Progressivity, Inequality Reduction and Merging-Proofness in Taxation*

Biung-Ghi Ju† Juan D. Moreno-Ternero‡

February 6, 2006

Abstract

Progressivity, *inequality reduction* and *merging-proofness* are three well-known axioms in taxation. We investigate implications of each of the three axioms through characterizations of several families of taxation rules and their logical relations. We also study the preservation of these axioms under two operators on taxation rules, the so-called convexity operator and minimal-burden operator, which give intuitive procedures of determining a tax schedules.

Keywords: taxation, progressivity, inequality reduction, merging-proofness, convexity operator, minimal-burden operator.

JEL Codes: C70, D63, D70, H20

1 Introduction

In modern welfare states, income tax is a major source of state funds and is an essential policy measure for the enhancement of distributive justice. In the framework introduced by O'Neill (1982), Aumann and Maschler (1985) and Young (1988),[1] we study two principles of distributive justice, known as *progressivity* (tax rates are in the order of income) and *inequality reduction* (taxation reduces income inequality). We investigate how the two principles are related to each other and to another principle that prevents any gain from strategic merging among taxpayers. This third principle, called *merging-proofness*, is studied by de Frutos (1999) and Ju (2003). We also study the robustness of the three principles, or axioms, of taxation under the application of two operators, known as convexity operator and minimal-burden operator (to be explained later).

*We thank William Thomson for detailed comments. All remaining errors are ours.

†Department of Economics, University of Kansas, 1300 Sunnyside Avenue, Lawrence, KS 66045, USA. e-mail: bgju@ku.edu

‡CORE, Université catholique de Louvain, 34 Voie du Roman Pays, 1348 Louvain-la-Neuve, Belgium. e-mail: moreno@core.ucl.ac.be

[1]We refer readers to Young (1994), Moulin (2002) and Thomson (2003, 2005) for extensive treatments of taxation problems and other related problems such as bankruptcy, cost sharing, surplus sharing, etc.

Merging-proofness and its motivation seem to have no bearing on the two principles of distributive justice. However, we find that they are in fact related. Based on two characterization results imposing *merging-proofness* or *progressivity* as well as some standard axioms in the literature, we show that any *progressive* taxation rule is *merging-proof*. This gives an extra advantage of imposing *progressivity*.

We establish a close connection between *progressivity* and *inequality reduction* that has long been perceived by a number of authors in the literature of tax function, which is a function from $\mathbb{R}$ (the set of real numbers) to $\mathbb{R}$. A formal proof in the tax function framework, however, is provided rather recently by Eichhorn et al. (1984). As far as we know, no earlier work provides a parallel result in our framework.

A recent study by Thomson and Yeh (2001) gives a novel classification of rules and axioms based on *operators* that map a rule into another, possibly the same, rule. Two types of operators we consider here capture intuitive proposals of determining tax schedules. When two rules compete, a natural compromise is mixing the two rules by taking a convex combination of them, which is what a *convexity operator* (Thomson and Yeh 2001) does. In this way, we are able to mix two different ideas of taxation embedded in two rules. The *minimal-burden operator* (Thomson and Yeh 2001) gives us an intuitive procedure of identifying tax schedules. If the aggregate income of all agents except, say, agent i is lower than the tax revenue to be collected, this difference can be interpreted as the minimal tax burden imposed on agent i (he is the only person who can contribute for this portion because the maximum aggregate tax payment by the remaining agents cannot cover it). Thus, the following two-step procedure, as suggested by the minimal-burden operator, seems interesting. First, let each agent pay his minimal burden. Second, the remainder of the tax revenue is collected by considering the remaining income profile.

The application of an operator may be problematic if it fails to preserve some appealing axioms, in particular, our three main axioms, *progressivity*, *inequality reduction* and *merging-proofness* (preservation of an axiom means that if a rule satisfies an axiom so does the rule obtained by applying the operator). We show that the two types of operators preserve *progressivity* and *inequality reduction*. Regarding *merging-proofness*, the minimal-burden operator is slightly disruptive as it requires an additional, but mild, axiom to preserve it.

The rest of the paper is organized as follows. In Section 2, we present the model and basic concepts. In Section 3, we define axioms. In Section 4, we state and prove the characterization results. In Section 5, we state and prove our results on operators. For a smooth passage, we defer some proofs and provide them in the appendix.

2 Model and basic concepts

We study taxation problems in a variable population model. The set of potential taxpayers, or *agents*, is identified by the set of natural numbers $\mathbb{N}$. Let $\mathcal{N}$ be the set of finite subsets of $\mathbb{N}$, with generic element N. For each $i \in N$, let $y_i \in \mathbb{R}_+$ be i's (taxable) *income* and $y \equiv (y_i)_{i \in N}$ the income profile. A (taxation) *problem* is a triple consisting of a population $N \in \mathcal{N}$, an income profile $y \in \mathbb{R}^N_+$, and a tax revenue $T \in \mathbb{R}_+$ such that $\sum_{i \in N} y_i \geq T$. Let $Y \equiv \sum_{i \in N} y_i$. To avoid unnecessary complication, we assume $Y = \sum_{i \in N} y_i > 0$. Let $\mathscr{D}^N$ be the set of taxation problems with population N and $\mathscr{D} \equiv \cup_{N \in \mathscr{N}} \mathscr{D}^N$.

