

hence the financial structure $\mathcal{F}$ *is arbitrage-free.*

The proof of Theorem 3.1 will be given in several steps in the following section.

The next proposition gives sufficient conditions for Assumption (B_λ) to hold.

Proposition 3.1 *Let* $\lambda \in \mathbb{R}^S_{++}$ *be fixed and assume that, for every* $i \in I$, X_i *is bounded from below. Then Assumption* (B_λ) *is satisfied if one of the following conditions holds:*

(i)[Bounded below portfolios] for every $i \in I$, *the portfolio set* Z_i *is bounded below, namely there exists* $\underline{z}_i \in \mathbb{R}^J$ *such that* $Z_i \subset \underline{z}_i + \mathbb{R}^J_+$*;*

(ii)[Rank condition for Long-Lived Assets] for every $(p, q, \eta) \in B_L(0,1) \times \mathbb{R}^J \times B_J(0,1)$ *such that* $^tW(p,q)\lambda = \eta$, *then* $\mathrm{rank} W(p,q) = \#J$.

(iii)[Rank condition for Short-Lived Assets] $\mathcal{F}$ *consists only of short-lived assets and* $\mathrm{rank} V(p) = \#J$ *for every* $p \in \mathbb{R}^L$.

The proof of Proposition 3.1 is given in the Appendix.

3.2 Existence for various financial models

We now give some consequences of the main existence Theorem 3.1, to get the existence of financial equilibria in important cases such as bounded portfolios (as in

Radner [32]), nominal financial assets, and numéraire assets; our approach however does not cover the general case of real assets which needs a different treatment.

In the case of unconstrained portfolios the existence result is of the following form.

Corollary 3.1 *[Unconstrained portfolio case] Let* $(\mathcal{E}, \mathcal{F})$ *be a financial economy and let* $\lambda \in \mathbb{R}^S_{++}$ *be such that Assumptions* (C), (F) *and* (B_λ) *hold and* $Z_i = \mathbb{R}^J$ *for some* $i \in I$*. Then* $(\mathcal{E}, \mathcal{F})$ *admits a financial equilibrium* $\big((x_i^*, z_i^*)_{i\in I}, p^*, q^*\big) \in \prod_{i\in I}(X_i \times Z_i) \times \mathbb{R}^L \times \mathbb{R}^J$ *such that, for every* $s \in S$, $p^*(s) \neq 0$ *and* q^* *is the no-arbitrage price associated to* λ*, that is*

$$^tW(p^*, q^*)\lambda = 0,$$

or, equivalently,

$$\lambda(s(j))q^{*j} = \sum_{s>s(j)} \lambda(s)v(p^*, s, j) \text{ for every } j \in J.$$

We now turn to the case of bounded from below portfolio sets.

Corollary 3.2 *[Bounded from below portfolio sets] Let* $(\mathcal{E}, \mathcal{F})$ *and* $\lambda \in \mathbb{R}^S_{++}$ *satisfy Assumptions* (C), (F), (B_λ)*, and assume that, for some agent* i_0, $Z_{i_0} = \underline{z} + \mathbb{R}^J_+$*, where* $\underline{z} \in -\mathbb{R}^J_+$*. Then there exists a financial equilibrium* $\big((x_i^*, z_i^*)_{i\in I}, p^*, q^*\big) \in \prod_{i\in I}(X_i \times Z_i) \times \mathbb{R}^L \times \mathbb{R}^J$ *of* $(\mathcal{E}, \mathcal{F})$*, such that, for every* $s \in S$, $p^*(s) \neq 0$ *and* $^tW(p^*, q^*)\lambda \leq 0$ *and the equality holds for each component* j *such that* $(z_{i_0}^*)^j > \underline{z}^j$*, or, equivalently,*

$$\text{for every } j \in J,\ \lambda(s(j))q^{*j} \geq \sum_{s>s(j)} \lambda(s)v(p^*, s, j), \text{with equality if } (z_{i_0}^*)^j > \underline{z}^j.$$

3.2.1 Short-lived financial structures

We first treat the case of nominal financial assets.

Corollary 3.3 *[Short-lived nominal assets] Let us assume that the economy* $(\mathcal{E}, \mathcal{F})$ *satisfies Assumption* (C), X_i *is bounded from below, for every* $i \in I$, $\mathcal{F}$ *consists of nominal short-lived assets and assume that one of the following conditions holds:*

(i)[unconstrained case] $Z_i = \mathbb{R}^J$ *for every* $i \in I$*;*

(ii)[constrained case] - Z_i *is a closed and convex subset of* $\mathbb{R}^J$ *containing 0;*
- $0 \in \mathrm{int} Z_{i_0}$ *for some* $i_0 \in I$*;*
-rank$V = \sharp J$.

For every $\lambda \in \mathbb{R}^S_{++}$, $(\mathcal{E}, \mathcal{F})$ *admits a financial equilibrium* $\big((x_i^*, z_i^*)_{i \in I}, p^*, q^*\big) \in \prod_{i \in I}(X_i \times \mathbb{R}^J) \times \mathbb{R}^L \times \mathbb{R}^J$ *such that, for every* $s \in S$, $p^*(s) \neq 0$ *and* q^* *is the no-arbitrage price associated to* λ*, that is*

$$^tW(q^*)\lambda \in N_{Z_{i_0}}(z^*_{i_0}) \text{ (resp. } ^tW(q^*)\lambda = 0, \text{ under } (i)),$$

or, equivalently, there exists $\eta^* \in N_{Z_{i_0}}(z^*_{i_0})$ *(resp.* $\eta^* = 0$*, under* (i)*) such that*

$$\lambda(s(j))q^{*j} = \sum_{s \in s(j)^+} \lambda(s)v(s,j) - \eta^{*j} \text{ for every } j \in J.$$

Proof. Let $r := rankV$. We can define a new financial structure $\mathcal{F}'$ with r nominal assets by eliminating the redundant assets. Formally, we let $J' \subset J$ be the set of r assets such that the columns $(V(j))_{j \in J'}$ are independent and V' the associated return matrix. The new financial structure is

$$\mathcal{F}' := (J', (s(j))_{j \in J'}, V').$$

Then $rankW_{\mathcal{F}'}(q) = r$ since, by Proposition 2.1, $r = rankV' \leq rankW_{\mathcal{F}'}(q)(\leq \min\{r, S\})$. Consequently, by Proposition 3.1, the set $B(\lambda)$ is bounded.

From the existence theorem (Corollary 3.1), for every $\lambda \in \mathbb{R}^S_{++}$ there exists an equilibrium $((x_i^*, z_i')_{i \in I}, p^*, q')$ of $(\mathcal{E}, \mathcal{F}')$ (where q' and z_i' are in $\mathbb{R}^{J'}$) such that $^tW_{\mathcal{F}'}(q')\lambda = 0$ or, equivalently,

$$\lambda(s(j))q'^j = \sum_{\sigma \in s(j)^+} \lambda(\sigma)v(\sigma, s(j)),$$

for every $j \in J'$. Now it is easy to see that $((x_i^*, z_i^*)_{i \in I}, p^*, q^*)$ is an equilibrium of $(\mathcal{E}, \mathcal{F})$, by defining $q^* \in \mathbb{R}^J$ as $^tW(q^*)\lambda = 0$, that is

$$\lambda(s(j))q^{*j} = \sum_{\sigma \in s(j)^+} \lambda(\sigma)v(\sigma, s(j)),$$

for every $j \in J$, and $z_i^* \in \mathbb{R}^J$ as $(z_i^*)^j = (z_i')^j$, if $j \in J'$, and $(z_i^*)^j = 0$, if $j \in J \setminus J'$. □

4 Proof of the main result

4.1 Proof under additional assumptions

In this section, we shall prove Theorem 3.1 under the additional assumption

Assumption (K) *For every* $i \in I$,

(i) X_i *and* Z_i *are compact;*

(ii)[Local Non-Satiation] for every $x^* = (x_i^*) \in \prod_{i \in I} X_i$, *for every* $x_i \in P_i(x^*)$ *then* $[x_i, x_i^*[\subset P_i(x^*)$.

