Robustness of Inferences to Singularity Bifurcations

Yijun He
School of Economic Sciences
Washington State University
Pullman, WA 99164-4741
yijun@wsu.edu

William A. Barnett
Department of Economics
University of Kansas
Lawrence, Kansas 66045-7585
barnett@ku.edu

Abstract

In this paper, we consider new bifurcation phenomena in a class of stochastic dynamic macroeconometric models as represented by the stochastic model developed by Leeper and Sims (1994). This model is a well known Euler equations macroeconometric model designed to be suitable for monetary policy analysis although the complexity of the model makes any attempt of analytical analysis a difficult task. Our analysis reveals that a singularity boundary exist within a small neighborhood of the estimated parameter values. That singularity boundary is located and displayed. As parameter values approach a singularity boundary, one eigenvalue of the linearized part of the model rapidly moves to infinity while others remain bounded, implying nearly instantaneous response of some variables to changes of other variables. On the singularity boundary, the number of differential equations will decrease while the number of algebraic constraints will increase. Such change in the order of dynamics is a new phenomenon in macroeconometric models. We determine the singularity-induced bifurcation and its effect on model behavior.

The primary concern is the loss of robustness in dynamic inferences, when a singularity bifurcation boundary occurs within a confidence region of the parameter inferences. The nature of the model's dynamics then can be very different in different subsets of that confidence region. Dynamic simulations produced at the parameters' point estimates can display dynamics that are very different from the dynamics that would be produced at nearby points in the parameter space. We believe that the appearance of singularity bifurcation boundaries in this model is associated with the fact that the model is based upon Euler equations, and we anticipate that the same problem may arise in other macroeconometric models.

1 Introduction

There is great interest in the rigorous analysis of macroeconomics through the study of their mathematical models. For this purpose, various macroeconometric models have been established in the literature. Among those models that have direct relevance to this research include high dimension continuous time macroeconometric models in Bergstrom, Nowman and Wymer (1992), Bergstrom, Norman, and Wandasiewicz (1994), Bergstrom and Wymer (1976), Grandmont (1998), Leeper and Sims (1994), Powell and Murphy (1997) and Kim (2000). Surveys of macroeconomical models are available in Bergstrom (1996) and in several textbooks such as Gandolfo (1992) and Medio (1992). The general theory of economic dynamics is provided for example in Boldrin and Woodford (1990), and Gandolfo (1992). With mathematical models available, it is natural to investigate their dynamical properties through which insights on macroeconomics could be obtained. Topics that have received great attention include stability/instability study and bifurcation analysis on which an extensive list of papers is available in the literature. Particularly, various bifurcation phenomena are reported in Bala (1997), Benhabib (1979), Medio (1992), Gandolfo (1992), Nishimura and Takahashi (1992). Focused studies of stability are conducted in Grandmont (1998), Scarf (1960), and Nieuwenhuis and Schoonbeek (1997). Barnett and Chen (1988) discovers chaotic behaviors in economics. Bergstrom, Nowman, and Wandasiewicz (1994) investigates stabilization of macroeconomic models using policy control. Wymer (1997) describes several mathematical frameworks for the study of structural properties of macroeconometrc models.

This paper is concerned with the analysis of bifurcation phenomena in stochastic dynamic macroeconometric models. Bifurcation analysis is useful in understanding dynamic properties of macroeconometric models. Barnett and He (1999) recently investigated bifurcation in high dimension macroeconometric models, particularly the UK model established in Bergstrom, Norman, and Wymer (1992). This practical model is characterized by large state space, which makes pure analytical study impossible. The main issue of concern is the change of structural properties such as stability when parameter values vary. A computational approach was adopted in the analysis of Bergstrom, Norman, and Wymer model in Barnett and He (1999). It was discovered that macroeconometric models could exhibit fascinating bifurcation phenomena including various types of bifurcation such as transcritical bifurcation, Hopf bifurcation, and codimension two bifurcation.

Recently, Leeper and Sims (1994) proposed a new macroeconometric model. The new model, while addressing issues such as Lucas critique (Lucas, 1996), is intended for use in policy analysis. Similar models are developed in Kim (2000). Such models are motivated by the need of analyzing behaviors of the macroeconomy. With such a goal in mind, the dimension of state space in the Leeper and Sims model is substantially lower than that in Bergstrom, Norman, and Wymer UK model. However, the reduction of state space is not much helpful to analytical study for the following reasons. First, the dimension (with seven state variables after reduction in our analysis) is still high for generally available analytical approaches. Second, the dynamics of the Leeper and Sims model is in fact considerably complex as we shall show later in the paper, because it includes not only the usual differential equations but also algebraic constraints. Such a combination of differential equations and algebraic constraints raise new issues as we explore further. It should be pointed out that the Leeper and Sims model, though we have not been aware of its use in policy analysis, indeed have several attractive features such as the integration of several factors of interest such as monetary stock, price level, wage level, and interest rate. It also simultaneously considers behaviors of consumers, firms, and the government. Furthermore, the model treats monetary and fiscal policies explicitly.

