Individual Powers and Social Consent: An Axiomatic Approach

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Abstract

We formalize a notion of conditionally decisive powers of which the exercise depends on social consent. Decisive powers, or the so-called libertarian rights, are examples and much weaker forms of powers are covered by our notion. Main results provide an axiomatic characterization for existence of a system of powers and its uniqueness as well as characterizations of various families of rules represented by systems of powers. In particular, we show that a rule satisfies *monotonicity*, *independence*, and *symmetric linkage* (person i and i's issues should be treated symmetrically to person j and j's issues for at least one linkage between issues and persons) if and only if there is a system of powers representing the rule and that the system is unique up to a natural equivalence relation. Considering a domain of simple preference relations (trichotomous or dichotomous preferences), we show that a rule satisfies *Pareto efficiency*, *independence*, and *symmetry* (the symmetric treatment condition in a model with an exogenous linkage between issues and persons) if and only if it is represented by a "quasi-plurality system of powers". For the exercise of a power under a quasi-plurality system, at least either a majority (or $(n+1)/2$) consent or a 50% (or $(n-1)/2$) consent is needed.

Keywords: Powers; Consent; Libertarian Rights; Monotonicity; Independence; Symmetric linkage; Symmetry; Pareto efficiency; Plurality

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1 Introduction

Some sorts of individual or positional powers feature common decision rules in numerous social or political institutions. Exercising these powers is often associated with a requirement of obtaining sufficient social consent and the level of the sufficiency may vary across powers. To take an example, the Constitution of United States describes powers of the President and how much degree of social consent is required for exercising presidential powers; for instance, '*power*, by and with the advice and *consent* of the Senate, to make treaties, provided two thirds of the Senators present concur'.[1] The main objective of this paper is to formalize a notion of individual powers of which the exercise depends on social consent and to give axiomatic characterizations of some families of rules represented by a system of powers.

We consider the following extension of the model by Samet and Schmeidler (2003). There is a society consisting of at least two members. There are a finite number of issues. The society needs to decide on each issue either positively (acceptance) or negatively (rejection). The social decision should reflect members' opinions that are expressed in one of the three ways, positively or negatively or neutrally (we also consider separately the case when opinions are either positive or negative). The systematic relationship between social decisions and members' opinions are described by a (decision) *rule*. It is a function associating with each list of members' opinions, namely, a *problem*, a single decision.

Building on Samet and Schmeidler (2003),[2] we say that person i has the power on the k^{th} issue if social decision on the k^{th} issue is made according to i's opinion when and only when i's opinion obtains sufficient social consent. The sufficiency means that the number of persons with the same opinion on the k^{th} issue as i's is greater than or equal to a certain level, called a *consent quota*.[3] For example, decisive powers, or the so-called libertarian rights by Sen (1970, 1976) and Gibbard (1974), are associated with the minimum consent quota of 1. The above mentioned Presidential power is associated with the consent quota of 2/3 of the number of the Senators. A *system of powers* is a function mapping each issue a person who has the power on this issue and the associated consent quotas.

Our main results show that existence of a system of powers is closely related with the following axioms for social choice. *Monotonicity* says that the rule should respond non-negatively whenever the set of members with positive opinion expands and the set of members with negative opinion shrinks. *Independence* says that the decision on each issue should be based only on members' opinions on this issue. These two axioms are also studied by Rubinstein and Fishburn (1986), Kasher and Rubinstein (1997), Samet and Schmeidler (2003), and Ju (2003, 2005). We also consider a symmetry axiom, called *symmetric linkage*. This axiom is motivated in an environment where issues have some connections with persons. For example, each person has his own areas of specialty and each issue falls on an area of at least one person. *Symmetric linkage* says that the rule should treat a person i and i's areas symmetrically to any other person j and j's areas, under at least one linkage between issues and persons (a function from the set of issues into the set of persons). In a specialized model

[1] The United States Constitution, Article II, Section 2, Clause 2.

[2] "Consent rules" by Samet and Schmeidler (2003) give each person the power of deciding his own qualification.

[3] When i's opinion is neutral, this description does not match exactly to our definition. This is because decision on each issue cannot be neutral. In this case, we require the k^{th} issue to be decided positively (acceptance) when the number of persons with the positive opinion is sufficiently large.

with a single exogenous linkage λ, *symmetric linkage associated with* λ is called *symmetry* as in Samet and Schmeidler (2003).

We show that a rule satisfies *monotonicity*, *independence*, and *symmetric linkage* if and only if there is a system of powers representing the rule and that the system is unique up to a natural equivalence relation. Adding *anonymity* (names of opinion holders should not matter), we establish a necessary and sufficient condition for existence of a *non-exclusive system of powers*, under which everyone has the equal power on every issue. Adding *neutrality* (names of issues should not matter either) instead of *anonymity*, we characterize rules represented either by a constant non-exclusive system of powers (constant consent quotas across issues) or by a *monocentric system of powers* (one and only one person has powers on all issues). Finally, considering simple preference relations called dichotomous and trichotomous preference relations, we show that rules represented by "quasi-plurality systems of powers" are the only rules satisfying *Pareto efficiency*, *independence*, and *symmetry* in a model with an exogenous linkage between issues and persons (e.g. the model in Samet and Schmeidler 2003). Under a *quasi-plurality system*, exercising a power needs at least either majority (or $(n+1)/2$) consent or 50% (or $(n-1)/2$) consent. We establish a similar result in the general model adding *neutrality*.

Related Literature

When issues are associated with personal matters such as believing in a religion, planting a tree in one's own backyard, etc., our powers and systems of powers can be interpreted as a weak notion of rights and systems of rights. In the Arrovian framework, Sen (1970, 1976, 1983) and many of his critics formulate individual rights based on (i) existence of the so-called recognized personal spheres (Gaertner, Pattanaik, and Suzumura 1992), and (ii) individuals' *decisiveness* on personal spheres (social decision on an issue in someone's sphere is decided by the person himself). Despite some fundamental differences between our model and the Arrovian framework (see Samet and Schmeidler 2003 for the details), our definition of a system of powers is similar to this formulation with regard to aspect (i). This is because a system of powers links issues with persons who have the powers on these issues. However, with regard to aspect (ii), our definition is substantially weaker and flexible. Our powers, interpreted as rights, are just rights to *influence* social decision, not necessarily decisive but conditionally decisive (decisiveness is one extreme case in our definition). They are alienable as in Blau (1975) and Gibbard (1974). But, alienation of rights in this paper relies on degree of social consent.

Motivation for our weakening decisiveness component in the earlier definition comes, first of all, from realistic rights that are often conditionally decisive. For example, consider rights for smoking or for clean air. There are some places where smoking is prohibited and also other places where smoking is allowed. A person's desire is not decisive in his own smoking. In order for a person to exercise his right, he needs to find a place where his desire can get sufficient consent from others. Motivation comes also from the so-called paradox of Paretian liberal. As pointed out by Sen (1970, 1976, 1983), Gibbard (1974) and other subsequent works,[4] existence of decisive rights is incompatible with *Pareto efficiency*. Sen (1983, p.14)

[4] See also Deb, Pattanaik, and Razzolini (1997) for the paradox in a framework where rights are represented as a game form.

proposed studying this compatibility issue in restricted preferences domains. However, we show that the paradox prevails even on the extremely restricted domains of trichotomous preferences (or dichotomous preferences). Thus, unless we are not going to abandon *Pareto efficiency*, it is inevitable to think about weakening "decisiveness" component in the definition of rights. How much weakening is necessary to escape from the paradox? Our answer is quasi-plurality systems of powers.

The major difference between our model and the qualification problem in Samet and Schmeidler (2003) lies in the following two extensions. First, in our model, the set of issues may differ from the set of persons both in terms of elements and cardinality. There is no exogenous linkage between issues and persons, while Samet and Schmeidler (2003) consider a model where the set of issues equals the set of persons and so the identity mapping is the exogenous linkage. This generalization enables us to have much wider variety of applications. Second, we allow for "neutral opinion" and consider trichotomous opinions as well as dichotomous opinions considered by Samet and Schmeidler (2003).

Our definition of "consent rules" is much weaker than Samet and Schmeidler's. *Consent rules* are those rules represented by a system of powers. Thus, we allow for a wide spectrum of systems of powers, while Samet and Schmeidler's definition allows for systems conforming to the exogenous linkage. On the one extreme, we have monocentric systems of powers giving only a single person powers on all issues. On the other extreme, we have non-exclusive systems of powers giving everyone the equal power on every issue. We also find that on the trichotomous domain, consent rules may quite differ from plurality rule, while, on the dichotomous domain, they are close to plurality (or majority) rule. Much richer variety of consent rules emerge after admitting neutral opinions. Neutral opinions, we think, are prevalent in realistic decision procedures (abstention can be viewed as an expression of a neutral opinion).

The rest of the paper is organized as follows. In Section 2, we define the model and basic concepts. In Section 3, we define main axioms. In Section 4, we state preliminary results. In Section 5, we state main results. Some proofs are collected in Section A.

2 The Model and Basic Concepts

Let $N \equiv \{1, \cdots, n\}$, $n \geq 2$, be the set of *persons* and $M \equiv \{1, \cdots, m\}$ the set of *issues*. Each person $i \in N$ has his *opinion* on issues in M, represented by an $1 \times m$ row vector P_i consisting of 1, 0, or -1.[5] A *problem* is an $n \times m$ opinion matrix P consisting of n row vectors $P_1, \cdots, P_n$. Let $\mathcal{P}_{\text{Tri}}$ be the set of problems, called, the *trichotomous domain*. An *alternative* is a list of either positive or negative decisions on all issues, formally, a vector of 1 and -1, $x \equiv (x_1, \cdots, x_k) \in \{-1, 1\}^M$, where 1 (resp. -1) in the k^{th} component means accepting the k^{th} issue (resp. rejecting the k^{th} issue). For each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$, P^k denotes the k^{th} column vector of P. Let

$$||P_+^k|| \equiv \sum_{\{i \in N : P_{ik} = 1\}} P_{ik} \text{ and } ||P_-^k|| \equiv \sum_{\{i \in N : P_{ik} = -1\}} -P_{ik}$$

[5] Notation 'P' for 'o**P**inion'.

be the number of 1's in P^k and the number of -1's in P^k respectively. Let

$$||P^k_{+,-}|| \equiv ||P^k_+|| + ||P^k_-||$$

be the total number of "votes".

Let $\mathcal{P}_{\mathrm{Di}}$ be the subset of $\mathcal{P}_{\mathrm{Tri}}$, consisting of the opinion matrices whose entries are either 1 or -1, called the *dichotomous domain.* Let $\mathcal{D}$ be either one of the two domains. The dichotomous domain is considered by Samet and Schmeidler (2003) in a special model of qualification problems.[6]

A decision rule, or briefly, a *rule*, on $\mathcal{D}$ is a function $f\colon \mathcal{D} \to \{-1,1\}^M$ associating with each problem in the domain a single alternative. We are interested in rules that are represented by a "system of powers" defined as follows. We introduce our definitions, first, in the dichotomous domain, and then, in the trichotomous domain.

Given a rule f defined on the dichotomous domain $\mathcal{P}_{\mathrm{Di}}$, person $i \in N$ has the "power to influence the social decision on the k^{th} issue", briefly, the *power on the k^{th} issue* if the decision on the k^{th} issue is made following person i's opinion whenever person i's opinion obtains sufficient consent from society: formally, there exist $q_+, q_- \in \{1, \cdots, n+1\}$ such that for each $P \in \mathcal{P}_{\mathrm{Di}}$,

$$\begin{array}{l} \text{(i) when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow ||P^k_+|| \geq q_+ \ ; \\ \text{(ii) when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow ||P^k_-|| \geq q_- \ . \end{array} \tag{1}$$

The two numbers q_+ and q_- are called *consent-quotas.* The greater q_+ or q_- is, the higher social consent is required for the exercise of the power. There are two extreme cases. When $q_+ = q_- = 1$, i's opinion determines social decision independently of social consent. Thus we call it *decisive.* When $q_+ = n+1$ and $q_- = n+1$, i's power is *void* because i's opinion is never reflected in social decision.

The total number of positive or negative votes always equals n on the dichotomous domain. However, on the trichotomous domain, it is variable. Thus, we allow consent-quotas to vary relative to the total number of votes. Given a rule f defined on $\mathcal{P}_{\mathrm{Tri}}$, a person $i \in N$ has the *power on the k^{th} issue* if there exist three functions $q_+\colon \{1,\ldots,n\} \to \{1,\ldots,n+1\}$, $q_0\colon \{0,1,\ldots,n-1\} \to \{0,1,\ldots,n\}$, and $q_-\colon \{1,\ldots,n\} \to \{1,\ldots,n+1\}$ such that for each $\nu \in \{0,1,\ldots,n\}$, $q_+(\nu)$, $q_0(\nu)$ and $q_-(\nu)$ are in $\{0,1,\ldots,\nu+1\}$, and for each $P \in \mathcal{P}_{\mathrm{Tri}}$ with $||P^k_{+,-}|| = \nu$,

$$\begin{array}{l} \text{(i) when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow ||P^k_+|| \geq q_+(\nu) \ ; \\ \text{(ii) when } P_{ik} = 0,\ f_k(P) = 1 \Leftrightarrow ||P^k_+|| \geq q_0(\nu) \ ; \\ \text{(iii) when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow ||P^k_-|| \geq q_-(\nu) \ . \end{array} \tag{2}$$

Let $q(\cdot) \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ be the *consent-quotas function* (with a slight abuse of notation),[7] and Q the family of consent-quota functions.