4.1.1 Preliminary definitions

In the following we fix some agent i_0, say $i_0 = 1$, for whom the assumption $0 \in \mathrm{int} Z_{i_0}$ is satisfied and we fix $\lambda = (\lambda(s))_{s \in S} \in \mathbb{R}^S_{++}$. We recall that for $(p, \eta) \in \mathbb{R}^L \times \mathbb{R}^J$, the vector $q = q(p, \eta) \in \mathbb{R}^J$ is uniquely defined by the equation

$$^t W_{\mathcal{F}}(p, q)\lambda - \eta = 0,$$

which, from Theorem 2.1 is equivalent to saying that

$$q^j(p, \eta) = \frac{1}{\lambda(s(j))} \Big(\sum_{s > s(j)} \lambda(s) v(p, s, j) - \eta^j \Big) \text{ for every } j \in J,$$

and, from Assumption (F), the mapping $(p, \eta) \mapsto q(p, \eta)$ is continuous. For (p, η) in the set $B := \{(p, \eta) \in \mathbb{R}^L \times \mathbb{R}^J : \ \|\lambda \,\Box\, p\| \leq 1, \ \|\eta\| \leq 1\}$, we define

$$\rho(p, \eta) = \max\{0, 1 - \|\lambda \,\Box\, p\| - \|\eta\|\}.$$

Following the so-called Cass' trick, hereafter, we shall distinguish Consumer 1 from the other agents, and we shall extend the budget sets as in Bergstrom ([2]). In the following, we let $\mathbb{1} = (1, \ldots, 1)$ denote the element in $\mathbb{R}^S$, whose coordinates are all equal to one. For $(p, \eta) \in B$, we define the following augmented budget sets: first, for $i = 1$,

$$\beta^1(p, \eta) = \Big\{ x_1 \in X_1 : \ (\lambda \,\Box\, p) \cdot (x_1 - e_1) \leq \sup_{z \in Z_1} \eta \cdot z + \rho(p, \eta) \sum_{s \in S} \lambda(s) \Big\},$$

$$\alpha^1(p, \eta) = \Big\{ x_1 \in X_1 : \ (\lambda \,\Box\, p) \cdot (x_1 - e_1) < \sup_{z \in Z_1} \eta \cdot z + \rho(p, \eta) \sum_{s \in S} \lambda(s) \Big\},$$

and for $i \neq 1$

$$\beta^i(p,\eta) = \Big\{(x_i, z_i) \in X_i \times Z_i : \ p \,\square\, (x_i - e_i) \leq W_{\mathcal{F}}(p, q(p,\eta))z_i + \rho(p,\eta)\mathbb{1}\Big\},$$

$$\alpha^i(p,\eta) = \Big\{(x_i, z_i) \in X_i \times Z_i : \ p \,\square\, (x_i - e_i) << W_{\mathcal{F}}(p, q(p,\eta))z_i + \rho(p,\eta)\mathbb{1}\Big\}.$$

We now define the following enlarged set of agents denoted I_0, by considering all the agents in $i \in I \setminus \{1\}$, by counting twice the agent 1, denoted by $i = (1,1)$ and $i = (1,2)$ and by considering an additional agent denoted $i = 0$. The additional and fictitious agent $i = 0$ is traditional and will fix the equilibrium prices (p^*, q^*) and the agent $i = 1$ has been disaggregated so that $i = (1,1)$ will fix the equilibrium consumption x_1^* and $i = (1,2)$ will fix the equilibrium portfolio z_1^* (which thus can be chosen by two independent maximization problems). For $(x, z, (p,\eta)) \in \prod_{i\in I} X_i \times \prod_{i \in I} Z_i \times B$, we define the correspondences Φ_i for $i \in I_0$ as follows:

$$\Phi_0(x,z,(p,\eta)) = \Big\{(p',\eta') \in B \ \mid \sum_{s\in S}\Big[\lambda(s)(p'(s)-p(s))\cdot\sum_{i\in I}(x_i(s)-e_i(s))\Big] - (\eta'-\eta)\cdot\sum_{i\in I} z_i > 0\Big\},$$

$$\Phi_{1,1}(x,z,(p,\eta)) = \begin{cases} \beta^1(p,\eta) & \text{if } x_1 \notin \beta^1(p,\eta) \\ \alpha^1(p,\eta) \cap P_1(x) & \text{if } x_1 \in \beta^1(p,\eta), \end{cases}$$

$$\Phi_{1,2}(x,z,p,\eta) = \{z_1' \in Z_1 \mid \eta \cdot z_1' > \eta \cdot z_1\},$$

and for every $i \in I$, $i \neq 1$

$$\Phi_i(x,z,(p,\eta)) = \begin{cases} \{(e_i, 0)\} & \text{if } (x_i, z_i) \notin \beta^i(p,\eta) \text{ and } \alpha^i(p,\eta) = \emptyset, \\ \beta^i(p,\eta) & \text{if } (x_i, z_i) \notin \beta^i(p,\eta) \text{ and } \alpha^i(p,\eta) \neq \emptyset, \\ \alpha^i(p,\eta) \cap (P_i(x) \times Z_i) & \text{if } (x_i, z_i) \in \beta^i(p,\eta). \end{cases}$$

4.1.2 The fixed-point argument

The existence proof relies on the following fixed-point-type theorem due to Gale and Mas Colell ([17]).

Theorem 4.1 *Let I_0 be a finite set, let C_i $(i \in I_0)$ be a nonempty, compact, convex subset of some Euclidean space, let $C = \prod_{i \in I} C_i$ and let Φ_i $(i \in I_0)$ be a correspondence from C to C_i, which is lower semicontinuous and convex-valued. Then, there exists $c^* = (c_i^*) \in C$ such that, for every $i \in I_0$ [either $c_i^* \in \Phi_i(c^*)$ or $\Phi_i(c^*) = \emptyset$].*

We now show that, for $i \in I_0$, the sets $C_0 = B$, $C_{1,1} = X_1$, $C_{1,2} = Z_1$, $C_i = X_i \times Z_i$ and the above defined correspondences Φ_i ($i \in I_0$) satisfy the assumptions of Theorem 4.1.

Claim 4.1 *For every $c^* := (x^*, z^*, (p^*, \eta^*)) \in \prod_{i \in I} X_i \times \prod_{i \in I} Z_i \times B$, for every $i \in I_0$, the correspondence Φ_i is lower semicontinuous at c^*, the set $\Phi_i(c^*)$ is convex (possibly empty) and $(p^*, \eta^*) \notin \Phi_0(c^*)$, $x_1^* \notin \Phi_{1,1}(c^*)$, $z_1^* \notin \Phi_{1,2}(c^*)$, $(x_i^*, z_i^*) \notin \Phi_i(c^*)$ for $i > 1$.*

Proof. Let $c^* := (x^*, z^*, (p^*, \eta^*)) \in \prod_{i \in I} X_i \times \prod_{i \in I} Z_i \times B$ be given. We first notice that $\Phi_i(c^*)$ is convex for every $i \in I_0$, recalling that $P_i(x^*)$ is convex, by Assumption (C). Clearly, $(p^*, \eta^*) \notin \Phi_0(c^*)$ and $z_1^* \notin \Phi_{1,2}(c^*)$ from the definition of these two sets; the two last properties $x_1^* \notin \Phi_{1,1}(c^*)$ and $(x_i^*, z_i^*) \notin \Phi_i(c^*)$ follow from the definitions of these sets and the fact that $x_i^* \notin P_i(x^*)$ from Assumption (C).

We now show that Φ_i is lower semicontinuous at c^*.