In this paper, we are interested in how parameter changes could affect dynamic behaviors of the model and subsequently the macroeconomy it represents. We discover that the order of the dynamics of the Leeper and Sims model could change within a small neighborhood of the estimated parameter values. More specifically, one eigenvalue of the linearized part of the model could go from finite quickly to infinite and again quickly comes back to finite. Such phenomenon characterizes the instability of the structure inherent in the model. A large stable eigenvalue represents the case in which some variables could respond rapidly to changes of other variables while a large unstable eigenvalue corresponds to rapid diversion of the variable from other variables. The infinity case of the eigenvalue implies the existence of pure algebraic relations among the variables of interests. In this sense, the change in the order of dynamic part of the system shows a fundamental property of Leeper and Sims model. It's not clear, however, whether such a change indicates an inherent instability of the model or simply the nature of macroeconomy that the model correctly and dutifully captures. To the best of our knowledge, this is a new bifurcation phenomenon in macroeconometric models.

The paper is organized as the following. Section 2 introduces Leeper and Sims model. Section 3 analyzes the model based on linearization around the equilibrium. Structural properties of the model are examined in detail. Section 4 uses numerical examples to illustrate the results. Implications of the singularity-induced bifurcation are also discussed.

2 A Stochastic Growth Model

In an effort to provide a macroeconomic model that one can rely on for policy analysis, Leeper and Sims (1994) developed a stochastic macroeconomic model. The model, among other attractive features, captures the dynamic behavior of consumers, firms, and the government. Several similar models were also developed in Kim (2000) and in Binder and Pesaran (1999). One special feature of those models is that they consist of dynamic subsystems described by ordinary differential equations (ODE) and algebraic constraints. Such systems are differential/algebraic systems. differential/algebraic systems (Dai 1989, Aplevich 1991?).

Leeper and Sims (1994) derive their macroeconometric model by considering consumers, firms, and the government. Both consumers and firms operate so as to maximize their respective utility functions. The government provides monetary and tax policies "to satisfy intertemporal governement budget identity and the pursuit of countercyclical policy objectives." The detailed derivation of the models is available in Leeper and Sims (1994) and will not be repeated in this paper as we are interested in the structural properties of the model. In the next we simply introduce the mathematical equations of the model and investigate its structural properties.

The Leeper and Sims model consists of the following 12 state variables.

L:	work
C^*:	consumption net of transactions cost
M:	non-interest-bearing money
D:	interesting-bearing government debt
K:	capital
Y:	factor income
C:	gross consumption
Z:	investment
X:	consumption goods price
Q:	investment goods price
V:	velocity of transactions costs
P:	general price level

The state variables satisfy the following differential equations.

$$\frac{1}{P}(\dot{M}+\dot{D}) = Y - XC - QL + \frac{iD}{P} + \tau \tag{1}$$

$$\dot{K} = Z - \delta K \tag{2}$$

$$(1-\pi(1-\gamma))\frac{\dot{C}^*}{C^*} + (1-\gamma)(1-\pi)\frac{\dot{L}}{1-L} + \frac{\dot{X}}{X} + \frac{\dot{P}}{P}$$

$$= i - \beta + \frac{\dot{\pi}}{\pi} + \dot{\pi}(1-\gamma)\log(\frac{C^*}{1-L}) \tag{3}$$

$$\frac{\dot{P}}{P} + \frac{\dot{Q}}{Q} = i + \delta - (1-2\phi V)\frac{r}{Q} \tag{4}$$

where (1) represents the consumers' budget constraint, (2) is the law of motion for capital, and (3) is first-order condition from optimizing consumers' objective function. In addition to the three dynamic equations, the state variables also satisfy the following algebraic constraints.

$$X = \left(\frac{Y}{C+g}\right)^{1-\mu} \tag{5}$$

$$Q = \theta\left(\frac{Y}{Z+nK}\right)^{1-\mu} \tag{6}$$

$$r = a^{\sigma}\alpha\left(\frac{Y}{K}\right)^{1-\delta} \tag{7}$$

$$w = a^{\sigma}\left(\frac{Y}{L}\right)^{1-\delta} \tag{8}$$

$$XC^* + \phi VY = XC \tag{9}$$

$$Y = rK + wL + s \tag{10}$$

$$V = \frac{PY}{M} \tag{11}$$

$$X(C+g) + Q(Z+nK) = Y \tag{12}$$

$$(1-2\phi V)\frac{w}{X} = \frac{1-\pi}{\pi}\frac{C^*}{1-L} \tag{13}$$

$$i = \phi V^2 \tag{14}$$

The relations (4)-(8) are obtained from the first-order condition by maximizing firms' objective function. Equation (9) defines consumption net of transactions costs, with total output serving as a measure of the level of transactions at a given point in time. Equation (10) defines income. Equation (11) is the income velocity of money. Equation (12) is the social resource constraints. Equations (13)-(14) are also obtained from the first-order condition for maximizing consumers' objective function.

Control variables are the following government policies.

- i: nominal rate of return of government bonds
- τ: the level of lump-sum taxes

Leeper and Sims (1994) introduces the following monetary policy and tax policy into the model. The monetary policy is

$$\frac{\dot{i}}{i} = a_p\log(\frac{P}{\bar{P}}) + a_{int}\frac{\dot{P}}{P} + a_i\log(\frac{i}{\bar{\beta}}) + a_L\log(\frac{L}{\bar{L}}) + \epsilon_i \tag{15}$$

and the tax policy is

$$\frac{d}{dt}\frac{\tau}{C} = b_\tau\left(\frac{\tau}{C} - \frac{\bar{\tau}}{\bar{C}}\right) + b_L\log(\frac{L}{\bar{L}}) + b_{inf}\frac{\dot{P}}{P}$$

$$+b_x\left(\frac{D}{PY} - \frac{\bar{D}}{\bar{P}\bar{Y}}\right) + \epsilon_\tau \tag{16}$$

The overscored variables, according to our convention, denote steady state values. $\bar{D}/\bar{Y}$ is the steady state debt-to-GNP level. The free parameters are the steady state price level $\bar{P}$, $\bar{D}/\bar{Y}$, the a's, and the b's. The noises are ϵ_i and ϵ_τ.