Definition 1 (System of Powers). A *system of powers* representing a rule f on $\mathcal{P}_{\mathrm{Tri}}$ is a

[6] Samet Schmeidler (2003) consider dichotomous opinions that are described by vectors of 1 and 0. Number 0 in their paper has the same meaning as -1 in this paper.

[7] Since the three component functions q_+, q_0, q_- have different domains, q cannot be described as a function. But, including 0 in the domain of q_+ and q_- and defining the values at 0 arbitrarily will not make any difference and, this way, the problem can be avoided.

function $W\colon M \to N \times Q$ mapping each issue $k \in M$ a pair of the person, $W_1(k)$, who has the power on the k^{th} issue, and the consent-quotas function, $W_2(k) = (q_+(\cdot), q_0(\cdot), q_-(\cdot))$, associated with the power.[8][9] That is, when $W_1(k) = i$, for each $\nu \in \{0, 1, \ldots, n\}$ and each $P \in \mathcal{P}_{\text{Tri}}$ with $||P^k_{+,-}|| = \nu$,

$$\begin{aligned}
&\text{(i) when } P_{ik} = 1,\;\; f_k(P) = 1 \Leftrightarrow ||P^k_+|| \geq q_+(\nu)\,;\\
&\text{(ii) when } P_{ik} = 0,\;\; f_k(P) = 1 \Leftrightarrow ||P^k_+|| \geq q_0(\nu)\,;\\
&\text{(iii) when } P_{ik} = -1,\;\; f_k(P) = -1 \Leftrightarrow ||P^k_-|| \geq q_-(\nu)\,.
\end{aligned}$$

A rule may be represented by multiple systems of powers, although all these systems will be shown to be equivalent under a natural equivalence relation to be defined in Section 4.

3 Axioms

In this section, we define axioms for rules, which are crucial in this paper.

The first axiom says that rules should not respond negatively when the opinion matrix increases.

Monotonicity. For each $P, P' \in \mathcal{D}$, if $P \geqq P'$, $f(P) \geqq f(P')$.

The second axiom says that decisions on different issues should be made independently: decision on the k^{th} issue should rely only on the k^{th} column of the opinion matrix.

Independence. For each $P, P' \in \mathcal{D}$ and each $k \in M$, if $P^k = P'^k$, $f_k(P) = f_k(P')$.

We refer readers to Rubinstein and Fishburn (1986), Kasher and Rubinstein (1997), and Samet and Schmeidler (2003) for more discussion on the two axioms.

We next introduce an axiom that is a generalization of “symmetry” by Samet and Schmeidler (2003). Suppose that members of society have their own areas of specialty and each issue lies in some of these areas. Ideally, it is important that society treats all members and their areas of specialty in a symmetric manner. To illustrate this idea, suppose that the first issue is in John’s area and the second issue is in Paul’s. Consider the case when both John and Paul have positive opinions on their own issues and John is negative on Paul’s issue while Paul is positive on John’s issue, as depicted in Table 1-(a). As in the table, suppose that the social decision on the second issue (Paul’s), in this case, is against Paul’s opinion (negative), while the decision on the first issue (John’s) follows John’s opinion. Now consider another case when John and Paul face the reverse situation, that is, John faces the same situation regarding his area as Paul faced in the earlier case, as depicted in Table 1-(b). If the social decision on the first issue (John’s) in this case follows John’s opinion (so it differs from the decision on the second issue in the earlier case), one could argue that the rule favors John and John’s area relative to Paul and Paul’s area. Our next axiom prevents such an asymmetric treatment.

An issue may lie in multiple areas and so there may exist multiple linkages between issues and persons (multiple functions from the set of issues to the set of persons). Requiring

[8] Notation ‘W’ for ‘po**W**er’.

[9] In the model with $M = N$, when $W_1(\cdot)$ is the identity function, Samet and Schmeidler (2003) call the rule represented by W a *consent rule*.

	Issue 1	Issue 2
John	1	-1
Paul	1	1
Others	1	-1
Decision	1	-1

(a)

	Issue 1	Issue 2
John	1	1
Paul	-1	1
Others	-1	1
Decision	1	1

(b)

Table 1: When issue 1 is in John's area and issue 2 is in Paul's, the social decisions in the two cases exhibit a violation of symmetric linkage.

symmetric treatment with respect to all possible linkages can be too strong. The next axiom requires symmetric treatment for at least one linkage. That is, it says that there be a linkage between issues and persons and that given this linkage, the rule should treat each person i and i's issues symmetrically to any other person j and j's issues. Technically, when names of person i and all i's issues are switched simultaneously to names of person j and all j's issues, social decision should also be switched accordingly.

To define this axiom formally, let $\lambda\colon M \to N$ be a function between issues and persons, called a *linkage*. For each $i \in N$, let us call elements in $\lambda^{-1}(i)$ person i's issues. Let $\pi\colon N \to N$ and $\delta\colon M \to M$ are permutations on N and on M such that δ maps the set of each person i's issues *onto* the set of person $\pi(i)$'s issues. Let ${}^{\delta}_{\pi}P$ be the matrix such that for each $i \in N$ and each $k \in M$, ${}^{\delta}_{\pi}P_{ik} \equiv P_{\pi(i)\delta(k)}$. Then each person i and his issue k play the same role in ${}^{\delta}_{\pi}P$ as person $\pi(i)$ and his issue $\delta(k)$ do in P.

Symmetric Linkage. There is $\lambda\colon M \to N$ such that for each permutation $\pi\colon N \to N$ and each permutation $\delta\colon M \to M$, if for each $i \in N$, δ maps the set of i's issues $\lambda^{-1}(i)$ *onto* the set of $\pi(i)$'s issues $\lambda^{-1}(\pi(i))$, then for each $k \in M$, $f_k\left({}^{\delta}_{\pi}P\right) = f_{\delta(k)}(P)$.

Given a function $\lambda\colon M \to N$, we say that a rule f satisfies λ-*symmetry* if f satisfies *symmetric linkage* with respect to $\lambda\colon M \to N$. Note that if $\pi(i) = j$ and $|\lambda^{-1}(i)| \neq |\lambda^{-1}(j)|$, then there is no permutation $\delta\colon M \to M$ satisfying the ontoness condition for δ stated in the definition of *symmetric linkage*. Thus, λ-*symmetry* does not impose any restriction for such π. In particular, if $\lambda^{-1}(i) = M$, λ-*symmetry* applies to only those permutations on N not changing the name of i and all permutations on M.[10]

Next are two standard axioms of social choice, known as *anonymity* and *neutrality*. The former says that social decision should not depend on how persons are named and the latter says that social decision should not depend on how issues are labeled. For each permutation π on N, let ${}_{\pi}P \in \mathcal{P}_{\text{Tri}}$ be such that for each $i \in N$ and each $k \in M$, ${}_{\pi}P_{ik} \equiv P_{\pi(i)k}$. For each permutation δ on M, let ${}^{\delta}P \in \mathcal{P}_{\text{Tri}}$ be such that for each $i \in N$ and each $k \in M$, ${}^{\delta}P_{ik} \equiv P_{i\delta(k)}$.

Anonymity. For each $P \in \mathcal{P}_{\text{Tri}}$ and each permutation $\pi\colon N \to N$, $f({}_{\pi}P) = f(P)$.

[10]In the qualification problems considered by Samet and Schmeidler (2003), $M = N$. Thus there is an exogenous one-to-one correspondence between M and N, namely the identity function $\lambda^{\text{ID}}(i) = i$, for each $i \in M$. Their symmetry axiom coincides with λ^{ID}-*symmetry*.

Neutrality. For each $P \in \mathcal{P}_{\text{Tri}}$, each permutation $\delta\colon M \to M$, and each $k \in M$, $f_k\left({}^{\delta}P\right) = f_{\delta(k)}\left(P\right)$.

Clearly, the combination of *anonymity* and *neutrality* implies *symmetric linkage* but the converse does not hold.

4 Preliminary Results

We distinguish powers into two types. The power on the k^{th} issue is (fully) *exclusive* if there is a person i who has the power on the k^{th} issue and no one else has the power on the k^{th} issue. It is (fully) *non-exclusive* if all agents have the "equal" power on the k^{th} issue associated with a single consent-quotas function (or, on the dichotomous domain, a list of consent-quotas). In proving Proposition 2, we will show that the power on an issue is either exclusive or non-exclusive: see Remark 2. Either one and only one person has the power or all persons have the equal power.

Given a system of powers W, when the power on the k^{th} issue is non-exclusive, who has the power on this issue is not essential. Thus by changing $W_1\left(k\right)$, we may find other systems representing the same rule. Thus the following equivalence relation on systems of powers is natural. Two systems of powers W and W' are *equivalent*, denoted by $W \sim W'$, if for each k with $W_1\left(k\right) \neq W_1'\left(k\right)$, the power on the k^{th} issue is *non-exclusive* (so, $W_2\left(k\right) = W_2'\left(k\right)$).

The following two extreme systems are notable. Under a *non-exclusive system of powers*, everyone has the non-exclusive power on every issue. Under a *monocentric system of powers*, one and only one agent has the exclusive power on every issue.

Lemma 1. *Assume that a rule f is represented by a system of powers W. Let $k \in M$, $i \equiv W_1\left(k\right)$, and $q\left(\cdot\right) \equiv W_2\left(k\right)$. Then for each $\nu \in \{1, \ldots, n\}$,* (i) $q_+\left(\nu\right) + q_-\left(\nu\right) = \nu + 1$ *and when $\nu \leq n-1$, $q_+\left(\nu\right) = q_0\left(\nu\right)$ if and only if for each $P \in \mathcal{P}_{Tri}$ with $||P^k_{+,-}|| = \nu$,*

$$f_k\left(P\right) = 1 \Leftrightarrow ||P^k_+|| \geq q_+\left(\nu\right). \tag{3}$$

(ii) $q\left(\nu\right) = \left(\nu+1, \nu+1, 1\right)$ *if and only if for each $P \in \mathcal{P}_{Tri}$ with $||P^k_{+,-}|| = \nu$, $f_k(P) = -1$.*
(iii) $q\left(\nu\right) = \left(1, 0, \nu+1\right)$ *if and only if for each $P \in \mathcal{P}_{Tri}$ with $||P^k_{+,-}|| = \nu$, $f_k(P) = 1$.*
Thus, if for each $\nu \in \{1, \ldots, n\}$, one of the three cases holds, then the power on the k^{th} issue is non-exclusive.

Proof. Let $\nu \in \{1, \ldots, n\}$. Assume $q_+\left(\nu\right) + q_-\left(\nu\right) = \nu + 1$ and $q_+\left(\nu\right) = q_0\left(\nu\right)$. Then the three parts (i)-(iii) in (2) collapse into (3). Conversely, if (3) holds, then from parts (ii) and (iii) in (2), $q_0(\nu) = q_+(\nu)$ and $q_+(\nu) = \nu + 1 - q_-(\nu)$.

Parts (ii) and (iii) are straightforward. Note that if any of the three cases (i)-(iii) holds, who has the power on the k^{th} issue is not essential. Changing $W_1\left(k\right)$ into any other person does not affect the rule the system represents, which means everyone has the power on the k^{th} issue associated with the same consent-quotas function. Thus the power is non-exclusive. $\square$

We now show that the three cases of non-exclusive powers in Lemma 1 characterize non-exclusive powers.

Proposition 1. *The power on an issue associated with* $(q_+(\cdot), q_0(\cdot), q_-(\cdot))$ *is non-exclusive if and only if for each* $\nu \in \{1, \ldots, n\}$, (i) $q_+(\nu) \leq \nu$, $q_-(\nu) \leq \nu$, $q_+(\nu) + q_-(\nu) = \nu + 1$, *and when* $\nu \leq n-1$, $q_0(\nu) = q_+(\nu)$, *or* (ii) $(q_+(\nu), q_-(\nu)) \in \{(\nu+1, 1), (1, \nu+1)\}$ *and when* $\nu \leq n-1$, $(q_+(\nu), q_0(\nu), q_-(\nu)) \in \{(\nu+1, \nu+1, 1), (1, 0, \nu+1)\}$.

The proof is in Appendix A.1.

The next result is uniqueness of systems of powers representing a rule.

Proposition 2. *Assume* $n \geq 4$. *If a rule is represented by a system of powers, then the system is unique up to the equivalence relation* $\sim$.