Step 1: $i \in I$, $i > 1$. Let U be an open subset of $X_i \times Z_i$ such that $\Phi_i(c^*) \cap U \neq \emptyset$. We will distinguish three cases:

Case(i) : $(x_i^*, z_i^*) \notin \beta^i(p^*, \eta^*)$ and $\alpha^i(p^*, \eta^*) = \emptyset$. Then $\Phi_i(c^*) = \{(e_i, 0)\} \subset U$. Since the set $\{(x_i, z_i, (p, \eta)) \mid (x_i, z_i) \notin \beta^i(p, \eta)\}$ is an open subset of $X_i \times Z_i \times B$ (by Assumptions (C) and (F)), it contains an open neighborhood O of c^*. Now, let $c = (x, z, (p, \eta)) \in O$. If $\alpha^i(p, \eta) = \emptyset$ then $\Phi_i(c) = \{(e_i, 0)\} \subset U$ and so $\Phi_i(c) \cap U$ is nonempty. If $\alpha^i(p, \eta) \neq \emptyset$ then $\Phi_i(c) = \beta^i(p, \eta)$. But Assumptions (C) and (F) imply that $(e_i, 0) \in X_i \times Z_i$, hence $(e_i, 0) \in \beta^i(p, \eta)$ (noticing that $\rho(p, q) \geq 0$). So $\{(e_i, 0)\} \subset \Phi_i(c) \cap U$ which is also nonempty.

Case(ii) : $c^* = (x_i^*, z_i^*, (p^*, \eta^*)) \in \Omega_i := \{c = (x_i, z_i, (p, \eta)) : \ (x_i, z_i) \notin \beta^i(p, \eta)$ and $\alpha^i(p, \eta) \neq \emptyset\}$. Then the set Ω_i is clearly open and on the set Ω_i one has $\Phi_i(c) = \beta^i(p, \eta)$. We recall that $\emptyset \neq \Phi_i(c^*) \cap U = \beta^i(p^*, \eta^*) \cap U$. We notice that $\beta^i(p^*, \eta^*) = \mathrm{cl}\ \alpha^i(p^*, \eta^*)$ since $\alpha^i(p^*, \eta^*) \neq \emptyset$. Consequently, $\alpha^i(p^*, \eta^*) \cap U \neq \emptyset$ and we chose a point $(x_i, z_i) \in \alpha^i(p^*, \eta^*) \cap U$, that is, $(x_i, z_i) \in [X_i \times Z_i] \cap U$ and

$$p^* \,\square\, (x_i - e_i) << W_{\mathcal{F}}(p^*, q(p^*, \eta^*)) z_i + \rho(p^*, \eta^*) \mathbb{1}.$$

Clearly the above inequality is also satisfied for the same point point (x_i, z_i) when (p, η) belongs to a neighborhood O of (p^*, η^*) small enough (using the continuity of

$q(\cdot,\cdot)$ and $p(\cdot,\cdot)$. This shows that on O one has $\emptyset \neq \alpha^i(p,\eta) \cap U \subset \beta^i(p,\eta) \cap U = \Phi(c) \cap U$.

$Case(iii)$: $(x_i^*, z_i^*) \in \beta^i(p^*, \eta^*)$. By assumption we have

$$\emptyset \neq \Phi_i(c^*) \cap U = \alpha_i(p^*, q^*) \cap [P_i(x^*) \times Z_i] \cap U.$$

By an argument similar to what is done above, one shows that there exists an open neighborhood N of (p^*, q^*) and an open set M such that, for every $(p,\eta) \in N$, one has $\emptyset \neq M \subset \alpha_i(p,\eta) \cap U$. Since P_i is lower semicontinuous at c^* (by Assumption (C)), there exists an open neighborhood Ω of c^* such that, for every $c \in \Omega$, $\emptyset \neq [P_i(x) \times Z_i] \cap M$, hence

$$\emptyset \neq [P_i(x) \times Z_i] \cap \alpha_i(p,\eta) \cap U \subset \beta_i(p,\eta) \cap U, \text{ for every } c \in \Omega.$$

Consequently, from the definition of Φ_i, we get $\emptyset \neq \Phi_i(c) \cap U$, for every $c \in \Omega$.

The correspondence $\Psi_i := \alpha^i \cap (P_i \times Z_i)$ is lower semicontinuous on the whole set, being the intersection of an open graph correspondence and a lower semicontinuous correspondence. Then there exists an open neighborhood O of $c^* := (x^*, z^*, (p^*, \eta^*))$ such that, for every $(x, z, (p,\eta)) \in O$, then $U \cap \Psi_i(x, z, (p,\eta)) \neq \emptyset$ hence $\emptyset \neq U \cap \Phi_i(x, z, (p,\eta))$ (since we always have $\Psi_i(x, z, (p,\eta)) \subset \Phi_i(x, z, (p,\eta))$).

Step 2: $i = (1,1)$. The proof is similar to the first step and more standard. We only check hereafter that the case $\alpha^1(p,\eta) = \emptyset$ never holds. Indeed, we will consider three cases. If $\eta \neq 0$ then $0 < \max\{\eta \cdot z_1 | z_1 \in Z_1\}$ since $0 \in \text{int} Z_1$ (by Assumption (F)). So $e_1 \in \alpha^1(p,\eta)$ since $e_1 \in X_1$ (by Assumption (C)). If $\eta = 0$ and $p = 0$, then $\rho(p,\eta) = 1$ and again $e_1 \in \alpha^1(p,\eta)$. Finally, if $\eta = 0$ and $p \neq 0$, then $e_1 - t(\lambda \,\square\, p) \in \alpha^1(p,\eta)$ for $t > 0$ small enough since $e_1 \in \text{int } X_1$ (by Assumption (C)).

Step 3: $i = 0$ and $i = (1,2)$. Obvious. □

Step 4: For $i = 0$, for every $(p,\eta) \in B$, in view of Claim 4.1, we can now apply the fixed-point Theorem 4.1. Hence there exists $c^* := (x^*, z^*, (p^*, \eta^*)) \in \prod_{i \in I} X_i \times \prod_{i \in I} Z_i \times B$ such that, for every $i \in I_0$, $\Phi_i(x^*, z^*, (p^*, \eta^*)) = \emptyset$. Written coordinatewise, this is equivalent to saying that:

$$(\lambda \,\square\, p) \cdot \sum_{i \in I} (x_i^* - e_i) - \eta \cdot \sum_{i \in I} z_i^* \leq (\lambda \,\square\, p^*) \cdot \sum_{i \in I} (x_i^*(s) - e_i(s)) - \eta^* \cdot \sum_{i \in I} z_i^*, \quad (4.1)$$

for $i = (1,1)$

$$x_1^* \in \beta^1(p^*, \eta^*) \text{ and } \alpha^1(p^*, \eta^*) \cap P_1(x^*) = \emptyset, \tag{4.2}$$

for $i = (1,2)$

$$\eta^* \cdot z_1^* = \max\{\eta^* \cdot z_1 | z_1 \in Z_1\}; \tag{4.3}$$

for the remaining i

$$(x_i^*, z_i^*) \in \beta^i(p^*, \eta^*) \text{ and } \alpha^i(p^*, \eta^*) \cap (P_i(x^*) \times Z_i) = \emptyset. \tag{4.4}$$

From now on we shall denote simply by W the full matrix of returns $W_{\mathcal{F}}(p^*, q^*)$ associated to the spot price p^* and to the asset price $q^* = q(p^*, \eta^*)$.

4.1.3 The vector $((x_i^*, z_i^*)_{i\in I}, p^*, q^*)$ is a financial equilibrium

We recall that, from Theorem 2.1, $q^* = q(p^*, \eta^*)$ is the unique vector $q^* \in \mathbb{R}^J$ satisfying

$$^tW(p^*, q^*)\lambda = \eta^*.$$

Since, by 4.2, $x_1^* \in \beta^1(p^*, \eta^*)$, using 4.3, one deduces that

$$(\lambda \Box p^*) \cdot (x_1^* - e_1) = \sum_{s\in S} \lambda(s) p^*(s) \cdot (x_1^*(s) - e_1(s)) \leq \eta^* \cdot z_1^* + \rho(p^*, \eta^*)(\sum_{s\in S} \lambda(s)), \tag{4.5}$$

and, for every $i \neq 1$, since $(x_i^*, z_i^*) \in \beta^i(p^*, \eta^*)$, by 4.4,

$$p^* \Box (x_i^* - e_i) \leq W z_i^* + \rho(p^*, \eta^*)\mathbb{1}. \tag{4.6}$$

Taking the scalar product with λ and recalling that $^tW\lambda = \eta^*$ from the definition of W, we conclude that, for $i \neq 1$,

$$\sum_{s\in S} \lambda(s) p^*(s) \cdot (x_i^*(s) - e_i(s)) - \rho(p^*, \eta^*)(\sum_{s\in S} \lambda(s)) \leq \lambda \cdot [W z_i^*] = [^tW\lambda] \cdot z_i^* = \eta^* \cdot z_i^*;$$

Hence, summing over $i \in I$ we have proved the following claim:

Claim 4.2 $(\lambda \Box p^*) \cdot \sum_{i\in I}(x_i^* - e_i) \leq \eta^* \cdot \sum_{i\in I} z_i^* + \#I(\sum_{s\in S} \lambda(s))\rho(p^*, \eta^*)$, *and the equality holds if the equality holds in (4.5) and (4.6).*

Claim 4.3 $\sum_{i\in I} z_i^* = 0$ *and* $\sum_{i\in I} x_i^* = \sum_{i\in I} e_i$.