In this model, it is conventional to use τ/C, rather than τ, as a control. Therefore, the control variables are i, $\tau_c = \tau/C$.

In Leeper and Sims model, the following parameters and exogenous variables n, g, π, δ, θ, α, A, and ϕ, are logrithmic first-order AR in continuous time. The variable β is a logrithmic first-order AR in unlogged form. We analyze the structural properties of (1)-(14) without any external disturbance, or equivalently, the exogenous variable are set at their nominal values. At the nominal value, we have $\dot{\pi} = 0$. Further analysis is part of an on-going research.

The original form (1)-(14) has 12 states variables and 14 equations. τ_C equations to For analytical investigation, we would like to have as few state variables as possible. For this purpose, we next reduce the dimension of the problem by temporarily eliminating some state variables. We consider the following state variables

$$x = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \end{bmatrix} \tag{17}$$

The rest of state variables can be written as unique functions of x.

By eliminating M, C^*, V, Q, X, direct verification from (1)-(14) shows that x satisfies the following equations.

$$\frac{1}{P}\dot{D} + \frac{Y\sqrt{\phi/i}}{P}\dot{P} + (\sqrt{\phi/i})\dot{Y}$$

$$= Y + \frac{iD}{P} - \left(\frac{Y}{C+g}\right)^{1-\mu} C - \theta\left(\frac{Y}{Z+nK}\right)^{1-\mu} L$$

$$-\tau_c C + \frac{Y\sqrt{i/\phi}}{2V^2\phi}\dot{i} \tag{18}$$

$$(1-\pi(1-\gamma))(\frac{1-\phi V Y^{\mu}(1-\mu)(C+g)^{-\mu}}{C-\phi V Y^{\mu}(C+g)^{1-\mu}}$$

$$-\frac{1-\mu}{C+g})\dot{C} - (\frac{(1-\pi(1-\gamma))\phi V \mu Y^{\mu-1}(C+g)^{1-\mu}}{C-\phi V Y^{\mu}(C+g)^{1-\mu}}$$

$$+\frac{1-\mu}{Y})\dot{Y} + \frac{\dot{P}}{P} + \frac{(1-\gamma)(1-\pi)}{1-L}\dot{L}$$

$$= i - \beta + \frac{Y^{\mu}(C+g)^{1-\mu}}{C-\phi V Y^{\mu}(C+g)^{1-\mu}}\frac{1}{2\sqrt{i\phi}}\dot{i} \tag{19}$$

$$\frac{\dot{P}}{P} + (1-\mu)(\frac{\dot{Y}}{Y} - \frac{\dot{Z}+n\dot{K}}{Z+nK})$$

$$= -(1-2\phi V)\frac{a^{\sigma}\alpha}{\theta}Y^{\mu-\sigma}(Z+nK)^{1-\mu}K^{\sigma-1} + i + \delta \tag{20}$$

$$\dot{K} = Z - \delta K \tag{21}$$

$$0 = (C+g)^{\mu} + \theta(Z+nK)^{\mu} - Y^{\mu} \tag{22}$$

$$0 = \alpha K^{\sigma} + L^{\sigma} - a^{-\sigma}Y^{-\sigma} \tag{23}$$

$$0 = (1-2\phi V)\frac{a^{\sigma}Y^{\mu-\sigma}(C+g)^{1-\mu}}{L^{1-\sigma}}$$

$$+\frac{1-\pi}{\pi}\frac{\phi V}{1-L}Y^{\mu}(C+g)^{1-\mu} - \frac{1-\pi}{\pi}\frac{C}{1-L} \tag{24}$$

For the ease of notation, we denote equations (18)-(24) as

$$h(x,u)\dot{x} = f(x,u) \tag{25}$$

$$0 = g(x,u) \tag{26}$$

in which the dimensions of $h(x,u), f(x,u)$ are 4×7. The dimension of $g(x,u)$ is 3×7. Equation (25) describes the nonlinear dynamical behavior of the model and (26) represents the algebraic (and nonlinear) constraints. Many systems can be described in the form of (25) and (26) which have been known as nonlinear descriptor systems. The model developed in Kim (2000) is also in this form. Thus we will investigate this class of systems. We shall use n, n_1, n_2, $n_1 + n_2 = n$, and m to denote respectively the dimension of x, the number of difference equations in (25), the number of algebraic constraints in (26), and the dimension of control variable. For Leeper and Sims model, $n = 7$, $n_1 = 4$, $n_2 = 3$, and $m = 3$.

The steady state of (25)-(26) can be solved from the following equations.

$$0 = f(x,u) \tag{27}$$

$$0 = g(x,u) \tag{28}$$

Let us denote the steady states of x and u by $\bar{x}$ and $\bar{u}$, respectively. The $\bar{u}$ is the solution of (15) and (16) in steady state when external noises are zero, i.e.,

$$\begin{array}{l} \bar{i} = \beta \\ \bar{\dot{i}} = 0 \\ \bar{\tau}_C = \frac{\bar{\tau}}{\bar{C}} \end{array} \tag{29}$$

The values $\bar{x}$ and $\bar{u}$ are solutions to (27)-(28), and (29). This steady state is an equilibrium of (25)-(26) when the control variables are set at their steady state.