The proof is in Appendix A.1.

We next state necessary and sufficient conditions on a system of powers which guarantee *monotonicity* or *symmetric linkage* of the rule the system represents.

A consent-quotas function $q(\cdot) \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ has *component ladder property* if for each $\nu \in \{1, \ldots, n\}$, the following three inequalities hold whenever they are well-defined

$$\begin{array}{l} \text{(i) } q_+(\nu - 1) \leq q_+(\nu) \leq q_+(\nu - 1) + 1; \\ \text{(ii) } q_0(\nu - 1) \leq q_0(\nu) \leq q_0(\nu - 1) + 1; \\ \text{(iii) } q_-(\nu - 1) \leq q_-(\nu) \leq q_-(\nu - 1) + 1. \end{array} \tag{4}$$

The function has *intercomponent ladder property* if for each $\nu \in \{1, \ldots, n\}$,

$$q_+(\nu) \leq q_0(\nu - 1) + 1 \leq \nu - q_-(\nu) + 2. \tag{5}$$

The function has *ladder property* if it has the above two properties. We also say that a system of powers W has *ladder property* if its consent-quotas functions have ladder property.

Proposition 3. *A rule represented by a system of powers satisfies* monotonicity *if and only if the system of powers has* ladder property.

The proof is given in Appendix A.2.

A system of powers $W : M \rightarrow N \times Q$ satisfies *horizontal equality* if for each pair of persons i and $j \in N$ with the same number of issues under W_1, that is, $|W_1^{-1}(i)| = |W_1^{-1}(j)|$, their powers are associated with the same consent-quotas function, that is, for each $k \in W_1^{-1}(i)$ and each $l \in W_1^{-1}(j)$, $W_2(k) = W_2(l)$. When $i = j$, this property says that person i's powers on two different issues are associated with the same consent-quotas function.

Proposition 4. *A rule represented by a system of powers satisfies* symmetric linkage *if and only if the system of powers satisfies* horizontal equality.

The proof is given in Appendix A.2.

5 Main Results

5.1 Monotonicity, Independence, and Symmetric Linkage

In this section, we state our results imposing the three axioms, *monotonicity*, *independence*, and *symmetric linkage*.

If a rule is represented by a system of powers, decisions on different issues are made independently and so the rule satisfies *independence*. By Propositions 3 and 4, if the system of powers satisfies both ladder property and horizontal equality, the rule also satisfies *monotonicity* and *symmetric linkage*. Our first main result tell us about the converse. It says that the combination of the three axioms is sufficient for existence of a system of powers.

Theorem 1. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$. A rule on $\mathcal{D}$ satisfies monotonicity, independence, and symmetric linkage if and only if it is represented by a system of powers satisfying ladder property and horizontal equality. Moreover, the system is unique up to the equivalence relation $\sim$.*

The proof is in the Appendix A.3. We show independence of the three axioms later.

Adding *anonymity*, we obtain:

Theorem 2. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$. The following are equivalent.*
(i) *A rule on $\mathcal{D}$ satisfies monotonicity, independence, symmetric linkage, and anonymity.*
(ii) *A rule on $\mathcal{D}$ is represented by a non-exclusive system of powers satisfying ladder property and horizontal equality.*
(iii) *A rule on $\mathcal{D}$ is represented by a system of powers W satisfying ladder property and horizontal equality such that for each $k \in M$, letting $(q_+(\cdot), q_0(\cdot), q_-(\cdot)) \equiv W_2(k)$, for each $\nu \in \{1, \ldots, n\}$,* (iii.1) *$q_+(\nu) \leq \nu$, $q_-(\nu) \leq \nu$, $q_+(\nu) + q_-(\nu) = \nu + 1$, and when $\nu \leq n-1$, $q_0(\nu) = q_+(\nu)$, or* (iii.2) *$(q_+(\nu), q_-(\nu)) \in \{(\nu+1, 1), (1, \nu+1)\}$ and when $\nu \leq n-1$, $(q_+(\nu), q_0(\nu), q_-(\nu)) \in \{(\nu+1, \nu+1, 1), (1, 0, \nu+1)\}$.*

Proof. Let $k \in M$ and $i \equiv W_1(k)$. By *anonymity*, when i has the power on the k^{th} issue, then every other agent should have the same power too. Thus the power on the k^{th} issue is non-exclusive. The proof for the reverse direction is straightforward. This proves the equivalence between (i) and (ii). We obtain the remaining equivalence from Proposition 1. □

On the dichotomous domain $\mathcal{P}_{\text{Di}}$, (iii) of Theorem 2 can be simplified into the following: for each $k \in M$, letting $(q_+, q_-) \equiv W_2(k)$, (iii.1) $q_+ \leq n$, $q_- \leq n$ and $q_+ + q_- = n + 1$ or (iii.2) $(q_+, q_-) \in \{(n+1, 1), (1, n+1)\}$.

Adding *neutrality* to the three axioms of Theorem 1, we characterize two extreme types of systems of powers, monocentric systems and non-exclusive systems.

Theorem 3. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$. A rule on $\mathcal{D}$ satisfies monotonicity, independence, symmetric linkage, and neutrality if and only if it is represented either by a monocentric system of powers or by a constant non-exclusive system of powers satisfying ladder property and horizontal equality, in either case.*

Proof. If f is represented by a monocentric system of powers, then one and only one agent has the power on each issue. By horizontal equality, the consent-quotas functions for all issues are identical. Hence decisions on different issues are made neutrally. If f is represented by a constant non-exclusive system of powers, then because of the constancy condition, f satisfies *neutrality*.

To prove the converse, let f be a rule satisfying the stated axioms. By Theorem 1, there is a system of powers W representing f. Suppose that there is $i \in N$ who has an exclusive power on the k^{th} issue. Then by *neutrality*, i should have the same exclusive power on every other issue. Thus, the system is monocentric. If there is no exclusive power, then by Proposition 2, the system is non-exclusive. And by *neutrality*, it is constant. □

We next consider *duality* (Samet and Schmeidler 2003). Each issue may be defined as representing a certain statement (a proposal) or its negation (the antiproposal): for example, qualification or disqualification. Which representation is taken does not matter if the rule satisfies *duality*.

Duality. For each $P \in \mathcal{P}_{\text{Tri}}$, $f(-P) = -f(P)$.

On the trichotomous domain $\mathcal{P}_{\text{Tri}}$, *duality* is incompatible with the combination of the three axioms in Theorem 1. For example, if f is a rule satisfying the three axioms in Theorem 1, then for each $i \in N$, each $k \in \lambda^{-1}(i)$, and each $P \in \mathcal{P}_{\text{Tri}}$ with $P_{ik} = 0$ and $||P_+^k|| = ||P_-^k||$, $f_k(-P) = f_k(P)$, violating *duality*. However, on the dichotomous domain $\mathcal{P}_{\text{Di}}$, adding *duality*, we are able to pin down a smaller family of rules. A system of powers W has *quotas duality* if for each issue $k \in M$, the consent-quotas function $(q_+(\cdot), q_0(\cdot), q_-(\cdot)) \equiv W_2(k)$ satisfies $q_+(\cdot) = q_-(\cdot)$.

Theorem 4. *On the dichotomous domain $\mathcal{P}_{Di}$, a rule satisfies monotonicity, independence, symmetric linkage, and duality if and only if it is represented by a system of powers satisfying ladder property, horizontal equality and quotas duality.*

Proof. Let f be a rule and W a system of powers of f such that for each $k \in M$, if we let $(q_+, q_-) \equiv W_2(k)$, $q_+ = q_-$. Let $i \in N$ and $k \in W_1^{-1}(i)$. Let $P \in \mathcal{P}_{\text{Di}}$. Note $(-P)_{ik} = -P_{ik}$, $||(-P)_+^k|| = ||P_-^k||$, and $||(-P)_-^k|| = ||P_+^k||$. Therefore, $||(-P)_-^k|| \geq q_- \Leftrightarrow ||P_+^k|| \geq q_+$ and $||(-P)_+^k|| \geq q_+ \Leftrightarrow ||P_-^k|| \geq q_-$. Then $f(-P) = -f(P)$. Hence f satisfies *duality*.

Conversely, let f be a rule satisfying the four axioms. By Theorem 1, there exists a system of powers W representing f. Let $k \in M$, $i \equiv W_1(k)$, and $(q_+, q_-) \equiv W_2(k)$. Suppose, by contradiction, that $q_+ \neq q_-$, say, $q_+ > q_-$ (the same argument applies when $q_+ < q_-$). Let r be the number such that $q_+ > r \geq q_-$. Then there exists $P \in \mathcal{P}_{\text{Di}}$ such that $P_{ik} = -1$ and $||P_-^k|| = r$. Then $f_k(P) = -1$. Since $(-P)_{ik} = 1$ and $||(-P)_+^k|| = ||P_-^k|| = r < q_+$, $f_k(-P) = -1$, contradicting *duality*. □

When n is even, there is no system of powers W satisfying (iii.1) of Theorem 2 and quotas duality. However, when n is odd, these two properties imply majority rule. Thus we obtain:

Corollary 1. *Assume that n is odd. On the dichotomous domain $\mathcal{P}_{Di}$, majority rule is the only rule satisfying monotonicity, independence, symmetric linkage, anonymity, and duality.*

When we consider *neutrality* instead of *anonymity*, we obtain a characterization of the family consisting of majority rule and rules represented by monocentric systems of powers with quotas duality.

Corollary 2. *Assume that n is odd. On the dichotomous domain $\mathcal{P}_{Di}$, a rule satisfies* monotonicity, independence, symmetric linkage, neutrality, *and* duality *if and only if it is majority rule or a rule that is represented by a monocentric system of powers satisfying ladder property, horizontal equality and quotas duality.*

Proof. To prove the nontrivial direction, let f be a rule satisfying the stated axioms. Then by Theorem 3, it is represented either by a monocentric system of powers or by a constant non-exclusive system of powers. In the former case, we are done. In the latter case, the rule satisfies *anonymity*. Thus it follows from Corollary 1 that f is majority rule. $\square$

We now investigate consequences of dropping any one of the three main axioms, *monotonicity*, *independence* and *symmetric linkage*.

Dropping Symmetric Linkage

We characterize the following rules satisfying *monotonicity* and *independence*. These rules can be described by "decisive structures" between subgroups of N (Ju 2003).[11] Let $\mathfrak{C}^* \equiv \{(C_1, C_2) \in 2^N \times 2^N : C_1 \cap C_2 = \varnothing\}$ be the set of all pairs of disjoint subgroups of N. For each $k \in M$, a *decisive structure for the k^{th}-issue*, denoted by $\mathfrak{C}_k \subseteq \mathfrak{C}^*$, is a subset of $\mathfrak{C}^*$. It satisfies *monotonicity* if for each $(C_1, C_2) \in \mathfrak{C}_k$, if $(C_1', C_2') \in \mathfrak{C}^*$ is such that $C_1' \supseteq C_1$ and $C_2' \subseteq C_2$, then $(C_1', C_2') \in \mathfrak{C}_k$. For each $P \in \mathcal{P}_{\mathrm{Tri}}$ and each $k \in M$, let

$$N\left(P_+^k\right) \equiv \{i \in N : P_{ik} = 1\} \text{ and } N\left(P_-^k\right) \equiv \{i \in N : P_{ik} = -1\}.$$

A rule f is *represented by a profile of decisive structures* $(\mathfrak{C}_k)_{k \in M}$ if for each $P \in \mathcal{D}$ and each $k \in M$, $f_k(P) = 1$ if and only if $(N(P_+^k), N(P_-^k)) \in \mathfrak{C}_k$. Any rule represented by a profile of decisive structures satisfies *independence*, since it makes decisions issue by issue. Conversely, if a rule satisfies *independence*, the decision on the k^{th} issue relies only on the pair of the set of persons in favor of k and the set of persons against k. Thus, it is represented by a profile of decisive structures. Monotonicity of decisive structures is a necessary and sufficient condition for *monotonicity* of the rule. Therefore we obtain:

Proposition 5. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$.* (i) *A rule on $\mathcal{D}$ satisfies* independence *if and only if it is represented by a profile of decisive structures.* (ii) *A rule on $\mathcal{D}$ satisfies* independence *and* monotonicity *if and only if it is represented by a profile of monotonic decisive structures.*

The formal proof is left for readers.