Proof of Claim 4.3. From Assertion (4.1) (taking successively $p = p^*$ and $\eta = \eta^*$), we get:

$$\eta^* \cdot \sum_{i \in I} z_i^* \leq \eta \cdot \sum_{i \in I} z_i^* \quad \text{for every } \eta \in \mathbb{R}^J,\ \|\eta\| \leq 1, \tag{4.7}$$

$$(\lambda \,\square\, p) \cdot \sum_{i \in I} (x_i^* - e_i) \leq (\lambda \,\square\, p^*) \cdot \sum_{i \in I} (x_i^* - e_i) \quad \text{for every } p \in \mathbb{R}^L,\ \|\lambda \,\square\, p\| \leq 1. \tag{4.8}$$

We first prove that $\sum_{i \in I} z_i^* = 0$ by contradiction. Suppose it is not true, from (4.7), we deduce that $\eta^* = -\frac{\sum_{i \in I} z_i^*}{\|z_i^*\|}$. Hence $\|\eta^*\| = 1$, $\rho(p^*, \eta^*) := \max\{0, 1 - \|\lambda \square p^*\| - \|\eta^*\|\} = 0$ and $\eta^* \cdot \sum_{i \in I} z_i^* < 0$. Consequently, from Claim 4.2 one gets:

$$(\lambda \,\square\, p^*) \cdot \sum_{i \in I} (x_i^* - e_i) \leq \eta^* \cdot \sum_{i \in I} z_i^* + 0 < 0,$$

But, from the above inequality (4.8), (taking $p = 0$) one gets

$$0 \leq (\lambda \,\square\, p^*) \cdot \sum_{i \in I} (x_i^* - e_i),$$

a contradiction with the above inequality. □

In the same way we now prove the second equality $\sum_{i \in I} (x_i^* - e_i) = 0$ by contradiction. Suppose it is not true, from (4.7), we deduce that $0 < (\lambda \,\square\, p^*) \cdot \sum_{i \in I} (x_i^* - e_i)$, $\|\lambda \,\square\, p^*\| = 1$ and so $\rho(p^*, \eta^*) := \max\{0, 1 - \|\lambda \square p^*\| - \|\eta^*\|\} = 0$. Consequently, from Claim 4.2, recalling from above that $\sum_{i \in I} z_i^* = 0$ one gets the contradiction:

$0 < (\lambda \,\square\, p^*) \cdot \sum_{i \in I} (x_i^* - e_i) \leq \eta^* \cdot \sum_{i \in I} z_i^* + 0 = 0.$ □

Claim 4.4 $x_1^* \in \beta^1(p^*, \eta^*)$ *and* $\beta^1(p^*, \eta^*) \cap P_1(x^*) = \emptyset$.

Proof of Claim 4.4. From the fixed-point condition (4.2), $x_1^* \in \beta^1(p^*, \eta^*)$. Now suppose that $\beta^1(p^*, \eta^*) \cap P_1(x^*) \neq \emptyset$ and choose $x_1 \in \beta^1(p^*, \eta^*) \cap P_1(x^*)$.

We know that $\alpha^1(p^*, \eta^*) \neq \emptyset$ (see the second step in the proof of Claim 4.1), and we choose $\bar{x}_1 \in \alpha^1(p^*, \eta^*)$. Suppose first that $\bar{x}_1 = x_1$; then, from above $x_1 \in P_1(x^*) \cap \alpha^1(p^*, \eta^*)$, which contradict the fact that this set is empty by Assertion (4.2). Suppose now that $\bar{x}_1 \neq x_1$, from Assumption $(C.iii)$, $[\bar{x}_1, x_1[\cap P_1(x^*) \neq \emptyset$ (recalling that $x_1 \in P_1(x^*)$) and clearly $[\bar{x}_1, x_1[\subset \alpha^1(p^*, \eta^*)$ (since $x_1 \in \beta^1(p^*, \eta^*)$ and $\bar{x}_1 \in \alpha^1(p^*, \eta^*)$). Consequently, $P_1(x^*) \cap \alpha^1(p^*, \eta^*) \neq \emptyset$, which contradicts again Assertion (4.2). □

Claim 4.5 *(a) For every $s \in S$, $p^*(s) \neq 0$.*

(b) For all $i \neq 1$, $(x_i^*, z_i^*) \in \beta^i(p^*, \eta^*)$ and $\beta^i(p^*, \eta^*) \cap (P_i(x^*) \times Z_i) = \emptyset$.

Proof of Claim 4.5. (a) Indeed, suppose that $p^*(s) = 0$, for some $s \in S$. From Claim 4.3, $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$, and from the Non Satiation Assumption at node s (for Consumer 1) there exists $x_1 \in P_1(x^*)$ such that $x_1(s') = x_1^*(s')$ for every $s' \neq s$; from Assertion (4.2), $x_1^* \in \beta^1(p^*, \eta^*)$ and, recalling that $p^*(s) = 0$, one deduces that $x_1 \in \beta^1(p^*, \eta^*)$. Consequently,

$$\beta_1(p^*, \eta^*) \cap P_1(x^*) \neq \emptyset,$$

which contradicts Claim 4.4.

(b) From the fixed point condition (4.4), for $i \neq 1$ one has $(x_i^*, z_i^*) \in \beta^i(p^*, \eta^*)$. Now, suppose that there exists $i \neq 1$ such that $\beta^i(p^*, \eta^*) \cap (P_i(x^*) \times Z_i) \neq \emptyset$ and let $(x_i, z_i) \in \beta^i(p^*, \eta^*) \cap (P_i(x^*) \times Z_i)$. From the Survival Assumption and the fact that $p^*(s) \neq 0$ for every $s \in S$ (Claim 4.4), one deduces that $\alpha^i(p^*, \eta^*) \neq \emptyset$ and we let $(\bar{x}_i, \bar{z}_i) \in \alpha^i(p^*, \eta^*)$.[15]

Suppose first that $\bar{x}_i = x_i$, then, from above $(x_i, \bar{z}_i) \in [P_i(x^*) \times Z_i] \cap \alpha^i(p^*, \eta^*)$, which contradict the fact that this set is empty by Assertion (4.4). Suppose now that $\bar{x}_i \neq x_i$, from Assumption $(C.iii)$, (recalling that $x_i \in P_i(x^*)$) the set $[\bar{x}_i, x_i[\cap P_i(x^*)$ is nonempty, hence contains a point $x_i(\lambda) := (1-\lambda)\bar{x}_i + \lambda x_i$ for some $\lambda \in [0, 1[$. We let $z_i(\lambda) := (1-\lambda)\bar{z}_i + \lambda z_i$ and we check that $(x_i(\lambda), z_i(\lambda)) \in \alpha^i(p^*, \eta^*)$ (since $(x_i, z_i) \in \beta^i(p^*, \eta^*)$ and $(\bar{x}_i, \bar{z}_i) \in \alpha^i(p^*, \eta^*)$). Consequently, $\alpha^i(p^*, \eta^*) \cap (P_i(x^*) \times Z_i) \neq \emptyset$, which contradicts again Assertion (4.4). □

Claim 4.6 $\rho(p^*, \eta^*) = 0$.