The vector of parameters of concern is

$$p = [g\ n\ \pi\ \beta\ \theta\ \alpha\ a\ \phi\ \delta\ \mu\ \gamma\ \sigma]^T,$$

where the superscript T denotes vector (or matrix) transpose. More parameters will be introduced later when needs arise. The nominal values of the parameters are estimated in Leeper and Sims (1994) from quarterly data from 1959 to 1992. Details of the estimation are available in that paper.

The constraints of the parameter values are:

$$\begin{array}{l} 0<\pi<1 \\ \gamma>0 \\ 0\leq\sigma\leq 1 \\ \mu\geq 1 \\ \delta\geq 0 \\ 0\leq\beta\leq 1 \\ g\geq 0 \\ n\geq 0 \\ \delta>0 \end{array} \tag{30}$$

In addition to the above constraints, the steady states of state variables need to exist and be practical. For example, C should be non-negative. Such constraints are included in the existence of equilibrium of the Leeper and Sims model.

3 Singularity in Leeper and Sims model

The structural properties of the Leeper and Sims model in a small neighborhood of the equilibrium $(\bar{x},\bar{u})$ can be studied using local linearization around this equilibrium. The linearized system of (25) and (26) is

$$E_1\dot{x}=A_1x+B_1u \tag{31}$$

$$0=A_2x+B_2u \tag{32}$$

where

$$E_1=h(\bar{x},\bar{u})\in R^{n_1\times n},$$

$$A_1=\frac{\partial f(x,u)}{\partial x}|_{x=\bar{x},u=\bar{u}}$$

$$\in R^{n_1\times n}, A_2=\frac{\partial g(x,u)}{\partial x}|_{x=\bar{x},u=\bar{u}}\in R^{n_2\times n}$$

$$B_1=\frac{\partial f(x,u)}{\partial u}|_{x=\bar{x},u=\bar{u}}$$

$$\in R^{n_1\times n}, B_2=\frac{\partial g(x,u)}{\partial u}|_{x=\bar{x},u=\bar{u}}\in R^{n_2\times n}$$

The linearized system (31)-(32) is solvable if it is regular (Gantmach 1974), i.e.,

$$\det(\left[\begin{array}{c} sE_1-A_1 \\ -A2 \end{array}\right])\not\equiv 0.$$

If the regularity condition is violated, the linearized system either has multiple solutions or no solution. We randomly chose parameter value within feasible region and observed that the Leeper and Sims model, as expected, is also regular. Because this is a practical model, it is reasonable to assume that it is always regular within the parameter feasibility region.

To study the structural properties of the Leeper and Sims model, we further transform the linearized system (31)-(32) into a form that is easy to work with.

Definition 3.1 Two systems

$$E\dot{x}=Ax+Bu \tag{33}$$

and

$$\tilde{E}\dot{y}=\tilde{A}y+\tilde{B}u \tag{34}$$

are said to be restricted system equivalent (r.s.e.) if there exit two nonsingular matrices T_1 and T_2 such that

$$T_1ET_2=\tilde{E},\ T_1AT_2=\tilde{A},\ T_1B=\tilde{B},\ y=T_2x.$$

The form (34) can be obtained by using coordinate transform $y=T_2x$ and by multiplying from left both sides of (33) by T_1. The relationship of r.s.e. allows one to transform a system into appropriate forms while preserving important properties of the system.

We next transform (31)-(32) into suitable r.s.e. forms. First, denote

$$r_E=rank(E_1).$$

Then there exist nonsingular matrices T_1 and T_2 such that

$$T_1E_1T_2=\left[\begin{array}{cc} I_{r_E} & 0 \\ 0 & 0 \end{array}\right].$$

Consider the following coordinate transform

$$\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]=T_2^{-1}x, x_1\in R^{r_E},\ x_2\in R^{n_1-r_E}$$

Then

$$x=T_2\left[\begin{array}{c} x_1 \\ x_2 \end{array}\right]$$

Substituting the form of x into (31)-(32) and also multiplying both sides of (31) by T_1, the we know that (31)-(32) is r.s.e. to

$$\begin{array}{l} \dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}u \\ 0 = A'_{21}x_1 + A'_{22}x_2 + B'_{12}u \\ 0 = A_{31}x_1 + A_{32}x_2 + B_2u \end{array} \qquad (35)$$

where

$$\begin{bmatrix} A_{11} & A_{12} \\ A'_{21} & A'_{22} \end{bmatrix} = T_1 A_1 T_2,$$

$$\begin{bmatrix} B_{11} \\ B'_{12} \end{bmatrix} = T_1 B_1, \begin{bmatrix} A_{31} & A_{32} \end{bmatrix} = A_2 T_2.$$

Combining the second and the third equation in (35), we have

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_{11}u \qquad (36)$$

$$0 = A_{21}x_1 + A_{22}x_2 + B_{12}u \qquad (37)$$

where

$$A_{21} = \begin{bmatrix} A'_{21} \\ A_{31} \end{bmatrix}, A_{22} = \begin{bmatrix} A'_{12} \\ A_{31} \end{bmatrix}, \quad B_{12} = \begin{bmatrix} B'_{12} \\ B_2 \end{bmatrix}.$$

If A_{22} is nonsingular (or invertible), it is possible to solve for x_2 from the algebraic constraint equation (37). In fact, in this case, we have

$$x_2 = -A_{22}^{-1}(A_{21}x_1 + B_{12}u).$$

Substituting the form of x_2 into (36), we obtain

$$\dot{x}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})x_1$$

$$+(B_{11} - A_{12}A_{22}^{-1}B_{12})u.$$

In other words, x_1 could be described by the usual ODE and an algebraic relationship between x_1 and x_2.