Let $\mathcal{I}^* \equiv \{(n_1, n_2) \in \mathbb{Z}_+ \times \mathbb{Z}_+ : n_1 + n_2 \leq n\}$, where $\mathbb{Z}_+$ is the set of non-negative integers. Any subset $\mathcal{I} \subseteq \mathcal{I}^*$ is called an *index set*. It is *comprehensive* if for each $(n_1, n_2) \in \mathcal{I}$ and each $(n_1', n_2') \in \mathcal{I}^*$, if $n_1' \geq n_1$ and $n_2' \leq n_2$, then $(n_1', n_2') \in \mathcal{I}$. Using Proposition 5, it is easy to characterize rules satisfying *independence* and *anonymity*. Decisive structures of each of these rules can be described by index sets. Formally, a *counting rule* is a rule that is represented by a profile of index sets, $(\mathcal{I}_k)_{k \in M}$, as follows: for each $P \in \mathcal{P}_{\mathrm{Tri}}$ and each $k \in M$, $f_k(P) = 1 \Leftrightarrow (||P_+^k||, ||P_-^k||) \in \mathcal{I}_k$. It is easy to show that a counting rule is *monotonic* if and only if all index sets in the profile $(\mathcal{I}_k)_{k \in M}$ are comprehensive. Thus, we obtain:

[11] Ju (2003) calls decisive structures "power structures". We use different name to avoid confusion with our stronger notion of power.

Proposition 6. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$.* (i) *A rule on $\mathcal{D}$ satisfies* independence *and* anonymity *if and only if it is a counting rule.* (ii) *A rule on $\mathcal{D}$ satisfies* monotonicity, independence, *and* anonymity *if and only if it is a counting rule represented by a profile of comprehensive index sets.*

The formal proof is left for readers.

Dropping Monotonicity

An *extended system of powers* ${}_eW$ maps each issue $k \in M$ into a person ${}_eW_1(k) \in N$ and a triple of index sets ${}_eW_2(k) = (\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k)$. A rule f is *represented by an extended system of powers* ${}_eW$ if for each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$,

$$\begin{array}{l} \text{(i) when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow \left(||P_+^k||, ||P_-^k||\right) \in \mathcal{I}_+^k; \\ \text{(ii) when } P_{ik} = 0,\ f_k(P) = 1 \Leftrightarrow \left(||P_+^k||, ||P_-^k||\right) \in \mathcal{I}_0^k; \\ \text{(iii) when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow \left(||P_-^k||, ||P_+^k||\right) \in \mathcal{I}_-^k; \end{array} \tag{6}$$

where $i \equiv {}_eW_1(k)$ and $(\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k) \equiv {}_eW_2(k)$.

Proposition 7. *Let $\mathcal{D} \in \{\mathcal{P}_{Di}, \mathcal{P}_{Tri}\}$. A rule over $\mathcal{D}$ satisfies* independence *and* symmetric linkage *if and only if it is represented by an extended system of power ${}_eW(\cdot)$ satisfying horizontal equality, that is, for each $i, j \in N$ with $|{}_eW_1^{-1}(i)| = |{}_eW_1^{-1}(j)|$, each $k \in {}_eW_1^{-1}(i)$, and each $l \in {}_eW_1^{-1}(j)$, ${}_eW_2(k) = {}_eW_2(l)$.*[12]

The proof is in Appendix A.3.

Dropping Independence

For each $P \in \mathcal{P}_{\text{Tri}}$, let $\chi(P) \equiv \sum_{k \in M} ||P_-^k||/|M|$. Let f be the rule represented by $\chi(\cdot)$ as follows: for each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$,

$$f_k(P) = 1 \Leftrightarrow ||P_+^k|| \geq \chi(P).$$

By definition, this rule treats agents anonymously and issues neutrally. Thus it satisfies *anonymity*, *neutrality*, and so *symmetric linkage*. If $P, P' \in \mathcal{P}_{\text{Tri}}$ are such that for each $k \in M$, $N\left(P_+^k\right) \subseteq N\left(P_+'^k\right)$ and $N\left(P_-^k\right) \supseteq N\left(P_-'^k\right)$, $\sum_{k \in M} ||P_-^k||/|M| \geq \sum_{k \in M} ||P_-'^k||/|M|$, that is, $\chi(P) \geq \chi(P')$. Then for each $k \in M$, if $f_k(P) = 1$ (that is, $||P_+^k|| \geq \chi(P)$), $||P_+'^k|| \geq ||P_+^k|| \geq \chi(P) \geq \chi(P')$ and so $f_k(P') = 1$. Thus f satisfies *monotonicity*. The threshold level $\chi(P)$ depends on opinions on all issues. So f violates *independence*. Using different $\chi(\cdot)$, we can define other examples of rules violating *independence* but satisfying other axioms. However, we leave it for future research to characterize the entire family of rules satisfying *monotonicity* and *symmetric linkage*.

Anonymity and Representation by A Non-Exclusive System of Powers

[12] This property of ${}_eR(\cdot)$ is needed to guarantee *symmetric linkage* like horizontal equality of a system of powers.

Consider a rule f satisfying *monotonicity* and *independence*. If f also satisfies *anonymity*, then f is a monotonic counting rule. Thus, there is a profile of comprehensive index sets $(\mathcal{I}_k)_{k\in M}$ representing f. For each $\nu \in \{0,1,\ldots,n\}$, if $\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}_k\} \neq \emptyset$, let $q_+^k(\nu) \equiv \min\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}_k\}$; otherwise, $q_+^k(\nu) \equiv \nu + 1$, and let $q_0^k(\nu) \equiv q_+^k(\nu)$ and $q_-^k(\nu) \equiv \nu + 1 - q_+^k(\nu)$. Then we obtain one of the three cases in Lemma 1 and so f is represented by a non-exclusive system. Conversely, if f is represented by a non-exclusive system, f satisfies *anonymity*. Therefore, we obtain:

Proposition 8. *Let f be a rule satisfying* monotonicity *and* independence*. Then the following are equivalent.*
(i) *Rule f satisfies* anonymity*.*
(ii) *Rule f is represented by a non-exclusive system of powers satisfying ladder property.*
(iii) *Rule f is a monotonic counting rule.*

If a monotonic counting rule f has at most n different index sets, then f can be represented by a system of powers satisfying horizontal equality. This is because there is a function $W_1(\cdot)$ that maps each pair $k, l \in M$ with the same index set $\mathcal{I}$ into the same person. Thus f satisfies *symmetric linkage*. If f has more than n different index sets, f violates *symmetric linkage*. Thus we obtain:

Corollary 3. *Assume $n \geq m$. Let f be a rule satisfying* monotonicity *and* independence*. Then the following are equivalent.*
(i) *Rule f satisfies* anonymity*.*
(ii) *Rule f is represented by a non-exclusive system of powers satisfying ladder property and horizontal equality.*
(iii) *Rule f is a monotonic counting rule.*

Remark 1. This proposition shows that if $n \geq m$, *monotonicity*, *independence* and *anonymity* together imply *symmetric linkage*. Therefore, in this case, *symmetric linkage* in Theorem 2 and Corollary 1 can be dropped.

Models with An Exogenous Linkage between Issues and Persons

We now turn to models considered by Samet and Schmeidler (2003) and its generalization. Assume that there is an exogenous linkage between issues and persons, denoted by $\lambda\colon M \to N$. In the model with λ, we skip λ in λ-*symmetry* and simply call this axiom *symmetry*. When $M = N$ and λ is the identity function, our *symmetry* coincides with the definition by Samet and Schmeidler (2003). Replacing *symmetric linkage* in all our results with *symmetry*, we obtain characterizations of subfamilies of rules represented by systems of powers $W(\cdot)$ *conforming to the exogenous linkage*, that is, $W_1(\cdot) = \lambda(\cdot)$.

Moreover, depending on $\lambda(\cdot)$, some results can be strengthened. For example, suppose that $\lambda(\cdot)$ is not constant. Then no system of powers conforming to $\lambda(\cdot)$ can be monocentric. Thus, it follows from Theorem 3 that a rule over $\mathcal{D} \in \{\mathcal{P}_{\mathrm{Tri}}, \mathcal{P}_{\mathrm{Di}}\}$ satisfies *monotonicity*, *independence*, *symmetry*, and *neutrality* if and only if it is represented by a constant non-exclusive system of powers conforming to the exogenous linkage λ and satisfying ladder property and horizontal equality. *Thus these four axioms together imply anonymity.* Also it follows from Corollary 2 that when n is odd, majority rule is the only rule on $\mathcal{P}_{\mathrm{Di}}$ satisfying *monotonicity*, *independence*, *symmetry*, *neutrality*, and *duality*.

5.2 Pareto Efficiency and Existence of A System of Powers

Compatibility of *Pareto efficiency* and existence of so-called libertarian rights (decisive powers) is widely studied by a number of authors followed by the celebrated work, Sen (1970). To discuss this issue in our framework, we now consider preference relations.

Opinions are partial description of the following preference relations. A *separable preference relation* R_0 orders social decisions in such a way that for each $k \in M$ and each quadruple $x, x', y, y' \in \{-1, 1\}^M$ with $x_k = y_k$, $x'_k = y'_k$, $x_{-k} = x'_{-k}$, and $y_{-k} = y'_{-k}$,

$$\begin{array}{ccc} x \succ_{R_0} x' & \Leftrightarrow & y \succ_{R_0} y' \ ; \\ x \sim_{R_0} x' & \Leftrightarrow & y \sim_{R_0} y' \ , \end{array}$$

where $\succ_{R_0}$ and $\sim_{R_0}$ are strict and indifference relations associated with R_0. Then issues are partitioned into goods, bads, and nulls depending on whether they have positive or negative or indifferent impacts on the person's well-being. Thus, each separable preference R_0 is associated with an opinion vector P_0, each positive (resp. negative or zero) component of P_0 representing the corresponding issue as a good (resp. a bad or a null). Obviously, there are a number of separable preference relations corresponding to a single opinion vector. Let $\mathcal{R}$ be the family of profiles of separable preference relations. A *rule* over the separable preferences domain $\mathcal{R}$ associates with each profile of preference relations a single alternative in $\{-1, 1\}^M$. With the above stated relationship between opinions and preferences, axioms and powers defined for the opinion domain are easily extended to the corresponding notions on the separable preferences domain.

5.2.1 Sen's Paradox of Paretian Liberal

Sen (1970) shows in the Arrovian social choice model that there is no *Pareto efficient* preference aggregation rule that gives at least two agents libertarian rights. This is so-called Sen's paradox of Paretian liberal. Sen's reasoning does not directly apply here because of the following differences between our model and his. The alternative space, here, is a product space and, associated with this structure, preference relations have the separability restriction. In addition, while Sen (1970) considers preference aggregation rules, we consider social choice functions. Despite these differences, our notion of decisive powers is a natural counterpart to Sen's libertarian rights. In fact, our decisive powers are the same as rights formulated by Gibbard (1974). Because we focus on separable preference relations, the so-called Gibbard paradox does not prevail in our model as pointed out by Sen (1983, p.14). Thus Sen's quest is still meaningful here. Does Sen's paradox prevail in our model? Not surprisingly, it does, as we show below. Furthermore, we show that the paradox prevails in a much stronger sense in a substantially restricted domain of preference relations.

We first show that the paradox prevails on the separable preferences domain. Sen's (1970) *minimal liberalism* postulates that there should be at least two persons who have decisive powers. Assume that persons 1 and 2 are given the decisive powers on the first and second issues respectively. Consider the following preference relations R_1 and R_2 of the two persons. The first issue is a bad for R_1 and any decision with the positive second component is preferred, under R_1, to any decision with the negative second component. The second issue is a bad for R_2 and any decision with the positive first component is preferred, under R_2, to any decision with the negative first component. Then by the decisive powers of the two persons, decisions on the first and second issues are both negative. But the two persons will be better off at any decision with positive components for both issues. This confirms that *minimal liberalism and Pareto efficiency are incompatible on the separable preferences domain.*[13]

Preference relations in the above example are "meddlesome"' (Blau 1975); person 1 cares so much about person 2's issue that positive decision on this issue is preferred to the negative decision no matter what decisions are made on the other issues. Without such meddlesome preference relations, the paradox of Paretian liberal may not apply.

Unfortunately, the paradox prevails even in a substantially restricted environment where only "trichotomous" or "dichotomous" preference relations are admissible. A *trichotomous preference relation* R_0 is a separable preference relation represented by a function $U_0 \colon \{-1,1\}^M \to \mathbb{R}$ such that for each $x \in \{-1,1\}^M$, $U_0(x) = \sum_{k \in M : x_k = 1} P_{0k}$, where $P_0 \in \{-1,0,1\}^M$ is the opinion vector corresponding to R_0.[14] A *dichotomous preference relation* is a trichotomous preference relation for which each issue is either a good or a bad. Let $\mathcal{R}_{\text{Tri}}$ be the family of profiles of trichotomous preference relations and $\mathcal{R}_{\text{Di}}$ the family of profiles of dichotomous preference relations. Note that there are one-to-one correspondences between $\mathcal{R}_{\text{Tri}}$ and $\mathcal{P}_{\text{Tri}}$ and between $\mathcal{R}_{\text{Di}}$ and $\mathcal{P}_{\text{Di}}$.