Proof of Claim 4.6. We first prove that the budget constraints of consumers $i \in I$, $i \neq 1$, are binded, that is:

$$p^* \,\Box\, (x_i^* - e_i) = W z_i^* + \rho(p^*, \eta^*)\mathbb{1}, \text{ for every } i \neq 1. \tag{4.9}$$

Indeed, if it is not true, there exist $i \in I$, $i \neq 1$ such that

$$p^* \,\Box\, (x_i^* - e_i) \leq W z_i^* + \rho(p^*, \eta^*)\mathbb{1},$$

[15] Take $\bar{z}_i = 0$ and $\bar{x}_i = e_i - tp^*$ for $t > 0$ small enough, so that $\bar{x}_i \in X_i$ (from the Survival Assumption). Then notice that $p^* \,\Box\, (\bar{x}_i - e_i) = -t(p^* \,\Box\, p^*) \ll 0 \leq 0 + \rho(p^*, \eta^*)\mathbb{1}$.

with a strict inequality for some component $s \in S$. But $\sum_{i \in I} x_i^* = \sum_{i \in I} e_i$ (Claim 4.3) and from the Non Satiation Assumption at node s (for consumer i), there exists $x_i \in P_i(x^*)$ such that $x_i(s') = x_i^*(s')$ for every $s' \neq s$. Consequently, we can choose $x \in [x_i, x_i^*[$ close enough to x_i^* so that $(x, z_i^*) \in \beta^i(p^*, \eta^*)$. But, from the local non-satiation (Assumption (K.ii)), $[x_i, x_i^*[\subset P_i(x^*)$. Consequently, $\beta^i(p^*, \eta^*) \cap (P_i(x^*) \times Z_i) \neq \emptyset$ which contradicts Claim 4.5.

In the same way, we prove that the budget constraint of Consumer 1 is binded. Consequently, from Claim 4.2, using the facts that $\sum_{i \in I}(x_i^* - e_i) = 0$ and $\sum_{i \in I} z_i^* = 0$ (by Claim 4.3) one has

$$0 = (\lambda \,\square\, p^*) \cdot \sum_{i \in I}(x_i^* - e_i) - \eta^* \cdot \sum_{i \in I} z_i^* = \#I\Big(\sum_{s \in S} \lambda(s)\Big)\rho(p^*, \eta^*).$$

Since $\sum_{s \in S} \lambda(s) > 0$, we conclude that $\rho(p^*, \eta^*) = 0$. □

Claim 4.7 *For every* $i \in I$, $(x_i^*, z_i^*) \in B^i_{\mathcal{F}}(p^*, q^*)$ *and* $[P_i(x^*) \times Z_i] \cap B^i_{\mathcal{F}}(p^*, q^*) = \emptyset$.

Proof of Claim 4.7. Since $\rho(p^*, \eta^*) = 0$ (From Claim 4.6), for every $i \neq 1$, $B^i_{\mathcal{F}}(p^*, q^*)) = \beta^i(p^*, q^*)$. Hence, from Claim 4.5 we deduce that Claim 4.7 is true for every consumer $i \neq 1$.

About the first consumer, we first notice that $B^1_{\mathcal{F}}(p^*, q^*) \subset \beta^1(p^*, \eta^*) \times Z_1$. So, in view of Claim 4.5, the proof will be complete if we show that $(x_1^*, z_1^*) \in B^1_{\mathcal{F}}(p^*, q^*)$. But since the budget constraints of agent $i \in I$, $i \neq 1$, are binded (see the proof of Claim 4.6), $\sum_{i \in I}(x_i^* - e_i) = 0$ and $\sum_{i \in I} z_i^* = 0$ (Claim 4.3), we conclude that

$$p^* \,\square\, (x_1^* - e_1) = -\sum_{i \neq 1} p^* \,\square\, (x_i^* - e_i) = -\sum_{i \neq 1} W z_i^* = W z_1^*,$$

which ends the proof of the Claim. □

4.2 Proof in the general case

We now give the proof of Theorem 3.1, without considering the additional Assumption (K), as in the previous section. We will first enlarge the strict preferred sets as in Gale-Mas Colell, and then truncate the economy $\mathcal{E}$ by a standard argument to define a new economy $\hat{\mathcal{E}}^r$, which satisfies all the assumptions of $\mathcal{E}$, together with the additional Assumption (K). From the previous section, there exists an equilibrium of $\hat{\mathcal{E}}^r$ and we will then check that it is also an equilibrium of $\mathcal{E}$.

4.2.1 Enlarging the preferences as in Gale-Mas Colell

The original preferences P_i are replaced by the "enlarged" ones $\hat{P}_i$ defined as follows. For every $i \in I$, $x^* = (x_i^*)_i \in \prod_i X_i$ we let

$$\hat{P}_i(x^*) := \bigcup_{x_i \in P_i(x^*)}]x_i^*, x_i] = \{x_i^* + t(x_i - x_i^*) \mid t \in]0,1],\ x_i \in P_i(x^*)\}.$$

The next proposition shows that $\hat{P}_i$ satisfies the same properties as P_i, for every $i \in I$, together with the additional Local Nonsatiation Assumption (K.ii).

Proposition 4.1 *Under* (C), *for every $i \in I$ and every $x^* = (x_i^*)_{i\in I} \in \prod_{i\in I} X_i$ one has:*

(i) $P_i(x^*) \subset \hat{P}_i(x^*) \subset X_i$;

(ii) the correspondence $\hat{P}_i$ is lower semicontinuous at x^* and $\hat{P}_i(x^*)$ is convex;

(iii) *for every $y_i \in \hat{P}_i(x^*)$ for every $x'_i \in X_i, x'_i \neq y_i$ then $[x'_i, y_i[\cap \hat{P}_i(x^*) \neq \emptyset$;*

(iv) $x_i^* \notin \hat{P}_i(x^*)$;

(v) *(Non-Satiation at Every Node) if $\sum_{i\in I} x_i^* = \sum_{i\in I} e_i$, for every $s \in S$, there exists $x \in \prod_{i\in I} X_i$ such that, for each $s' \neq s$, $x_i(s') = x_i^*(s')$ and $x_i \in \hat{P}_i(x^*)$;*

(vi) for every $y_i \in \hat{P}_i(x^*)$, then $[y_i, x_i^*[\subset \hat{P}_i(x^*)$.

Proof. Let $x^* \in \prod_{i\in I} X_i$ and let $i \in I$

Part (i). It follows by the convexity of X_i, for every $i \in I$.

Part (ii). Let $y_i \in \hat{P}_i(x^*)$ and consider a sequence $(x^{*n})_n \subset \prod_{i\in I} X_i$ converging to x^*. Since $y_i \in \hat{P}_i(x^*)$, then $y_i = x_i^* + t(x_i - x_i^*)$ for some $x_i \in P_i(x^*)$ and some $t \in]0,1]$. Since P_i is lower semicontinuous, there exists a sequence (x_i^n) converging to x_i such that $x_i^n \in P_i(x^{*n})$ for every $n \in \mathbb{N}$. Now define $y_i^n := x_i^{*n} + t(x_i^n - x_i^{*n}) \in]x_i^{*n}, x_i^n]$ then $y_i^n \in \hat{P}_i(x^{*n})$ and obviously the sequence (y_i^n) converges to y_i. This shows that $\hat{P}_i$ is lower semicontinuous at x^*.

To show that $\hat{P}_i(x^*)$ is convex, let $y_i^1, y_i^2 \in \hat{P}_i(x^*)$, let $\lambda^1 \geq 0, \lambda^2 \geq 0$, such that $\lambda^1 + \lambda^2 = 1$, we show that $\lambda^1 y_i^1 + \lambda^2 y_i^2 \in \hat{P}_i(x^*)$. Then $y_i^k = x_i^* + t^k(x_i^k - x_i^*)$ for some $t^k \in]0,1]$ and some $x_i^k \in P_i(x^*)$ $(k = 1,2)$. One has

$$\lambda^1 y_i^1 + \lambda^2 y_i^2 = x_i^* + (\lambda^1 t^1 + \lambda^2 t^2)(x_i - x_i^*),$$

where $x_i := (\lambda^1 t^1 x_i^1 + \lambda^2 t^2 x_i^2)/(\lambda^1 t^1 + \lambda^2 t^2) \in P_i(x^*)$ (since $P_i(x^*)$ is convex, by Assumption (C)) and $\lambda^1 t^1 + \lambda^2 t^2 \in]0,1]$. Hence $\lambda^1 y_i^1 + \lambda^2 y_i^2 \in \hat{P}_i(x^*)$.