However, the previously described transformation could not be finished if A_{22} is singular (or not invertible). In fact, as we shall explain soon, that the dynamics of (25)-(26) could be completely different from those of ordinary linear differential equations if A_{22} becomes singular.

To see what could happen when A_{22} is singular, let revisit the linearized system (36)-(37) which could be re-written as

$$\begin{bmatrix} I_{r_E} & 0 \\ 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} u. \qquad (38)$$

If the Leeper and Sims model is regular, so is the matrix pair

$$(\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix})$$

which is called a matrix pencil. For a regular matrix pencil, there exit nonsingular matrices $\tilde{T}_1, \tilde{T}_2$ such that (Gantmacher 1974)

$$\tilde{T}_1 \begin{bmatrix} I_{r_E} & 0 \\ 0 & 0 \end{bmatrix} \tilde{T}_2 = \begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & N \end{bmatrix},$$

$$\tilde{T}_1 \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tilde{T}_2 = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & I_{\tilde{n}_2} \end{bmatrix}$$

where $\tilde{n}_1 + \tilde{n}_2 = n$, N is a nilpotent matrix, i.e., there exists a positive integer $d \geq 1$ such that

$$N^d = 0$$

The smallest such integer d is called the nilpotent index of N. The following is an example of nilpotent matrix

$$N = \begin{bmatrix} 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ & .. & & ... & & \\ 0 & 0 & 0 & ... & 0 & 1 \\ 0 & 0 & 0 & ... & 0 & 0 \end{bmatrix}. \qquad (39)$$

A matrix is nilpotent if and only if it is similar to the following block diagonal matrix

$$diag(N_1, N_2, ..., N_p)$$

in which each N_i has the form of (39).

Consider the coordinate transform

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \tilde{T}_2 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

$$\text{or equivalently} \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \tilde{T}_2^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Multiplying both sides of (38) by $\tilde{T}_1$, we have another r.s.e. form of (31)-(32)

$$\dot{y}_1 = \tilde{A}_1 y_1 + \tilde{B}_1 u \tag{40}$$

$$N\dot{y}_2 = y_2 + \tilde{B}_2 u \tag{41}$$

where

$$\begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix} = \tilde{T}_1^{-1} \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}$$

The solution to (40) and (41) are respectively

$$y_1 = e^{\tilde{A}_1(t-t_0)} y_1(0) + \int_{t_0}^{t} e_1^{\tilde{A}}(t-\xi)\tilde{B}_1 u(\xi) d\xi$$

$$y_2 = -\sum_{k=1}^{d-1} \delta^{(k-1)}(t) N^k y_2(0) - \sum_{k=0}^{d-1} N^k \tilde{B}_{12} u^{(k)}(t)$$

where $t_0 \geq 0$ is the initial time and $u^{(k)}$ denotes that k-th order derivative of u.

Unless $N = 0$ or the initial state $y_2(0) = 0$, there exist impulsive terms and derivative terms of u in the solution y_2. Such a solution structure is totally different from those of ODE such as y_1. The derivative terms could produce shock effects to the state response y_2 if u is not smooth. For example, if u is a step function

$$u(t) = \begin{cases} 1 & \text{if } t \geq t_0 \\ 0 & \text{if } t < t_0 \end{cases}$$

the first-order derivative of such as step function is the well known Dirac function

$$\delta(t - t_0) = \frac{d}{dt} u(t).$$

On the other hand, if $N = 0$, we have

$$y_2 = -\tilde{B}_2 u$$

which is again an algebraic relationship between y_2 and u. This is not surprising as explained in the following theorem.

Theorem 3.1 If both (40)-(41) and (36)-(37) are r.s.e. forms of the same linearized system (31)-(32), then

$$N = 0$$

if and only if A_{22} is nonsingular, i.e.,

$$det(A_{22}) \neq 0.$$

Proof. If $N = 0$, then (40)-(41) and (36)-(37) have the same form with $A_{22} = I_{\tilde{n}_2}$ which is nonsingular.

On the other hand, if A_{22} is nonsingular, choose

$$\tilde{T}_1 = \begin{bmatrix} I_{\tilde{n}_1} & -A_{12}A_{22}^{-1} \\ 0 & I_{\tilde{n}_2} \end{bmatrix},$$

$$\tilde{T}_2 = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & A_{12}A_{22}^{-1} \\ 0 & I_{\tilde{n}_2} \end{bmatrix}, \tag{42}$$

Direct verification confirms that

$$\tilde{T}_1 \begin{bmatrix} I_{r_E} & 0 \\ 0 & 0 \end{bmatrix} \tilde{T}_2 = \begin{bmatrix} I_{\tilde{n}_1} & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{T}_1 \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tilde{T}_2 = \begin{bmatrix} \tilde{A}_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}$$

with

$$\tilde{A}_1 = A_{11} - A_{12}A_{22}^{-1}A_{21}.$$

Therefore, we have $N = 0$. This completes the proof. □

Since the linearized model (31)-(32) singular A_{22} results in completely different behaviors, we say a singularity occurs when A_{22} becomes singular. The condition for singularity is

$$det(A_{22}) = 0 \tag{43}$$

The preceding condition has another form in terms of original coefficient matrices. In fact, we can prove the following theorem.