To show the paradox, suppose that there are at least three persons, $n \geq 3$ and that persons 1 and 2 have the decisive powers respectively on issues 1 and 2. Consider the profile of dichotomous preference relations $(R_i)_{i \in N}$ given by the following opinion vectors: $P_1 \equiv (1,-1,-1,\ldots,-1)$, $P_2 \equiv (-1,1,-1,\ldots,-1)$, and for each $i \in N \backslash \{1,2\}$, $P_i \equiv (-1,\ldots,-1)$. Then by the decisive powers of persons 1 and 2, $f_1(R) = f_2(R) = 1$. If the rule is *Pareto efficient*, for each $k \in M \backslash \{1,2\}$, $f_k(R) = -1$. Thus $f(R) = (1,1,-1,\ldots,-1)$. Note that this alternative is indifferent to $x \equiv (-1,\ldots,-1)$ for both person 1 and person 2 and x is preferred to $f(R)$ by all others. This contradicts *Pareto efficiency*. Therefore, *when there are at least three persons, no Pareto efficient rule on the dichotomous preferences domain satisfies minimal liberalism.*

Note that unlike the previous paradox on the separable preferences domain, we need the assumption $n \geq 3$. The case with two persons ruled out by this assumption is very limited. However, it should be noted that the paradox does not apply in the two-person case (then decisiveness is quite close to plurality principle since one person's opinion accounts for 50%). This is an implication of our results in the next section.

5.2.2 Quasi-Plurality Systems of Powers

The observations made in Section 5.2.1 show that decisiveness component in the definition of libertarian rights is too strong to be compatible with *Pareto efficiency*. They force us to consider non-decisive powers instead. Is it, then, possible to have non-decisive powers and at the same time to satisfy *Pareto efficiency*? It is indeed possible on the trichotomous preferences domain $\mathcal{R}_{\text{Tri}}$ and also on the dichotomous preferences domain $\mathcal{R}_{\text{Di}}$ as we show in this section. Moreover, we provide a characterization of plurality-like rules on the basis of *Pareto efficiency*, *independence*, and *symmetry* (or *symmetric linkage*). Since we only

[13] This was originally proven by Gibbard (1974, Theorem 2).

[14] That is, $U_0(x) = |\{k \in M : x_k = 1 \text{ and } P_{0k} = 1\}| - |\{k \in M : x_k = 1 \text{ and } P_{0k} = -1\}|$.

consider trichotomous or dichotomous preference relations, throughout this section, we use opinion vectors to refer to the corresponding trichotomous preference relations.

We begin with a definition of important systems of powers in our results.

Definition 2 (Quasi-Plurality Systems of Powers). A system of powers W is called a *quasi-plurality system* if there is a consent-quotas function $q(\cdot) \equiv (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ such that for each $k \in M$, $W_2(k) = (q_+(\cdot), q_0(\cdot), q_-(\cdot))$ and for each $\nu \in \{1, \ldots, n\}$,

$$q_+(\nu), q_-(\nu) \in \{\frac{\nu-1}{2}, \frac{\nu+1}{2}\}, \tag{7}$$

for each $\nu \in \{0, \ldots, n-1\}$,

$$q_0(\nu) \in \{\frac{\nu-1}{2}, \frac{\nu+1}{2}\}. \tag{8}$$

The rule represented by a quasi-plurality system is called a *quasi-plurality rule*.

Clearly, quasi-plurality systems satisfy horizontal equality and thus quasi-plurality rules satisfy *symmetric linkage*. Obviously, plurality rule is an example; it is represented by a non-exclusive quasi-plurality system. Quasi-plurality systems are not always non-exclusive. For example, for each $\nu \in \{1, \ldots, n\}$, let $q_+(\nu) = q_-(\nu) \equiv (\nu-1)/2$ and for each $\nu \in \{0, \ldots, n-1\}$, let $q_0(\nu) \equiv (\nu-1)/2$. Then the power on each issue is exclusive by Proposition 1. However, note that for each $k \in M$, if $||P_+^k|| \neq ||P_-^k||$, $f_k(P)$ equals the decision made by plurality rule and that if $||P_+^k|| = ||P_-^k||$, $f_k(P)$ is determined by the opinion of the person, say i, who has the power on the k^{th} issue (that is, $f_k(P) = 1$ if $P_{ik} = 1$ or 0; $f_k(P) = -1$ if $P_{ik} = -1$). Thus "exclusiveness" feature, if it exists, in a quasi-plurality system plays a role only when there is a tie between the group of persons with the positive opinion and the group of persons with the negative opinion.

Any quasi-plurality rule f has the following property: for each $k \in M$,

$$\begin{array}{l} f_k(P) = 1 \ \Rightarrow \ ||P_+^k|| \geq ||P_-^k||; \\ ||P_+^k|| > ||P_-^k|| \ \Rightarrow \ f_k(P) = 1. \end{array} \tag{9}$$

Note that $\sum_{i \in N} U_i(x) = \sum_{i \in N} \sum_{\{k \in M : x_k = 1\}} P_{ik} = \sum_{\{k \in M : x_k = 1\}} (||P_+^k|| - ||P_-^k||)$. Therefore, by (9), any quasi-plurality rule maximizes the sum of utilities. Thus it satisfies *Pareto efficiency*. Moreover, our next result shows that quasi-plurality rules are the only rules satisfying *Pareto efficiency*, *independence*, and *symmetry*.

Theorem 5. *Assume that there is an exogenous linkage λ between issues and persons and that all persons are linked to the same number of issues under λ. Then a rule on $\mathcal{D} \in \{\mathcal{R}_{Tri}, \mathcal{R}_{Di}\}$ satisfies Pareto efficiency, independence, and symmetry if and only if it is represented by a quasi-plurality system of powers conforming to λ.*[15]

The proof is in Appendix A.4. Note that this result holds in the model considered by Samet and Schmeidler (2003) because in their model $N = M$ and λ is the identity function.

[15]This result is similar to the efficiency characterization of plurality social choice in Ju (2005). However, there are crucial differences. Ju (2005) imposes *anonymity* instead of *symmetry* and his result holds only on $\mathcal{R}_{\text{Tri}}$, while Theorem 5 holds both on $\mathcal{R}_{\text{Tri}}$ and on $\mathcal{R}_{\text{Di}}$. The family of quasi-plurality rules is larger than the family of "semi-plurality rules" characterized in Ju (2005).

Not all quasi-plurality systems satisfy intercomponent ladder property. This extra property is obtained after adding *monotonicity* to the three axioms in the theorem.

Next is a direct corollary to Theorem 5.

Corollary 4. *Given the assumption in Theorem 5, a rule on $\mathcal{D} \in \{\mathcal{R}_{Tri}, \mathcal{R}_{Di}\}$, represented by a system of powers conforming to the exogenous linkage λ, satisfies* Pareto efficiency *if and only if the system of powers is a quasi-plurality system.*

Adding *neutrality* allows us to establish the same characterization in the model without any exogenous linkage.

Theorem 6. *Suppose $m \geq n$. A rule on $\mathcal{D} \in \{\mathcal{R}_{Tri}, \mathcal{R}_{Di}\}$ satisfies* Pareto efficiency, independence, symmetric linkage, *and* neutrality *if and only if it is represented either by a non-exclusive quasi-plurality system of powers or by a monocentric quasi-plurality system of powers.*

The proof is in Appendix A.4.

A Proofs

A.1 Proofs of Propositions 1 and 2

Let $\mathfrak{W}_f$ be the set of systems of powers representing a rule f.

Claim 1. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W_1'(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W_2'(k)$. Then for each $\nu \geq 2$, we have*

$$\begin{aligned} &\text{(i)} \;\; q_+(\nu) \geq 2 \text{ and } q'_+(\nu) \geq 2 \quad \Rightarrow \quad q_+(\nu) = q'_+(\nu)\,; \\ &\text{(ii)} \;\; q_-(\nu) \geq 2 \text{ and } q'_-(\nu) \geq 2 \quad \Rightarrow \quad q_-(\nu) = q'_-(\nu)\,. \end{aligned}$$

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. We prove (i) and skip the same proof of (ii). Suppose that $q_+(\nu) \neq \nu + 1$ and $q'_+(\nu) \neq \nu + 1$. Because $\nu \geq 2$, $q_+(\nu) \neq \nu + 1$, and $q_+(\nu) \geq 2$, there exists P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = P_{i'k} = 1$ and $||P^k_+|| = q_+(\nu)$. Then by i's power $W(k)$, $f_k(P) = 1$. Thus, by i''s power $W'(k)$, $q'_+(\nu) \leq q_+(\nu)$. Similarly, we show the reverse inequality.

If $q_+(\nu) = \nu + 1$, then consider P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = P_{i'k} = 1$ and $||P^k_+|| = \nu$. By i's power $W(k)$, $f_k(P) = -1$. Then by i''s power $W'(k)$, $q'_+(\nu) > ||P^k_+|| = \nu$, which implies $q'_+(\nu) = \nu + 1$. □

Claim 2. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W_1'(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W_2'(k)$. Then for each $\nu \geq 2$, we have*

$$\begin{aligned} &\text{(i)} \;\; q_+(\nu) = 1 \quad \Leftrightarrow \quad q'_-(\nu) \geq \nu\,; \\ &\text{(ii)} \;\; q_-(\nu) = 1 \quad \Leftrightarrow \quad q'_+(\nu) \geq \nu\,. \end{aligned}$$

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. We prove the first equivalence and skip the same proof of the second. Assume $q_+(\nu) = 1$. Since $\nu \geq 2$, there is P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = 1$, $P_{i'k} = -1$, and $||P^k_+|| = 1$ (so $||P^k_-|| = \nu - 1$). Then by i's power $W(k)$, $f_k(P) = 1$.

Hence by i''s power $W'(k)$, $q'_-(\nu) > \nu - 1$, that is, $q'_-(\nu) \geq \nu$. Conversely, if $q'_-(\nu) \geq \nu$, then using the same P as above and i''s power $W'(k)$, we have $f_k(P) = 1$. This means, by i's power $W(k)$, $q_+(\nu) \leq 1$. Thus $q_+(\nu) = 1$. $\square$

Claim 3. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W'_1(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W'_2(k)$. Then for each $\nu \in \{1, \ldots, n-1\}$,*

$$\begin{aligned} &\text{(i)} \;\; q_0(\nu) \geq 1 \;\; \Rightarrow \;\; q_0(\nu) = q'_+(\nu)\,; \\ &\text{(ii)} \;\; q_0(\nu) = 0 \;\; \Leftrightarrow \;\; q'_+(\nu) = 1 \textit{ and } q'_-(\nu) = \nu + 1\,. \end{aligned}$$

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W'_1(k)$.

Part (i). Suppose $q_0(\nu) \geq 1$. If $q_0(\nu) = \nu + 1$, then for each P with $||P^k_{+,-}|| = \nu$, $P_{ik} = 0$ and $P_{i'k} = 1$, by i's power $W(k)$, $f_k(P) = -1$. Thus by i''s power $W'(k)$, $q'_+(\nu) = \nu + 1$. If $q_0(\nu) \leq \nu$, there exists P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = 0$, $P_{i'k} = 1$, and $||P^k_+|| = q_0(\nu)$ (such P exists because $1 \leq q_0(\nu) \leq \nu \leq n-1$). Then by $W(k)$, $f_k(P) = 1$. And by $W'(k)$, $q'_+(\nu) \leq q_0(\nu)$. Thus if $q_0(\nu) = 1$, $q'_+(\nu) = 1$. Suppose $q_0(\nu) \geq 2$. In this case, if $q'_+(\nu) < q_0(\nu)$, there exists P with $||P^k_{+,-}|| = \nu$, $P_{ik} = 0$, $P_{i'k} = 1$, and $q'_+(\nu) \leq ||P^k_+|| < q_0(\nu)$ (such P exists because $q_0(\nu) \geq 2$). Then by $W'(k)$, $f_k(P) = 1$ and by $W(k)$, $f_k(P) = -1$, which is a contradiction.

Part (ii). Suppose $q_0(\nu) = 0$. Consider P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = 0$, $P_{i'k} = 1$, and $||P^k_+|| = 1$. By $W(k)$, $f_k(P) = 1$. Then by $W'(k)$, $q'_+(\nu) \leq 1$. Thus $q'_+(\nu) = 1$.

Next consider P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = 0$, $P_{i'k} = -1$, and $||P^k_-|| = \nu$. By $W(k)$, $f_k(P) = 1$. Then by $W'(k)$, $q'_-(\nu) > \nu$. Thus $q'_-(\nu) = \nu + 1$.

To prove the converse, suppose $q'_+(\nu) = 1$ and $q'_-(\nu) = \nu + 1$. If P is such that $||P^k_{+,-}|| = \nu$, $P_{ik} = 0$, $P_{i'k} = -1$, and $||P^k_-|| = \nu$, then by $W'(k)$, $f_k(P) = 1$. Thus by $W(k)$, $q_0(\nu) \leq 0$. $\square$

Claim 4. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W'_1(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W'_2(k)$. Then for each $\nu \in \{3, \ldots, n\}$, $q_+(\nu) = q'_+(\nu)$ and $q_-(\nu) = q'_-(\nu)$.*

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W'_1(k)$.