Part (iii). Let $y_i \in \hat{P}_i(x^*)$ and let $x'_i \in X_i, x'_i \neq y_i$. From the definition of $\hat{P}_i$, $y_i = x_i^* + t(x_i - x_i^*)$ for some $x_i \in P_i(x^*)$ and some $t \in]0,1]$. Suppose first that $x_i = x'_i$, then $y_i \in]x_i^*, x_i[\subset \hat{P}_i(x^*)$. Consequently, $[x'_i, y_i[\cap \hat{P}_i(x^*) \neq \emptyset$. Suppose now that $x_i \neq x'_i$; since P_i satisfies Assumption $(C.iii)$, there exists $\lambda \in [0,1[$ such that $x_i(\lambda) = x'_i + \lambda(x_i - x'_i) \in P_i(x^*)$. We let

$$z := [\lambda(1-t)x_i^* + t(1-\lambda)x'_i + t\lambda x_i]/\alpha \quad \text{with } \alpha := t + \lambda(1-t),$$

and we check that $z = [\lambda(1-t)x_i^* + t x_i(\lambda)]/\alpha \in]x_i^*, x_i(\lambda)]$, with $x_i(\lambda) \in P_i(x^*)$, hence $z \in \hat{P}_i(x^*)$. Moreover, $z := [\lambda y_i + t(1-\lambda)x'_i]/\alpha \in [x'_i, y_i[$. Consequently, $[x'_i, y_i[\cap \hat{P}_i(x^*) \neq \emptyset$, which ends the proof of (iii)).

Parts (iv), (v) and (vi). They follow immediately by the definition of $\hat{P}_i$ and the properties satisfied by P_i in (C). □

4.2.2 Truncating the economy

We now define the "truncated economy" as follows.

For every $i \in I$, $\lambda \in \mathbb{R}^S_{++}$, we let $\hat{X}_i(\lambda)$ and $\hat{Z}_i(\lambda)$ be the projections of $K(\lambda)$ on X_i and Z_i, respectively, namely

$$\hat{X}_i(\lambda) := \{x_i \in X_i : \exists (x_j)_{j\neq i} \in \prod_{j\neq i} X_j, \ \exists (z_i)_{i\in I} \in \prod_{i\in I} Z_i, \ (x_i, z_i)_{i\in I} \in K(\lambda)\}$$

and

$$\hat{Z}_i(\lambda) := \{z_i \in Z_i : \exists (z_j)_{j\neq i} \in \prod_{j\neq i} Z_j, \ \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i, \ (x_i, z_i)_{i\in I} \in K(\lambda)\}.$$

By Assumption (B_λ), the set $B(\lambda)$ is bounded, hence the sets $\hat{X}_i(\lambda)$ and $\hat{Z}_i(\lambda)$ are also bounded subsets of $\mathbb{R}^L$ and $\mathbb{R}^J$, respectively. So there exists a real number $r > 0$ such that, for every agent $i \in I$, $\hat{X}_i(\lambda) \subset \text{int}B(0,r)$ and $\hat{Z}_i(\lambda) \subset \text{int}B(0,r)$.

The truncated economy $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$ is the collection

$$(\hat{\mathcal{E}}^r, \mathcal{F}^r) = [\mathcal{S}, H, I, (X_i^r, \hat{P}_i^r, e_i)_{i\in I}, (J, (s(j))_{j\in J}, V, (Z_i^r)_{i\in I})],$$

where, for every $x = (x_i)_i \in \prod_i X_i$

$$X_i^r = X_i \cap B(0,r),\ Z_i^r = Z_i \cap B(0,r) \text{ and } \hat{P}_i^r(x) = \hat{P}_i(x) \cap \mathrm{int}B(0,r).$$

The existence of financial equilibria of $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$ is then a consequence of Section 4.1, that is, Theorem 3.1 with the additional Assumption (K). We just have to check that Assumption (K) and all the assumptions of Theorem 3.1, that is, Theorem are satisfied by $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$. In view of Proposition 4.1, this is clearly the case for all the assumptions but the Survival Assumptions $(C.vi)$ and $(F.iii)$, that are proved via a standard argument (that we recall hereafter).

Indeed we first notice that $(e_i, 0)_{i\in I}$ belongs to $B(\lambda)$, hence, for every $i \in I$, $e_i \in \hat{X}_i(\lambda) \subset \mathrm{int}B(0,r)$. Recalling that $e_i \in \mathrm{int}X_i$ (from the Survival Assumption), we deduce that $e_i \in \mathrm{int}X_i \cap \mathrm{int}B(0,r) \subset \mathrm{int}[X_i \cap B(0,r)] = \mathrm{int}X_i^r$. Similarly, for every $i \in I$, $0 \in \hat{Z}_i(\lambda) \subset \mathrm{int}B(0,r)$. Consequently $0 \in Z_i^r = Z_i \cap B(0,r)$. Moreover, for some $i_0 \in I$ one has $0 \in \mathrm{int}Z_{i_0}$ (by Assumption $(F.iii)$), and, as above, $0 \in \mathrm{int}B(0,r)$. Consequently, $0 \in \mathrm{int}[Z_{i_0} \cap B(0,r)] = \mathrm{int}Z_{i_0}^r$.

The end of the proof of Theorem 3.1 consists to show that financial equilibria of $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$ are in fact also financial equilibria of $(\mathcal{E}, \mathcal{F})$, which thus exist from above.

Proposition 4.2 *Under Assumption (B_λ), if $((x_i^*, z_i^*)_i, p^*, q^*)$ is a financial equilibrium of $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$ such that $p^* \in B_L(0,1)$ and ${}^tW(p^*, q^*)\lambda \in N_{Z_1^r \cap B_J(0,1)}(z_1^*)$, then it is also a financial equilibrium of $(\mathcal{E}, \mathcal{F})$ and ${}^tW_{\mathcal{F}}(p^*, q^*)\lambda \in N_{Z_1}(z_1^*)$.*

Proof. Let $((x_i^*, z_i^*)_i, p^*, q^*)$ be a financial equilibrium of the economy $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$. In view of the definition of a financial equilibrium, to prove that it is also an equilibrium of $(\mathcal{E}, \mathcal{F})$ we only have to check that $[P_i(x^*) \times Z_i] \cap B_{\mathcal{F}}^i(p^*, q^*) = \emptyset$ for every $i \in I$, where $B_{\mathcal{F}}^i(p^*, q^*)$ denotes the budget set of agent i in the economy $(\mathcal{E}, \mathcal{F})$.

Assume, on the contrary, that, for some $i \in I$ the set $[P_i(x^*) \times Z_i] \cap B_{\mathcal{F}}^i(p^*, q^*)$ is nonempty, hence contains a couple (x_i, z_i). Clearly the allocation $(x_i^*, z_i^*)_i$ belongs to the set $K(\lambda)$, hence for every $i \in I$, $x_i^* \in \hat{X}_i(\lambda) \subset \mathrm{int}B(0,r)$ and $z_i^* \in \hat{Z}_i(\lambda) \subset \mathrm{int}B(0,r)$. Thus, for $t \in]0,1]$ sufficiently small, $x_i(t) := x_i^* + t(x_i - x_i^*) \in \mathrm{int}B(0,r)$

and $z_i(t) := z_i^* + t(z_i - z_i^*) \in \mathrm{int}B(0,r)$. Clearly $(x_i(t), z_i(t))$ belongs to the budget set $B^i_{\mathcal{F}}(p^*, q^*)$ of agent i (for the economy $(\mathcal{E}, \mathcal{F})$) and since $x_i(t) \in X_i^r := X_i \cap B(0,r)$, $z_i(t) \in Z_i^r := Z_i \cap B(0,r)$, the couple $(x_i(t), z_i(t))$ belongs also to the budget set $B^{ir}(p^*, q^*)$ of agent i (in the economy $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$). From the definition of $\hat{P}_i$, we deduce that $x_i(t) \in \hat{P}_i(x^*)$ (since from above $x_i(t) := x_i^* + t(x_i - x_i^*)$ and $x_i \in P_i(x^*)$), hence $x_i(t) \in \hat{P}_i^r(x^*) := \hat{P}_i(x^*) \cap \mathrm{int}B_L(0,r)$. We have thus shown that, for $t \in]0,1]$ small enough, $(x_i(t), z_i(t)) \in [\hat{P}_i^r(x^*) \times Z_i^r] \cap B^{ir}(p^*, q^*)$. This contradicts the fact that this set is empty, since $((x_i^*, z_i^*)_i, p^*, q^*)$ is a financial equilibrium of the economy $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$.