Theorem 3.2 Assume that E_1 has full row rank, i.e.,

$$rank(E_1) = n_1 \tag{44}$$

Then A_{22} is nonsingular if and only if

$$\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}$$

is nonsingular, i.e.,

$$rank(\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}) = n \tag{45}$$

Proof. Denote

$$\hat{T}_1 = \begin{bmatrix} \tilde{T}_1 & 0 \\ 0 & \tilde{T}_1 \end{bmatrix}, \hat{T}_2 = \begin{bmatrix} \tilde{T}_2 & 0 \\ 0 & \tilde{T}_2 \end{bmatrix},$$

where $\tilde{T}_1$ and $\tilde{T}_2$ are defined in (42). Then both $\hat{T}_1$ and $\hat{T}_2$ are non-singular.

Consider the following matrix

$$\mathcal{A} = \begin{bmatrix} E_1 & A_1 \\ 0 & A_2 \\ 0 & E_1 \\ 0 & 0 \end{bmatrix}.$$

On one hand, we know that

$$\hat{T}_1 \mathcal{A} \hat{T}_2 = \begin{bmatrix} \tilde{T}_1 \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \tilde{T}_2 & \tilde{T}_1 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \tilde{T}_2 \\ 0 & \tilde{T}_1 \begin{bmatrix} E_1 \\ 0 \end{bmatrix} \tilde{T}_2 \end{bmatrix}$$

$$= \begin{bmatrix} I_{\tilde{n}_1} & 0 & A_{11} & A_{12} \\ 0 & 0 & A_{21} & A_{22} \\ 0 & 0 & I_{\tilde{n}_1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= 2\tilde{n}_1 + rank(A_{22}). \tag{46}$$

On the other hand, if E_1 has full row rank, $\tilde{n} - 1 = n_1$,

$$rank(E_1) = \tilde{n}_1 = n_1,$$

and

$$rank(\mathcal{A}) = rank(\begin{bmatrix} E_1 & A_1 \\ 0 & A_2 \\ 0 & E_1 \\ 0 & 0 \end{bmatrix})$$

$$= rank(E_1) + rank(\begin{bmatrix} A_2 \\ E_1 \end{bmatrix})$$

$$= n_1 + rank(\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}).$$

Combining the previous equation with (42), we obtain

$$rank(\begin{bmatrix} A_2 \\ E_1 \end{bmatrix}) = n_1 + rank(A_{22}). \tag{47}$$

Note that $A_{22} \in R^{\tilde{n}_2 \times \tilde{n}_2}$ and $\tilde{n}_2 = n_2$. Equation (47) says that A_{22} is nonsingular if and only

$$\begin{bmatrix} A_2 \\ E_1 \end{bmatrix}$$

is nonsingular, which is exactly what we need. □.

Therefore, another singularity condition is readily available from Theorem 3.2.

$$\det(\begin{bmatrix} E_1 \\ A_2 \end{bmatrix}) = 0 \tag{48}$$

Note that x_2 is solvable from (37) alone if A_{22} is non-singular. Therefore, singularity condition implies the case in which x_{22} is not readily solvable from the algebraic (37) alone. We need to take into account of the dynamic constraint (36).

We next introduce another property to have a closer look at the singularity condition.

Corollary 3.3 Consider the following system

$$\begin{aligned} E_1\dot{x} + E_{12}\dot{y} &= A_1 x + A_1 2y + B_1 u \\ \dot{y} &= A_y y + B_y u \\ 0 &= A_2 x + A_{22} y + B_2 u \end{aligned} \tag{49}$$

Then the singularity condition for (49) is the same as that for (31)-(32).

Proof. According to Theorem 4.2, the singularity condition for (49) is

$$\det(\begin{bmatrix} E_1 & E_{12} \\ 0 & I \\ A_2 & A_{22} \end{bmatrix}) = 0,$$

which,by eliminating the second column, is equivalent to (48), the singularity condition for (31)-(32). □

Corollary 3.1 says that adding (or deleting) state variables that can be described by ordinary differential equations does not change the singularity condition. This property is useful in reducing the dimension of the problem under consideration. For example, we could drop the state variable K in (31)-(32) without affecting the singularity condition.

It is easy to verify that, after dropping the state variable K, the singularity condition becomes

$$\det\left(\begin{bmatrix} E_1' \\ A_2' \end{bmatrix}\right) = 0 \tag{50}$$

in which

$$E_1' = \begin{bmatrix} \frac{1}{P} & \frac{Y}{PV} & 0 & 0 & 0 & \frac{1}{V} \\ 0 & \frac{1}{P} & e_{23} & \frac{(1-\gamma)(1-\pi)}{1-L} & 0 & e_{26} \\ 0 & \frac{1}{P} & 0 & 0 & -\frac{1-\mu}{Z+nK} & \frac{1-\mu}{Y} \end{bmatrix}$$

and

$$A_2' =$$

$$\begin{bmatrix} 0 & 0 & \mu(C+g)^{\mu-1} & 0 & \theta\mu(Z+nK)^{\mu-1} & \mu Y^{\mu-1} \\ 0 & 0 & a_{23} & a_{24} & 0 & a_{26} \\ 0 & 0 & 0 & \sigma L^{\sigma-1} & 0 & A^{-\sigma}\sigma Y^{\sigma-1} \end{bmatrix}$$