We first show $q_+(\nu) = q'_+(\nu)$. If both numbers are greater than or equal to 2, the result follows from Claim 1. Suppose $q_+(\nu) = 1$. Then by Claim 2, $q'_-(\nu) \geq \nu$. If $q_-(\nu) \neq q'_-(\nu)$ $(\geq \nu \geq 3)$, then $q_-(\nu) = 1$ (because otherwise, by Claim 1, $q_-(\nu) = q'_-(\nu)$). Let P be such that $||P^k_{+,-}|| = \nu$, $P_{ik} = P_{i'k} = -1$ and $||P^k_-|| = 2$. Since $q_-(\nu) = 1 < ||P^k_-|| < 3 \leq \nu \leq q'_-(\nu)$, then by $W(k)$, $f_k(P) = 1$ and by $W'(k)$, $f_k(P) = -1$, which is a contradiction. Therefore $q_-(\nu) = q'_-(\nu) \geq \nu$. Then by Claim 2, $q'_+(\nu) = 1$.

We next show $q_-(\nu) = q'_-(\nu)$. If both numbers are greater than or equal to 2, the result follows from Claim 1. Suppose $q_-(\nu) = 1$. Then by Claim 2, $q'_+(\nu) \geq \nu$. Since $q_+(\nu) = q'_+(\nu) \geq \nu$, by Claim 2 again, $q'_-(\nu) = 1$. $\square$

Claim 5. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W'_1(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W'_2(k)$. Then for each $\nu \in \{0, \ldots, n-2\}$, $q_0(\nu) = q'_0(\nu)$ and when $n \geq 4$, $q_0(n-1) = q'_0(n-1)$.*

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. Let $\nu \in \{0, \ldots, n-2\}$. Suppose $q_0(\nu) \neq q_0'(\nu)$, say, $q_0(\nu) < q_0'(\nu)$. Since $\nu \leq n-2$ and $q_0(\nu) \leq \nu$ (note $q_0(\nu) < q_0'(\nu) \leq \nu + 1$), then there is P be such that $P_{ik} = P_{i'k} = 0$, $||P_{+,-}^k|| = \nu$, and $||P_+^k|| = q_0(\nu)$. Then by $W(k)$, $f_k(P) = 1$ and by $W'(k)$, $f_k(P) = -1$, which is a contradiction.

Finally, $q_0(n-1) = q_0'(n-1)$ follows from Claim 3 and the fact that $q_+(n-1) = q_+'(n-1)$ and $q_-(n-1) = q_-'(n-1)$, which holds by Claim 4 (here we need the assumption of $n \geq 4$ in order to have $n-1 \geq 3$). □

Claim 6. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W_1'(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W_2'(k)$. Then for each $\nu \in \{1, \ldots, n-1\}$,*

$$q_-(\nu) = 1 \Leftrightarrow q_0'(\nu) \geq \nu\ .$$

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. Let $\nu \in \{1, \ldots, n-1\}$. Suppose $q_-(\nu) = 1$. Let P be such that $||P_{+,-}^k|| = \nu$, $P_{ik} = -1$, $P_{i'k} = 0$, and $||P_+^k|| = \nu - 1$ (so $||P_-^k|| = 1$). By $W(k)$, $f_k(P) = -1$. Thus by $W'(k)$, $q_0'(\nu) > \nu - 1$, that is, $q_0'(\nu) \geq \nu$.

Suppose $q_0'(\nu) \geq \nu$. Consider the same P as above. By $W'(k)$, $f_k(P) = -1$. Thus by $W(k)$, $q_-(\nu) \leq 1$ and so $q_-(\nu) = 1$. □

Lemma 2. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W_1'(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W_2'(k)$. Then for each $\nu \in \{0, 1, \ldots, n\}$, if $\nu \geq 1$, $q_+(\nu) = q_+'(\nu)$ and $q_-(\nu) = q_-'(\nu)$; if $\nu \leq n-1$, $q_0(\nu) = q_0'(\nu)$.*

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. By Claims 4 and 5, we only need to show that for each $\nu \in \{1, 2\}$, $q_+(\nu) = q_+'(\nu)$ and $q_-(\nu) = q_-'(\nu)$.

Consider $\nu = 2$. Then $q_0(\nu) = q_0'(\nu)$ by Claim 5. If $q_0(\nu) = 0$, then by Claim 3, $q_+'(\nu) = 1 = q_+(\nu)$. If $q_0(\nu) = q_0'(\nu) \geq 1$, then applying (i) of Claim 3 twice, $q_0'(\nu) = q_+(\nu)$ and $q_0(\nu) = q_+'(\nu)$. Thus $q_+(\nu) = q_+'(\nu)$.

We next show $q_-(\nu) = q_-'(\nu)$. If both numbers are greater than or equal to 2, the result follows from Claim 1. Suppose $q_-(\nu) = 1$. Then by Claim 2, $q_+'(\nu) \geq \nu$. Since $q_+(\nu) = q_+'(\nu) \geq \nu$, then by Claim 2 again, $q_-'(\nu) = 1$.

Now consider $\nu = 1$. By Claim 5, $q_0(1) = q_0'(1)$. Suppose $q_0(1) = q_0'(1) \geq 1$. Then by Claim 3, $q_+(1) = q_+'(1)$. And by Claim 6, $q_-(1) = q_-'(1)$. Suppose $q_0(1) = q_0'(1) = 0$. Then by Claim 3, $q_+(1) = q_+'(1) = 1$ and $q_-(1) = q_-'(1) = 2$. □

Claim 7. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W_1'(k)$. Let $q(\cdot) \equiv W_2(k)$ and $q'(\cdot) \equiv W_2'(k)$. Then for each $\nu \in \{1, \ldots, n\}$, $q_+(\nu) + q_-'(\nu) > \nu$ (and $q_+'(\nu) + q_-(\nu) > \nu$).*

Proof. The inequalities hold trivially for $\nu = 1$.

Let $\nu \in \{2, \ldots, n\}$. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. Suppose by contradiction $q_+(\nu) + q_-'(\nu) \leq \nu$. Then $q_+(\nu) < \nu$ or $q_-'(\nu) < \nu$. We consider the former case and skip the same proof for the latter case.

Suppose $q_+(\nu) < \nu$. Then there exists $P \in \mathcal{P}_{\text{Tri}}$ be such that $P_{ik} = 1$, $P_{i'k} = -1$, $||P_{+,-}^k|| = \nu$, and $||P_+^k|| = q_+(\nu)$ (such P exists because $\nu \geq 2$, $q_+(\nu) < \nu$, and so $||P_-^k|| = \nu - q_+(\nu) \geq 1$). Then $||P_-^k|| = \nu - q_+(\nu) \geq q_-'(\nu)$. Since $P_{ik} = 1$, $W(k) = (i, q(\cdot))$, and $||P_+^k|| = q_+(\nu)$, then $f_k(P) = 1$. On the other hand, since $P_{i'k} = -1$, $W'(k) = (i', q'(\cdot))$, and $||P_-^k|| = \nu - q_+(\nu) \geq q_-'(\nu)$, then $f_k(P) = -1$, contradicting $f_k(P) = 1$. □

Lemma 3. *Assume that $W, W' \in \mathfrak{W}_f$ and for some $k \in M$, $W_1(k) \neq W_1'(k)$ and $W_2(k) = W_2'(k) = q(\cdot)$. Then for each $\nu \in \{1, \ldots, n\}$,* (i) *$q_+(\nu) \leq \nu$, $q_-(\nu) \leq \nu$, $q_+(\nu) + q_-(\nu) = \nu + 1$, and when $\nu \leq n-1$, $q_0(\nu) = q_+(\nu)$, or* (ii) *$(q_+(\nu), q_-(\nu)) \in \{(\nu+1, 1), (1, \nu+1)\}$ and when $\nu \leq n-1$, $(q_+(\nu), q_0(\nu), q_-(\nu)) \in \{(\nu+1, \nu+1, 1), (1, 0, \nu+1)\}$.*

Proof. Let $i \equiv W_1(k)$ and $i' \equiv W_1'(k)$. The proof is in the following three steps.

Step 1. For each $\nu \in \{2, \ldots, n\}$, if $q_+(\nu) \leq \nu$ and $q_-(\nu) \leq \nu$, then $q_+(\nu) + q_-(\nu) = \nu + 1$ and when $\nu \leq n-1$, $q_0(\nu) = q_+(\nu)$.

By Claim 7, $q_+(\nu) + q_-(\nu) \geq \nu + 1$. In order to show $q_+(\nu) + q_-(\nu) = \nu + 1$, suppose $q_+(\nu) + q_-(\nu) \geq \nu + 2$. Let $P \in \mathcal{P}_{\text{Tri}}$ be such that $P_{ik} = 1$, $P_{i'k} = -1$, $||P^k_{+,-}|| = \nu$, and $||P^k_+|| = \nu - q_-(\nu) + 1$ (since $q_+(\nu), q_-(\nu) \leq \nu$ and $q_+(\nu) + q_-(\nu) \geq \nu + 2$, then $q_+(\nu), q_-(\nu) \geq 2$; thus $||P^k_+|| = \nu - q_-(\nu) + 1 = q_+(\nu) - 1 \geq 1$ and similarly $||P^k_-|| = q_-(\nu) - 1 \geq 1$; also note $||P^k_{+,-}|| = \nu \geq 2$; all these guarantee existence of such P). Then $||P^k_+|| = \nu - q_-(\nu) + 1 = q_+(\nu) - 1 < q_+(\nu)$ and $||P^k_-|| = q_-(\nu) - 1 < q_-(\nu)$. Since $P_{ik} = 1$, $W(k) = (i, q(\cdot))$, and $||P^k_+|| < q_+(\nu)$, then $f_k(P) = -1$. Since $P_{i'k} = -1$, $W'(k) = (i', q(\cdot))$, and $||P^k_-|| = \nu - ||P^k_+|| = q_-(\nu) - 1 < q_-(\nu)$, then $f_k(P) = 1$, contradicting $f_k(P) = -1$.

We now show that when $\nu \leq n-1$, $q_0(\nu) = q_+(\nu)$. By (ii) of Claim 3 and the assumption $q_-(\nu) \leq \nu$, $q_0(\nu) \geq 1$. Thus the equation follows directly from (i) of Claim 3.

Step 2. For each $\nu \in \{2, \ldots, n\}$, (i) *if $q_+(\nu) = \nu + 1$, $q_-(\nu) = 1$;* (ii) *if $q_-(\nu) = \nu + 1$, $q_+(\nu) = 1$.*

Suppose $q_+(\nu) = \nu + 1$. Since $\nu \geq 2$, there is P such that $||P^k_{+,-}|| = \nu$, $P_{ik} = 1$, $P_{i',k} = -1$, and $||P^k_-|| = 1$ (so $||P^k_+|| = \nu - 1$). Then by i's power $W(k)$, $f_k(P) = -1$. By i''s power $W'(k)$, $q_-(\nu) = 1$. The same argument applies to show the second part.

Step 3. For each $\nu \in \{1, \ldots n-1\}$, (i) *$q_0(\nu) = \nu + 1$ if and only if $q_+(\nu) = \nu + 1$ and $q_-(\nu) = 1$;* (ii) *$q_0(\nu) = 0$ if and only if $q_+(\nu) = 1$ and $q_-(\nu) = \nu + 1$.*

Part (ii) follows from Claim 3. To prove part (i), suppose $q_0(\nu) = \nu + 1$. Consider P and P' such that $||P^k_{+,-}|| = ||P'^k_{+,-}|| = \nu$, $P_{ik} = P'_{ik} = 0$, $P_{i'k} = 1$, $P'_{i'k} = -1$, $||P^k_+|| = \nu$, and $||P'^k_-|| = 1$. By i's power, $f_k(P) = f_k(P') = -1$. Since $f_k(P) = -1$, by i''s power, $q_+(\nu) > \nu$ and so $q_+(\nu) = \nu + 1$. Also since $f_k(P') = -1$, by i''s power, $q_-(\nu) \leq 1$ and so $q_-(\nu) = 1$. The converse is proven using the same argument in the reverse direction.

Step 4. If $q_0(1) = 0$, then $q_+(1) = 1$ and $q_-(1) = 2$; if $q_0(1) = 1$, then $q_+(1) = 1$ and $q_-(1) = 1$; if $q_0(1) = 2$, then $q_+(1) = 2$ and $q_-(1) = 1$. Thus $(q_+(1), q_0(1), q_-(1)) \in \{(1, 0, 2), (1, 1, 1), (2, 2, 1)\}$.

The two cases for $q_0(1) = 0$ or 2 are shown in Step 3. The remaining case with $q_0(1) = 1$ follows from (i) of Claim 3 and Claim 6. $\square$

Remark 2. Lemmas 1 and 3 show that the power on an issue can be either exclusive or non-exclusive. That is, either only one person has the power or everyone has the power. There is no power shared by more than one but not all persons.