We now prove that $\eta^* :=^t W_{\mathcal{F}}(p^*, q^*)\lambda \in N_{Z_1}(z_1^*)$. We let $z_1 \in Z_1$ and we show that $y^* \cdot z_1^* \geq \eta^* \cdot z_1$. We have seen above that $z_1^* \in \hat{Z}_1(\lambda) \subset \mathrm{int}B(0,r)$. Then, for $t > 0$ small enough, $z(t) := z_1^* + t(z_1 - z_1^*) \in \mathrm{int}B(0,r)$ and $z(t) \in Z_1$, by the convexity of Z_1. Consequently, for t small enough, $z(t) \in Z_1^r = Z_1 \cap B(0,r)$ and using the fact that $\eta^* \in N_{Z_1^r}(z_1^*)$, we deduce that

$$\eta^* \cdot z_1^* \geq \eta^* \cdot z(t) = \eta^* \cdot z_1^* + t\eta^* \cdot (z_1 - z_1^*),$$

hence $\eta^* \cdot z_1 \leq \eta^* z_1^*$. □

5 Appendix

5.0.3 Proof of Proposition 2.1 on the relationship between $rankV$ and $rankW$

Part (a) is straightforward. We prepare the proofs of Part (b) and (c) by introducing some notations and definitions. We let, for $t = 1, \ldots, T+1$, the set $J^t = \{j \in J | s(j) \in S^{t-1}\}$.

We give the proof under the additional assumption that $J^t \neq \emptyset$ for $t \in [1, T]$ and $J^{T+1} = \emptyset$ (and we let the reader adapt this proof to the general case). Then the sets J^t ($t \in [1, T]$) define a partition of the set J and we write every $z \in \mathbb{R}^J$ as $z = (z_t)$ with $z_t \in \mathbb{R}^{J^t}$. We also define the $S^t \times J^\tau$ sub-matrix $V_{t,\tau}(p)$ of $V(p)$ and the $S^t \times J^\tau$ sub-matrix $W_{t,\tau}(p,q)$ of $W(p,q)$, for $t \in \mathcal{T}$ and $\tau = 1, \ldots, T$.

In this case, the matrices $V(p)$ and $W(p,q)$ can be written as follows:

$$
\begin{array}{c@{}c}
\begin{array}{ccccc} J^1 & J^2 & \dots & J^{T-1} & J^T \end{array} & \\
V(p) = \begin{pmatrix}
0 & 0 & \dots & 0 & 0 \\
V_{1,1}(p) & 0 & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & V_{T-1,T-1}(p) & 0 \\
0 & 0 & \dots & 0 & V_{T,T}(p)
\end{pmatrix} &
\begin{array}{c} S^0 \\ S^1 \\ \dots \\ S^{T-1} \\ S^T \end{array},
\end{array}
$$

$$
W_{\mathcal{F}}(p,q) = \begin{pmatrix}
W_{0,1}(p,q) & 0 & \dots & 0 & 0 \\
V_{1,1}(p) & W_{1,2}(p,q) & \dots & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots \\
0 & 0 & \dots & V_{T-1,T-1}(p) & W_{T-1,T} \\
0 & 0 & \dots & 0 & V_{T,T}(p)
\end{pmatrix}
\begin{array}{c} S^0 \\ S^1 \\ \dots \\ S^{T-1} \\ S^T \end{array}.
$$

To see the above, it suffices to check that, for every (p, q), one has $V_{0,\tau}(p) = 0$ for every τ, $V_{t,\tau}(p) = 0$ if $t \neq \tau$, $W_{0,\tau}(p, q) = 0$, for every $\tau \neq 1$, $W_{t,\tau}(p, q) = 0$ if $\tau \geq t + 2$ and $W_{t,t}(p, q) = V_{t,t}(p)$ for every $t \geq 1$.

Part (b). We first prove it under the additional assumption that $rankV(p) = \#J$ (i.e., $V(p)$ is one-to-one). Let $z = (z_t) \in \prod_t \mathbb{R}^{J_t}$ be such that $W(p, q)z = 0$; then one has

$V_{1,1}(p)z_1 + W_{1,2}(p, q)z_2 = 0,$

$\dots$

$V_{T-1,T-1}(p)z_{T-1} + W_{T-1,T}(p, q)z_T = 0,$

$V_{T,T}(p)z_T = 0.$

One notices that $rankV(p) = \sum_{t=1}^{T} rankV_{t,t}(p)$. So, for every t, $rankV_{t,t}(p) = \sharp J_t$ (hence $rankV(p) = \sharp J$) and each matrix $V_{t,t}(p)$ is one-to-one. From above, by an easy backward induction argument, we deduce that $z_T = 0$, then $z_{T-1} = 0, \dots, z_1 = 0$. Thus $z = 0$ and we have proved that $W_{\mathcal{F}}(p, q)$ is also on-to-one, that is, $rankW_{\mathcal{F}}(p, q) = \sharp J$.

Suppose now that $rankV(p) < \#J$. By eliminating columns of the matrix $V(p)$ we can consider a set $\widetilde{J} \subset J$ and a $(S \times \widetilde{J})$-sub-matrix $\widetilde{V}(p)$ of $V(p)$ such that $rankV(p) = \#\widetilde{J} = rank\widetilde{V}(p)$ and the matrix $\widetilde{W}(p, q)$ is defined in a similar way. From the first part of the proof of Part (b), $rank\widetilde{V}(p) \leq rank\widetilde{W}(p, q)$, and clearly $rank\widetilde{W}(p, q) \leq rankW(p, q)$. Hence $rankV(p) \leq rankW(p, q)$.

Part (c). We denote by $V(p, s)$ and $W(p, q, s)$, respectively, the rows of the matrices $V(p)$

and $W(p,q)$. Since ${}^tW_{\mathcal{F}}(p,q)\lambda = 0$, from Theorem 2.1 we get

$$\lambda(s(j))q^j = \sum_{\sigma \in s(j)^+} \lambda(\sigma) v(p,\sigma,j), \text{ for every } j \in J.$$

Consequently, we have:

for $s \in S_T$, $W(p,q,s) = V(p,s)$ and

for $s \notin S_T$, $W(p,q,s) + [1/\lambda(s)] \sum_{\sigma \in s^+} \lambda(\sigma) V(p,\sigma) = V(p,s)$ (recalling that $V(p,s_0) = 0$).

Hence, for every $s \in S$, $W(p,q,s)$ belongs to the vector space spanned by the vectors $\{V(p,s) | s \in S\}$, thus we conclude that $rankW(p,q) \leq rankV(p)$. □

Remark 5.1 *(Long-lived assets)* The inequality $rankV(p) \leq rankW(p,q)$ (Assertion (b) of Proposition 2.1) may not be true in the case of long-lived assets. Consider a stochastic economy with $T = 2$ and three nodes, namely $S = \{0,1,2\}$, and two assets j_1, j_2, where j_1 is emitted at node 0 and pays -1 a node 1, 1 at node 2, j_2 is emitted at node 1 and gives 1 at node 2. Consider the asset price $q = (0,1)$; then the matrices of returns are

$$V = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad W(q) = \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 1 \end{pmatrix},$$

and $rankW(q) = 1 < rankV = 2$. □

Assertion (a) of Proposition 2.1 may not be true in the case of long-lived assets, that is, the payoff matrix may not suffice to describe the financial structure. Consider the above example: then V is also the return matrix of the financial structure $\mathcal{F}'$ consisting of two assets $\{j_1, j_2'\}$, where j_1 is defined as previously and j_2' has for emission node 0 and pays 1 at node 2. It is clear, however, that, for $q = (0,1)$, the full matrix of returns $W_{\mathcal{F}'}(q)$ is different from $W_{\mathcal{F}}(q)$. □

5.0.4 Proof of Proposition 3.1 on the Boundedness Assumption B

We will use the following lemma.