where

$$e_{23} = \frac{1-\pi(1-\gamma)}{C^*}[1-\phi V Y^{\mu}(\mu-1)(C+g)^{\mu-2}]$$

$$-\frac{1-\mu}{C+g}$$

$$e_{26} = \frac{1-\pi(1-\gamma)}{C^*}[-\phi V Y^{\mu}\mu(C+g)^{\mu-1}] + \frac{1-\mu}{Y},$$

$$a_{23} = (1-2\phi V)A^{\sigma}Y^{\mu-\sigma}L^{\sigma-1}(1-\mu)(C+g)^{-\mu}$$

$$-\frac{1-\pi}{\pi}\frac{1}{1-L}$$

$$a_{24} = (1-2\phi V)A^{\sigma}Y^{\mu-\sigma}(\sigma-1)L^{\sigma-2}(C+g)^{1-\mu}$$

$$-\frac{1-\pi}{\pi}\frac{C}{(1-L)^2}$$

$$a_{26} = (1-2\phi V)A^{\sigma}(\mu-\sigma)Y^{\mu-\sigma-1}L^{\sigma-1}(C+g)^{1-\mu}.$$

Direct calculation shows that (50) is equivalent to

$$\det\left(\begin{bmatrix} e_{23} & \frac{(1-\gamma)(1-\pi)}{1-L} & \frac{1-\mu}{Z+nK} & e_{26}' \\ \mu(C+g)^{\mu-1} & 0 & \theta\mu(Z+nK)^{\mu-1} & -\mu Y^{\mu-1} \\ a_{23} & a_{24} & 0 & a_{26} \\ 0 & \sigma L^{\sigma-1} & 0 & A^{-\sigma}\sigma Y^{\sigma-1} \end{bmatrix}\right) \tag{51}$$

where

$$e_{26}' = \frac{1-\pi(1-\gamma)}{C^*}[-\phi V Y^{\mu}\mu(C+g)^{\mu-1}].$$

As we shall demonstrate later, singularity does occur within feasible parameter regions.

In systems theory, bifurcation is said to occur if change of structural properties occurs when a parameter crosses a certain value. Such value is referred to as a bifurcation point. There have discovered many types of bifurcation such as saddle-node bifurcation, transcritical bifurcation, and Hopf bifurcation. Bifurcation analysis is useful to determining critical behaviors of a system such as limit cycle or stability.

Because the Leeper and Sims model has structural changes in its dynamics, the boundary determined by (51) will be referred to as singularity-induced bifurcation boundary. To the best of our knowledge, this is a new type of bifurcation in macroeconometrics models.

Leeper and Sims (1994) proposed the government policy control using the monetary policy (15) and tax policy (16). boundary. To investigate bifurcation of the closed-loop system under the control of government policies, let us expand the state variable to

$$x_c = \begin{bmatrix} D \\ P \\ C \\ L \\ K \\ Z \\ Y \\ i \\ \tau_C \end{bmatrix} \tag{52}$$

With this new state variable, the linearized system (31)-(32) now becomes

$$E_1^c \dot{x}_c = A_1^c x_c \tag{53}$$

$$0 = [A_2 \;\; 0]x_c \tag{54}$$

where $E_1^c \in R^{n_1^c \times n^c}$, $A_1^c \in R^{n_1^c \times n^c}$, $n_1^c = n_1 + 2$, $n^c = n + 2$.

4 Numerical Examples

In this section, we numerically find the singularity-induced bifurcation boundaries using the condition (51) but appliedtothe closed-loop system (54). Note that extra care is needed in the numerical calculation of the r.s.e. (40)-(41). The issue of numerical stability was indeed encountered in our numerical calculation of both (40)-(41) and (36)-(37) although we only followed the theoretical procedure mentioned earlier and did not try any special algorithms to ensure numerical stability. It's not our intention to divert the focus of this research to address the issue of stability. On the other hand, calculation using the condition (51) is numerically rather stable. We did not have any stability problem to finish the task using MatLab software.

We first tried to test all pairs of parameters to find out the pairs that yield bifurcation boundaries. Parameters are allowed to take values within the 95%th-percentile confidence intervals of their estimated values, i.e.,

$$p(i) \in [\bar{p}(i) - \bar{c}\sigma_i \;\; \bar{p}(i)) - \bar{c}\sigma_i]$$

where $\bar{p}(i)$ is the estimated value of parameter $p(i)$, σ_i the estimated variance, $\bar{c}$ the critical value of the 95%th-percentile confidence interval for N(0,1). For some parameters, variance information is not provided in Leeper and Sims (1994). In this case, parameter values are allowed to take values with 50% of the estimated values. Such a range of parameter values puts parameters well within the feasible region.

The estimate information for involved parameters μ, g, and β is shown in Table 1.

To find out what could happen when parameter values cross the singularity boundary, consider the parameter β. The following table shows that change of finite eigenvalues, $\lambda_1, ..., \lambda_8$, when β varies.

The first row in Table 2 are the values β takes. The second through the ninth rows are the corresponding finite eigenvalues of the linearized model. There are three more infinite eigenvalues which are not shown in the table. The table clearly shows, when the value of β increases (and crosses the bifurcation boundary), λ_8 decreases rapidly to $-\infty$ and then decrease from $+\infty$. Table 2 clearly shows that the Leeper and Sims model has a structural change when β crosses the singularity-induced bifurcation boundary and the two regions separated by the boundary exhibits drastically different dynamical behaviors.