Proof of Proposition 1. The characterization of non-exclusive powers in Proposition 1 follows from Lemmas 1 and 3. $\square$

Proof of Proposition 2. Uniqueness of systems of powers in Proposition 2 follows from Lemmas 2 and 3, and Proposition 1. $\square$

A.2 Proofs of Propositions 3 and 4

Lemma 4. *A rule f is represented by a system of powers $W(\cdot)$ satisfying ladder property if and only if it is represented by an extended system of powers ${}_eW(\cdot)$ such that for each issue $k \in M$, the three index sets in ${}_eW_2(k) \equiv (\mathcal{I}_+, \mathcal{I}_0, \mathcal{I}_-)$ are comprehensive and*

$$\begin{array}{l} \text{(i)}\ \ (n_1, n_2) \in \mathcal{I}_0 \Rightarrow (n_1+1, n_2) \in \mathcal{I}_+; \\ \text{(ii)}\ \ (n_1, n_2) \notin \mathcal{I}_- \Rightarrow (n_2, n_1-1) \in \mathcal{I}_0; \\ \text{(iii)}\ \ (n_1, n_2) \notin \mathcal{I}_- \Rightarrow (n_2+1, n_1-1) \in \mathcal{I}_+. \end{array} \tag{10}$$

Proof. Suppose that person $i \in N$ has the power on the k^{th} issue associated with a consent-quotas function $q(\cdot)$. Then we can construct three comprehensive index sets, $\mathcal{I}_+$, $\mathcal{I}_0$, and $\mathcal{I}_-$ as follows. For each $s \in \{+, 0, -\}$, let $\mathcal{I}_s \equiv \{(n_1, n_2) \in \mathcal{I}^* : n_1 \geq q_s(n_1+n_2)\}$. Then it is easy to show that (6) implies (2), comprehensiveness of $\mathcal{I}_s$ implies (4) of component ladder property and (10) implies (5) of intercomponent ladder property.

To explain the reverse construction, let $\mathcal{I}_+$, $\mathcal{I}_0$, and $\mathcal{I}_-$ be the three comprehensive sets satisfying (6) and (10). For each $\nu \in \{1, \ldots, n\}$ and each $s \in \{+, 0, -\}$, let

$$q_s(\nu) \equiv \begin{cases} \min\{n_1 : (n_1, \nu - n_1) \in \mathcal{I}_s\}, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}_s\} \neq \emptyset; \\ \nu + 1, \text{ if } \{n_1 : (n_1, \nu - n_1) \in \mathcal{I}_s\} = \emptyset. \end{cases}$$

Then this consent-quotas function satisfies the two ladder properties because of comprehensiveness of $\mathcal{I}_+$, $\mathcal{I}_0$, and $\mathcal{I}_-$ and (10). And (6) follows from (2).[16] $\square$

Lemma 5. *A rule f represented by an extended system of powers ${}_eW(\cdot)$ satisfies monotonicity if and only if ${}_eW(\cdot)$ satisfies the comprehensiveness property and* (10) *stated in Lemma 4.*

Proof. Let f be a rule represented by an extended system of powers ${}_eW$. Then clearly f satisfies *independence* and so by Proposition 5, f is represented by a profile of decisive structures $(\mathfrak{C}_k)_{k \in M}$.

Assume that f satisfies *monotonicity*. Then all decisive structures in $(\mathfrak{C}_k)_{k \in M}$ are monotonic. Let $k \in K$, $i \equiv {}_eW_1(k)$ and $(\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k) \equiv {}_eW_2(k)$. Then by (6), $\mathcal{I}_+^k = \{(|C_1|, |C_2|)\colon (C_1, C_2) \in \mathfrak{C}_k \text{ and } i \in C_1\}$, $\mathcal{I}_0^k = \{(|C_1|, |C_2|)\colon (C_1, C_2) \in \mathfrak{C}_k \text{ and } i \notin C_1 \cup C_2\}$, and $\mathcal{I}_-^k = \{(|C_2|, |C_1|)\colon (C_1, C_2) \notin \mathfrak{C}_k \text{ and } i \in C_2\}$. Comprehensiveness of the three index sets $\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k$ is a direct consequence of monotonicity of the decisive structure $\mathfrak{C}_k$. To show part (i) of (10), let $(n_1, n_2) \in \mathcal{I}_0^k$. Suppose to the contrary $(n_1+1, n_2) \notin \mathcal{I}_+^k$. Let $P \in \mathcal{P}_{\text{Tri}}$ be such that $P_{ik} = 0$, $||P_+^k|| = n_1$, and $||P_-^k|| = n_2$. Then $f_k(P) = 1$. Let $P' \in \mathcal{P}_{\text{Tri}}$ have the same components as P except $P'_{ik} \equiv 1$. Then $P' \geq P$, $||P'^k_+|| = n_1 + 1$, and $||P_-^k|| = n_2$. Since $(n_1+1, n_2) \notin \mathcal{I}_+^k$, $f_k(P') = -1$, contradicting *monotonicity* of f.

To show part (ii) of (10), suppose to the contrary that $(n_1, n_2) \notin \mathcal{I}_-^k$ and $(n_2, n_1 - 1) \notin \mathcal{I}_0^k$. Let $P \in \mathcal{P}_{\text{Tri}}$ be such that $P_{ik} = -1$, $||P_-^k|| = n_1$, and $||P_+^k|| = n_2$. Then $f_k(P) = 1$. Let $P' \in \mathcal{P}_{\text{Tri}}$ have the same components as P except $P'_{ik} \equiv 0$. Then $P' \geq P$, $||P'^k_+|| = n_2$, and $||P_-^k|| = n_1 - 1$. Since $(n_2, n_1 - 1) \notin \mathcal{I}_0^k$, $f_k(P') = -1$, contradicting *monotonicity* of f.

To show (iii) of (10), suppose to the contrary that $(n_1, n_2) \notin \mathcal{I}_-^k$ and $(n_2+1, n_1-1) \notin \mathcal{I}_+^k$. Let $P \in \mathcal{P}_{\text{Tri}}$ be such that $P_{ik} = -1$, $||P_-^k|| = n_1$, and $||P_+^k|| = n_2$. Then $f_k(P) = 1$. Let $P' \in \mathcal{P}_{\text{Tri}}$ have the same components as P except $P'_{ik} \equiv 1$. Then $P' \geq P$, $||P'^k_+|| = n_2 + 1$,

[16] The proof is available in the earlier version of this article, Ju (2004).

and $||P_-^k|| = n_1 - 1$. Since $(n_2+1, n_1-1) \notin \mathcal{I}_0^k$, $f_k(P') = -1$, contradicting *monotonicity* of f.

To prove the converse, assume that ${}_eW$ satisfies the comprehensiveness property and (10) stated in Lemma 4. In order to prove *monotonicity* of f, let $P' \geq P$ and $k \in M$ be such that $f_k(P) = 1$. We only have to show $f_k(P') = 1$. Let $i \equiv {}_eW(k)$ and $(\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k) \equiv {}_eW_2(k)$. When $P'_{ik} = P_{ik}$, it follows directly from the comprehensiveness condition of the three sets $\mathcal{I}_+^k, \mathcal{I}_0^k, \mathcal{I}_-^k$ that $f_k(P') = 1$. There are two remaining cases.

Case 1. $P_{ik} = 0 \neq P'_{ik}$ and $(||P_+^k||, ||P_-^k||) \in \mathcal{I}_0^k$. Then $P'_{ik} = 1$. Hence $||P_+'^k|| \geq ||P_+^k|| + 1$ and $||P_-'^k|| \leq ||P_-^k||$. By comprehensiveness of $\mathcal{I}_+^k$ and part (i) of (10), $(||P_+'^k||, ||P_-'^k||) \in \mathcal{I}_+^k$. Therefore $f_k(P') = 1$.

Case 2. $P_{ik} = -1 \neq P'_{ik}$ and $(||P_-^k||, ||P_+^k||) \notin \mathcal{I}_-^k$. Then either $P'_{ik} = 0$ or $P'_{ik} = 1$. If $P'_{ik} = 0$, $||P_+'^k|| \geq ||P_+^k||$ and $||P_-'^k|| \leq ||P_-^k|| - 1$. Then by comprehensiveness of $\mathcal{I}_-^k$ and part (ii) of (10), $(||P_+'^k||, ||P_-'^k||) \in \mathcal{I}_0^k$. Thus, $f_k(P') = 1$. If $P'_{ik} = 1$, $||P_+'^k|| \geq ||P_+^k|| + 1$ and $||P_-'^k|| \leq ||P_-^k|| - 1$. Then by comprehensiveness of $\mathcal{I}_-^k$ and part (iii) of (10), $(||P_+'^k||, ||P_-'^k||) \in \mathcal{I}_+^k$. Therefore $f_k(P') = 1$. $\square$

Proof of Proposition 3. Proposition 3 follows directly from Lemmas 4 and 5. $\square$

Proof of Proposition 4. Consider a rule f represented by a system of powers W. Let $\lambda(\cdot) \equiv W_1(\cdot)$. Let $\pi\colon N \to N$ be a permutation on N and $\delta\colon M \to M$ a permutation on M such that for each $i \in N$, δ maps $\lambda^{-1}(i)$ onto $\lambda^{-1}(\pi(i))$. Then because of the ontoness property of δ, $i \in N$ and $\pi(i)$ are associated with the same number of issues under λ. Thus by horizontal equality, for each $k \in \lambda^{-1}(i)$, i's power on the k^{th} issue and $\pi(i)$'s power on the $\delta(k)^{\text{th}}$ issue are associated with the same consent-quotas function, that is, $W_2(k) = W_2(\delta(k))$. Denote the common consent-quotas function by $q(\cdot)$. For each $P \in \mathcal{P}_{\text{Tri}}$, $||P_+^{\delta(k)}|| = ||{}_\pi^\delta P_+^k||$ and $||P_-^{\delta(k)}|| = ||{}_\pi^\delta P_-^k||$. Thus, $q(||P_{+,-}^{\delta(k)}||) = q(||{}_\pi^\delta P_+^k||)$ and ${}_\pi^\delta P_{ik} = P_{\pi(i)\delta(k)}$. Therefore, $f_k\left({}_\pi^\delta P\right) = f_{\delta(k)}(P)$. This shows that f satisfies *symmetric linkage* associated with λ. The converse can be proven similarly. $\square$

A.3 Proofs of Proposition 7 and Theorem 1

Proof of Proposition 7. Using the same argument as in the proof of Proposition 4, we can show that a rule represented by an extended system of powers satisfies *symmetric linkage* if and only if the extended system satisfies horizontal equality. Clearly, any rule represented by an extended system of powers satisfies *independence*.

To prove the converse, consider a rule f satisfying *independence* and *symmetric linkage*. Then by Proposition 5, f is represented by a profile of decisive structures $(\mathfrak{C}_k)_{k \in M}$. By *symmetric linkage*, there exists $\lambda\colon M \to N$ such that f satisfies λ-*symmetry*. We identify an extended system of powers of f and complete the proof in two steps.

Step 1. For each pair $i, j \in N$ with $|\lambda^{-1}(i)| = |\lambda^{-1}(j)|$, each $k \in \lambda^{-1}(i)$, each $l \in \lambda^{-1}(j)$, and each $(C_1, C_2), (C'_1, C'_2) \in \mathfrak{C}^*$ with $|C_1 \cap \{i\}| = |C'_1 \cap \{j\}|$ and $|C_2 \cap \{i\}| = |C'_2 \cap \{j\}|$ (or equivalently, $[i \in C_1 \Leftrightarrow j \in C'_1]$ and $[i \in C_2 \Leftrightarrow j \in C'_2]$), if $|C_1| = |C'_1|$ and $|C_2| = |C'_2|$, then

$$(C_1, C_2) \in \mathfrak{C}_k \Leftrightarrow (C'_1, C'_2) \in \mathfrak{C}_l\ .$$

Let $i,j \in N$, $k \in \lambda^{-1}(i)$, $l \in \lambda^{-1}(j)$, and $(C_1, C_2), (C_1', C_2') \in \mathfrak{C}^*$ be given as above. Consider the case $i \in C_1$ and $j \in C_1'$ (the proofs for the other cases are similar). Suppose $(C_1, C_2) \in \mathfrak{C}_k$. Let P be such that $N(P_+^k) \equiv C_1$ and $N(P_-^k) \equiv C_2$. So $f_k(P) = 1$. Since $|C_1| = |C_1'|$ and $|C_2| = |C_2'|$, there is a permutation π on N such that $\pi(i) = j$, $\pi(j) = i$, $\pi(C_1) = C_1'$, and $\pi(C_2) = C_2'$. Since $|\lambda^{-1}(i)| = |\lambda^{-1}(j)|$, there is a permutation δ on M such that $\delta\left(\lambda^{-1}(j)\right) = \lambda^{-1}(i)$, $\delta(\lambda^{-1}(i)) = \lambda^{-1}(j)$, $\delta(l) = k$, and for all other $k' \in M \backslash [\lambda^{-1}(i) \cup \lambda^{-1}(j)]$, $\delta(k') = k'$. Then $N({}^{\delta}_{\pi}P^l_+) = \pi^{-1}(N(P_+^{\delta(l)})) = \pi^{-1}(C_1) = C_1'$. Similarly, $N({}^{\delta}_{\pi}P^l_-) = C_2'$. By λ-*symmetry*, $f_l\left({}^{\delta}_{\pi}P\right) = f_{\delta(l)}(P) = f_k(P) = 1$. Therefore, $(C_1', C_2') \in \mathfrak{C}_l$. The proof of the opposite direction is similar.