Lemma 5.1 *Let A be a compact subset of $\mathbb{R}^n$ and let $W(\alpha) : \mathbb{R}^J \longrightarrow \mathbb{R}^S$ ($\alpha \in A$) be a linear mapping such that the application $\alpha \mapsto W(\alpha)$ is continuous and $rankW(\alpha) = \sharp J$.*

Then there exists $c > 0$ such that:

$$\|W(\alpha)z\| \geq c\|z\| \text{ for every } z \in \mathbb{R}^J \text{ and every } \alpha \in A.$$

Proof. By contradiction. Let us assume that, for every $n \in \mathbb{N}$, there exist $z_n \in \mathbb{R}^J$, $\alpha_n \in A$ such that $\|W(\alpha_n)z_n\| < \frac{1}{n}\|z_n\|$. Noticing that $z_n \neq 0$, without any loss of generality we can assume that $\left(\frac{z_n}{\|z_n\|}\right)_n$ (which is in the unit sphere of $\mathbb{R}^J$) converges to some element $v \neq 0$ and (α_n) converges to some element $\alpha \in A$ (since A is compact). By the continuity of the map W, taking the limit when $n \to \infty$, we get $\|W(\alpha)v\| \leq 0$, hence $W(\alpha)v = 0$, a contradiction with the hypothesis that $\mathrm{rank} W(\alpha) = \sharp J$. □

Proof of Proposition 3.1. Let $\lambda \in \mathbb{R}^S_{++}$ be fixed. We first show that, for every $i \in I$, the set $\hat{X}_i(\lambda)$ is bounded. Indeed, since the sets X_i are bounded below, there exist $\underline{x}_i \in \mathbb{R}^L$ such that $X_i \subset \underline{x}_i + \mathbb{R}^L_+$. If $x_i \in \hat{X}_i(\lambda)$, there exist $x_j \in X_j$, for every $j \neq i$, such that $\sum_{j \in J} x_j = \sum_{j \in J} e_j$. Consequently,

$$\underline{x}_i \leq x_i = -\sum_{j \neq i} x_j + \sum_{j \in J} e_j \leq -\sum_{j \neq i} \underline{x}_j + \sum_{j \in J} e_j$$

and so $\hat{X}_i(\lambda)$ is bounded.

We now show that $\hat{Z}_i(\lambda)$ is bounded under the three sufficient assumptions (i), (ii) or (iii) of Proposition 3.1. Indeed, for every $z_i \in \hat{Z}_i(\lambda)$ there exist $(z_j)_{j \neq i} \in \prod_{j \neq i} Z_j$, $(x_j)_j \in \prod_{j \in I} X_j$, $p \in B_L(0,1)$, $q \in \mathbb{R}^J$ such that ${}^tW(p,q)\lambda \in B_J(0,1)$, $\sum_{j \in J} z_j = 0$ and $(x_j, z_j) \in B^j_{\mathcal{F}}(p,q)$.

Under Assumption (i), for every $j \in I$ the portfolio set Z_j is bounded from below, that is there exists $\underline{z}_j \in \mathbb{R}^L$ such that $Z_j \subset \underline{z}_j + \mathbb{R}^L_+$. Using the fact that $\sum_{j \in I} z_j = 0$, we get

$$\underline{z}_i \leq z_i = -\sum_{j \neq i} z_j \leq -\sum_{j \neq i} \underline{z}_j \text{for every } \ z_i \in \hat{Z}_i(\lambda).$$

Under Assumption (ii), since $(x_i, z_i) \in B^i_{\mathcal{F}}(p,q)$ and $(x_i, p) \in \hat{X}_i(\lambda) \times B_L(0,1)$, a compact set from above, there exists $\alpha_i \in \mathbb{R}^S$ such that

$$\alpha_i \leq p \,\square\, (x_i - e_i) \leq W(p,q)z_i.$$

But (using the fact that $\sum_{i \in I} z_i = 0$) we also have

$$W(p,q)z_i = W(p,q)(-\sum_{j \neq i} z_j) \leq -\sum_{j \neq i} \alpha_j,$$

hence there exists $r > 0$ such that $W(p,q)z_i \subset B_S(0,r)$.

From Lemma 5.1, taking $W(\alpha) = W(p,q)$ for $\alpha = (p,q) \in A := \{(p,q) \in B_L(0,1) \times \mathbb{R}^J : {}^tW(p,q)\lambda \in B_J(0,1)\}$, which is compact, for fixed $\lambda \in \mathbb{R}^S_{++}$, there exists $c > 0$ such that, for every $(p,q) \in A$, $z_i \in \mathbb{R}^J$, $c\|z_i\| \leq \|W(p,q)z_i\|$. Hence,

$$c\|z_i\| \leq \|W(p,q)z_i\| \leq r \text{ for every } \; z_i \in \hat{Z}_i(\lambda),$$

which shows that the set $\hat{Z}_i(\lambda)$ is bounded.

Finally, under Assumption (iii) the case of *short-lived assets* is a consequence of Part (ii) and Proposition 2.1.b. □

5.0.5 Proof of the no-arbitrage characterization Theorem 2.1

The proof is a direct consequence of the following result by taking $W := W_{\mathcal{F}}(p,q)$, $c^* = z_i$ and $C = Z_i$.

Theorem 5.1 (Koopmans [19]) *Let $W : \mathbb{R}^n \to \mathbb{R}^m$ be linear, let $C \subset \mathbb{R}^n$ be convex, let $c^* \in C$, and consider the two following assertions:*

(i) there exists $\lambda \in \mathbb{R}^n_{++}$ such that ${}^tW\lambda \in N_C(c^*)$,

or equivalently, $\lambda \cdot Wc^* = [{}^tW\lambda] \cdot c^* \geq \lambda \cdot Wc = [{}^tW\lambda] \cdot c$ for every $c \in C$;

(ii) $W(C) \cap (Wc^* + \mathbb{R}^n_+) = \{0\}$.

The implication [$(i) \Rightarrow (ii)$] always holds and the converse is true under the additional assumption that C is a polyhedral set.

Proof of Theorem 5.1. [$(i) \Rightarrow (ii)$] By contradiction. Suppose that there exists $c \in C$ such that $Wc > Wc^*$. This implies that, for every $\lambda \in \mathbb{R}^n_{++}$, $\lambda \cdot Wc > \lambda \cdot Wc^*$ or equivalently $[{}^tW\lambda] \cdot c > [{}^tW\lambda] \cdot c^*$, that is, ${}^tW\lambda \notin N_C(c^*)$, which contradicts (i). □

For the proof of the implication [$(ii) \Rightarrow (i)$], see Koopmans ([19]), taking into account the following known result on polyhedral sets.

Lemma 5.2 *Let $C \subset \mathbb{R}^n$ be a convex set. (a) ([33] Theorem 19.1) Then C is polyhedral if and only if there exist finitely many vectors $c_1, \ldots, c_k, d_1, \cdots, d_r$ in R^n such that*

$$C = \mathrm{co}\{\mathrm{c}_1, \ldots, \mathrm{c}_\mathrm{k}\} + \Big\{ \sum_{\mathrm{j}=1}^{\mathrm{r}} \beta_\mathrm{j} \mathrm{d}^\mathrm{j} \mid \beta_\mathrm{j} \geq 0, \; \mathrm{j} = 1, \ldots, \mathrm{r} \Big\}.$$

(b) ([33] Theorem 19.3) Let $W : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. If $C \subset \mathbb{R}^n$ is polyhedral set, then $W(C)$ is also polyhedral.

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