Example 4.1. Figure 4.1 shows singularity-induced bifurcation boundaries

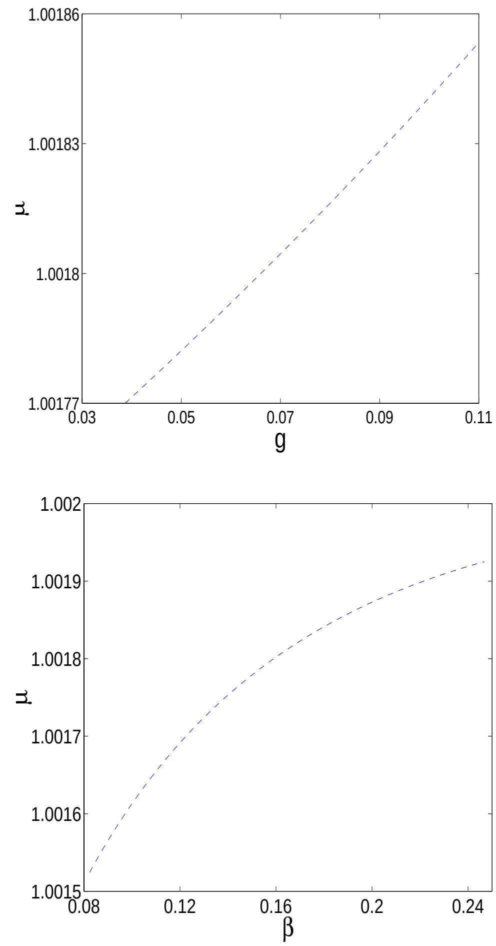

Figure 4.1. Singularity-induced bifurcation boundaries

The numerical examples show several issues related to Leeper and Sims model that deserves special attention. First, μ seems to play a critical role in the structural properties of the model. Parameter estimation should pay special attention on the accuracy of the estimation of μ. Second, the numerical example shows that the number of dynamic equations and the number of algebraic equations change when μ crosses the singularity-induced bifurcation boundary. One natural question is: Was such a change caused by estimation error or modeling error? Or this is a fundamental property of macroeconometric models? The reduction might have occurred in cases where an actual algebraic equation is modeled by a differential equation. At present we are unable to reach any conclusion, however. Further study is needed.

5 Conclusions

The Sims and Leeper model is representative of a larger class of systems. Another example of this class is the well-known fundamental dynamic Leontief model. The most distinguishing characteristic of this class of system is the form of the model

$$E\dot{x} = f(x)$$

in which the matrix E could become singular. In this paper, we have examined the basic properties of such model, proposed an approach for bifurcation analysis, and most importantly discovered the existence of singularity-induced bifurcations. Within a practically small region of estimated parameter values, we found and characterized the nature of the singularity-induced bifurcation. Of notable significance is the fact that the dynamic order of the system changes when parameter values cross the bifurcation boundary.

We believe that this effort is the first step toward achieving a better understanding of the dynamics of this class of systems. It's important to understand the implications of the bifurcation on system dynamics and stability.

Example 4.2. Figure 4.2 shows singularity-induced bifurcation boundaries for three parameters

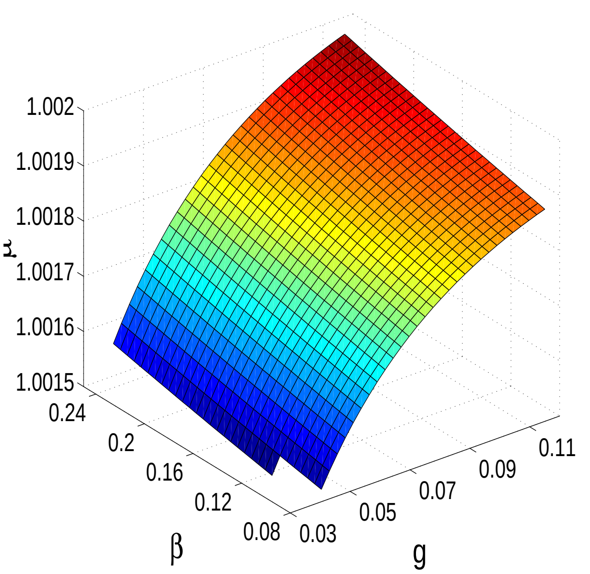

Figure 4.2. Examples of singularity-induced bifurcation boundaries for three parameters.

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parameter	estimate	standard deviation	95%th-percentile confidence interval
μ	1.0248	0.324	(1,1.6598]
g	0.0773	0.292	(0, 0.6496]
β	0.1645	0.288	(0, 0.7290]

Table 1: Estimation of μ, g, and β

β	0.080	0.120	0.160	0.165	0.170	0.200	0.240
λ_1	1.002	1.002	1.002	1.002	1.002	1.002	1.002
λ_2	0.080	0.120	0.160	0.165	0.170	0.200	0.240
λ_3	-0.303	-0.262	-0.220	-0.215	-0.210	-0.178	-0.135
λ_4	-3.558	-3.559	-3.561	-3.561	-3.561	-3.563	-3.566
λ_5	-0.098	-0.084	-0.077	-0.076	-0.075	-0.072	-0.069
λ_6	-0.002	-0.003	-0.003	-0.003	-0.003	-0.004	-0.004
λ_7	3.101	5.177	8.237	8.682	9.254	13.416	28.401
λ_8	-117.790	-204.703	-1811.413	∞	1456.294	195.888	58.059
note			boundary				

Table 2: Illustration of eignevalue changes

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