One notable implication of Step 1 is that for each $i \in N$ and each pair $k, l \in \lambda^{-1}(i)$, $\mathfrak{C}_k = \mathfrak{C}_l$.

Step 2. Rule f is represented by an extended system of powers satisfying horizontal equality.

Let N/λ be the partition of N such that for each pair $i,j \in N$, i and j are in the same set $G \in N/\lambda$ if and only if $|\lambda^{-1}(i)| = |\lambda^{-1}(j)|$. For each $G \in N/\lambda$, let $K_G \equiv \{k \in M : \lambda(k) \in G\}$ be the set of issues linked to a person in G under λ. Then $M/\lambda \equiv \{K_G : G \in N/\lambda\}$ is a partition of M. For each $K \in M/\lambda$, pick $k \in K$ and let $i \equiv \lambda(k)$. Let $\mathcal{I}_+^K \equiv \{(|C_1|, |C_2|) \colon (C_1, C_2) \in \mathfrak{C}_k \text{ and } i \in C_1\}$, $\mathcal{I}_0^K \equiv \{(|C_1|, |C_2|) \colon (C_1, C_2) \in \mathfrak{C}_k \text{ and } i \notin C_1 \cup C_2\}$, and $\mathcal{I}_-^K \equiv \{(|C_2|, |C_1|) \colon (C_1, C_2) \notin \mathfrak{C}_k \text{ and } i \in C_2\}$. For each $l \in K \in M/\lambda$, let ${}_eW_2(l) \equiv (\mathcal{I}_+^K, \mathcal{I}_0^K, \mathcal{I}_-^K)$. Let ${}_eW_1(\cdot) \equiv \lambda$ and ${}_eW(\cdot) \equiv ({}_eW_1(\cdot), {}_eW_2(\cdot))$. Then by construction, ${}_eW(\cdot)$ satisfies horizontal equality. We next show that for each $P \in \mathcal{P}_{\text{Tri}}$, each $K \in M/\lambda$, and each $l \in K$, if $\lambda(l) = j \in N$,

$$
\begin{aligned}
\text{when } P_{jl} &= 1, \; f_l(P) = 1 \Leftrightarrow (||P_+^l||, ||P_-^l||) \in \mathcal{I}_+^K \; ; &(11)\\
\text{when } P_{jl} &= 0, \; f_l(P) = 1 \Leftrightarrow (||P_+^l||, ||P_-^l||) \in \mathcal{I}_0^K \; ; &(12)\\
\text{when } P_{jl} &= -1, \; f_l(P) = -1 \Leftrightarrow (||P_-^l||, ||P_+^l||) \in \mathcal{I}_-^K \; . &(13)
\end{aligned}
$$

When $j = i$, Step 1 says that the decision on the k^{th} issue relies on person i's opinion, the number of agreeing persons, and the number of disagreeing persons. Therefore, since for each $l \in \lambda^{-1}(i)$, $\mathfrak{C}_l = \mathfrak{C}_k$, then (11)-(13) hold when $j = i$. When $j \in G \backslash \{i\}$, Step 1 says that for each $l \in \lambda^{-1}(j)$, the decision on the l^{th} issue is made symmetrically to the decision on the k^{th} issue. Therefore, (11)-(13) hold also for j and l. □

Proof of Theorem 1*.* Theorem 1 follows directly from Propositions 3, 4 and 7, and Lemmas 4 and 5. □

A.4 Proofs of Theorems 5 and 6

Proof of Theorem 5*.* Let $\lambda \colon M \to N$ be the exogenous linkage. Let f be a rule over $\mathcal{P}_{\text{Tri}}$ (or $\mathcal{R}_{\text{Tri}}$, recall that we will treat each opinion matrix as a profile of trichotomous preference relations) satisfying the three axioms (the proof for $\mathcal{P}_{\text{Di}}$ or $\mathcal{R}_{\text{Di}}$ is essentially the same). Without loss of generality, we assume $N \subseteq M$ (since the number of objects linked to a person is constant across persons, we may label at least one object by the label of the person linked to it) and for each $i \in \{1, \ldots, n\}$, $\lambda(i) = i$. By Proposition 7 and the assumption on

$$P \equiv \begin{pmatrix} 1 & 0 & -1 & -1 & 1 & 1 \\ 1 & 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ -1 & 1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}; \; {}^{\delta}_{\pi}P = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 & 1 & 1 \end{pmatrix}$$

Figure 1: Construction of P in the proof of Theorem 5. An example with $|N| = |M| = 6$, $t_1 = 3$, $t_2 = 2$, $i = 1$, and $j = 2$. Let $\pi\colon N \to N$ be the transposition of 1 and 2 and $\delta\colon M \to M$ the same transposition.

λ, there exist three index sets $\mathcal{I}_+$, $\mathcal{I}_0$, and $\mathcal{I}_-$ such that for each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$, if $i \equiv \lambda(k)$,

$$\begin{array}{l} \text{(i) when } P_{ik} = 1,\ f_k(R) = 1 \Leftrightarrow \left(||P^k_+||, ||P^k_-||\right) \in \mathcal{I}_+; \\ \text{(ii) when } P_{ik} = 0,\ f_k(R) = 1 \Leftrightarrow \left(||P^k_+||, ||P^k_-||\right) \in \mathcal{I}_0; \\ \text{(iii) when } P_{ik} = -1,\ f_k(R) = -1 \Leftrightarrow \left(||P^k_-||, ||P^k_+||\right) \in \mathcal{I}_-. \end{array} \tag{14}$$

Claim 1. For each $s \in \{+, 0, -\}$,

$$\begin{array}{l} \{(t_1, t_2) \in \mathcal{I}^* : t_1 > t_2\} \subseteq \mathcal{I}_s; \\ \{(t_1, t_2) \in \mathcal{I}^* : t_1 < t_2\} \cap \mathcal{I}_s = \emptyset. \end{array} \tag{15}$$

Proof. Let $(t_1, t_2) \in \mathcal{I}^*$ be such that $t_1 > t_2$. Suppose by contradiction $(t_1, t_2) \notin \mathcal{I}_+$. Let $[0] \equiv n$. For each $l \in \{1, \ldots, n\}$, let $[l] \equiv l$, $[n + l] \equiv l$, and $[-l] \equiv [n - l]$. Let P be the opinion matrix such that for each $i \in \{1, \ldots, n\}$,

$$\begin{aligned} l \in \{0, 1, \ldots, t_1 - 1\} &\;\Rightarrow\; P_{[i+l]i} = 1\,; \\ l = t_1, \ldots, t_1 + t_2 - 1 &\;\Rightarrow\; P_{[i+l]i} = -1\,; \\ l = t_1 + t_2, \ldots, n &\;\Rightarrow\; P_{[i+l]i} = 0\,; \end{aligned}$$

and for each $k \in M\backslash\{1, \ldots, n\}$ and each $i \in N$, $P_{ik} = -1$. See Figure 1 for an illustration of P. Then for each $i \in \{1, \ldots, n\}$, there are t_1 persons, $\{[i], [i + 1], \ldots, [i + t_1 - 1]\}$, who have the positive opinion on the i^{th} issue, t_2 persons, $\{[i + t_1], \ldots, [i + t_1 + t_2 - 1]\}$, who have the negative opinion, and $n - t_1 - t_2$ remaining persons with the null opinion. Hence for each $i \in \{1, \ldots, n\}$, $||P^i_+|| = t_1$ and $||P^i_-|| = t_2$. Let $i, j \in \{1, \ldots, n\}$. Let $\pi\colon N \to N$ and $\delta\colon M \to M$ be two permutations on N and on M transposing i and j. Then the i^{th} and the j^{th} columns in ${}^{\delta}_{\pi}P$ are obtained by making an one-to-one and onto switch between the i^{th} and the j^{th} columns in P, not necessarily preserving the row positions of entries.[17] Thus, $||{}^{\delta}_{\pi}P^i_+|| = ||P^j_+||$, $||{}^{\delta}_{\pi}P^i_-|| = ||P^j_+||$, $||{}^{\delta}_{\pi}P^j_+|| = ||P^i_+||$, and $||{}^{\delta}_{\pi}P^j_-|| = ||P^i_+||$. By *symmetry*, $f_i\left({}^{\delta}_{\pi}P\right) = f_j(P)$ and $f_j\left({}^{\delta}_{\pi}P\right) = f_i(P)$. Since $||P^i_+|| = ||P^j_+||$ and $||P^i_-|| = ||P^j_-||$, then $||P^i_+|| = ||{}^{\delta}_{\pi}P^j_+||$, $||P^i_-|| = ||{}^{\delta}_{\pi}P^i_-||$, $||P^j_+|| = ||{}^{\delta}_{\pi}P^j_+||$, and $||P^j_-|| = ||{}^{\delta}_{\pi}P^j_-||$. So $f_i(P) = f_i\left({}^{\delta}_{\pi}P\right)$

[17]Note that P_{ii} and P_{ji} in the i^{th} column are switched into P_{jj} and P_{ij} in the j^{th} column respectively. Other entries in the i^{th} column are switched into the entries in the j^{th} column in the same rows.

and $f_j(P) = f_j\left({}^{\delta}_{\pi}P\right)$. Hence $f_i(P) = f_j(P)$. Since $(t_1, t_2) \notin \mathcal{I}$, $f_N(P) = (-1, \ldots, -1)$. On the other hand, by *Pareto efficiency*, $f_{M \setminus N} = (-1, \ldots, -1)$. For each $i \in N$, let $U_i(\cdot)$ be the representation of the trichotomous preference relation P_i. Then for each $i \in N$, $U_i(f(P)) = 0$. Let x be such that $x_N \equiv (1, \ldots, 1)$ and $x_{M \setminus N} \equiv (-1, \ldots, -1)$. Then for each $i \in N$, $U_i(x) = t_1 - t_2 > 0$, contradicting *Pareto efficiency*.

Let $(t_1, t_2) \in \mathcal{I}^*$ be such that $t_1 < t_2$. Suppose by contradiction $(t_1, t_2) \in \mathcal{I}_+$. Then using the same argument as above, we show $f_N(P) = (1, \ldots, 1)$ and $f_{M \setminus N}(P) = (-1, \ldots, -1)$. Let $x \equiv (-1, \ldots, -1)$. Then for each $i \in N$, $U_i(f(P)) = t_1 - t_2 < 0 = U_i(x)$, contradicting *Pareto efficiency*.

Similar arguments can be used to prove the same properties for $\mathcal{I}_0$ and $\mathcal{I}_-$. ♦

Note that the properties stated in (15) imply comprehensiveness of the three index sets. Finally, for each $s \in \{+, 0, -\}$, let $q_s(\nu) \equiv \min\{t_1 : (t_1, \nu - t_1) \in \mathcal{I}_s\}$ for each ν. Then (15) implies (7) and (8). Because of comprehensiveness of the three index sets, (14) implies (2). □

Proof of Theorem 6. Let f be a rule over $\mathcal{P}_{\text{Tri}}$ satisfying the four axioms (the proof for $\mathcal{P}_{\text{Di}}$ or $\mathcal{R}_{\text{Di}}$ is essentially the same). By Proposition 7, f is represented by an extended system of powers ${}_eW(\cdot)$. Then by *neutrality*, for each pair $l, k \in M$, ${}_eW_2(l) = {}_eW_2(k)$. Thus there exist three index sets $\mathcal{I}_+$, $\mathcal{I}_0$, and $\mathcal{I}_-$ such that for each $P \in \mathcal{P}_{\text{Tri}}$ and each $k \in M$, if $i \equiv \lambda(k)$,

$$\begin{array}{l} \text{(i) when } P_{ik} = 1,\ f_k(P) = 1 \Leftrightarrow \left(||P_+^k||, ||P_-^k||\right) \in \mathcal{I}_+; \\ \text{(ii) when } P_{ik} = 0,\ f_k(P) = 1 \Leftrightarrow \left(||P_+^k||, ||P_-^k||\right) \in \mathcal{I}_0; \\ \text{(iii) when } P_{ik} = -1,\ f_k(P) = -1 \Leftrightarrow \left(||P_-^k||, ||P_+^k||\right) \in \mathcal{I}_-. \end{array} \tag{16}$$

Using essentially the same argument as in the proof of Theorem 5, we can show that f is represented by a quasi-plurality system of powers. Because of *neutrality*, the system is either non-exclusive or monocentric. □

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