Given a problem $(N, y, T) \in \mathscr{D}$, a *tax profile* is a vector $x \in \mathbb{R}^N$ satisfying the following two conditions: (i) for each $i \in N$, $0 \leq x_i \leq y_i$ and (ii) $\sum_{i \in N} x_i = T$. We refer to (i) as *boundedness* and (ii) as *balancedness*.[2] A (taxation) *rule* on $\mathscr{D}$, R: $\mathscr{D} \to \cup_{N \in \mathscr{N}} \mathbb{R}^N$, associates with each problem $(N, y, T) \in \mathscr{D}$ a tax profile $R(N, y, T)$ for the problem. Each rule R gives the associated *post-tax income function* $S^R(\cdot)$ defined as follows: for each $(N, y, T) \in \mathscr{D}$, $S^R(N, y, T) \equiv y - R(N, y, T)$. Throughout the paper, for each $N \in \mathscr{N}$, each $M \subseteq N$, and each $z \in \mathbb{R}^N$, let $z_M \equiv (z_i)_{i \in M}$.

We now provide some examples of rules. We start with three well-known rules. The *head tax* distributes the tax burden equally, provided no agent ends up paying more than her income. The *leveling tax* equalizes post-tax income across agents, provided no agent is subsidized. The *flat tax* equalizes tax rates across agents. These three rules are examples of rules in the following family introduced by Young (1987).

Definition 1 (Parametric Rules)**.** A rule R is a *parametric rule* if there is a function f : $[a, b] \times \mathbb{R}_+ \to \mathbb{R}$, where $a, b \in \mathbb{R} \cup \{\pm\infty\}$, such that (i) f is continuous and non-decreasing in the first variable; (ii) for each $x \in \mathbb{R}_+$, $f(a, x) = 0$ and $f(b, x) = x$; (iii) for each $(N, y, T) \in \mathscr{D}$ and each $i \in N$, $R_i(N, y, T) = f(\lambda, y_i)$, where $\lambda \in [a, b]$ satisfies $\sum_{i \in N} f(\lambda, y_i) = T$.[3] We call f a *parametric representation of* R.

The three rules mentioned earlier have the following parametric representations:

- Head tax: $f^H(\lambda, y) = \min\{-\frac{1}{\lambda}, y\}$, for each $\lambda \in \mathbb{R}_-$ and each $y \in \mathbb{R}_+$.
- Leveling tax: $f^L(\lambda, y) = \max\{y - \frac{1}{\lambda}, 0\}$, for each $\lambda \in \mathbb{R}_+$ and each $y \in \mathbb{R}_+$.
- Flat tax: $f^F(\lambda, y) = \lambda \cdot y$, for each $\lambda \in [0, 1]$ and each $y \in \mathbb{R}_+$.

3 Axioms

We now define our three main axioms of taxation.

Progressivity postulates that for any pair of agents, the one with higher income should pay at least as high a rate of tax as the other.

Progressivity. For each $(N, y, T) \in \mathscr{D}$ and each $i, j \in N$, if $0 < y_i \leq y_j$,

$$\frac{R_i(N, y, T)}{y_i} \leq \frac{R_j(N, y, T)}{y_j}.$$

[2] Note that *boundedness* implies that each agent with zero income pays zero tax.

[3] Existence of such λ is guaranteed by the first two conditions (i) and (ii).

Our second axiom requires that the post-tax income profile should have at least as low "income inequality" as the original income profile. This axiom is based on the following basic inequality relation over income profiles. For each population $N \equiv \{1,\ldots,n\}$ and each pair of income profiles $y, y' \in \mathbb{R}^N_+$, y *Lorenz dominates* y' if, for each $k = 1,\ldots,n-1$, the proportion of the sum of the k lowest incomes to the total income at y is greater than or equal to the same proportion at y': that is, when $y_1 \leq y_2 \leq \ldots \leq y_n$ and $y'_1 \leq y'_2 \leq \ldots \leq y'_n$, for each $k = 1,\ldots,n-1$,

$$\frac{\sum_{i=1}^k y_i}{\sum_{i=1}^n y_i} \geq \frac{\sum_{i=1}^k y'_i}{\sum_{i=1}^n y'_i}.$$

Inequality reduction. For each $(N,y,T) \in \mathscr{D}$, the post-tax income profile $S^R(N,y,T)$ Lorenz dominates y.

Our third axiom prevents a rule from being manipulated by a pair of agents through merging their incomes.

Merging-proofness. For each $(N,y,T) \in \mathscr{D}$ and each pair $i, j \in N$ with $i \neq j$, if $y' \in \mathbb{R}^{N\setminus\{j\}}_+$ is such that $y'_i = y_i + y_j$ and $y'_{N\setminus\{i\}} = y_{N\setminus\{i,j\}}$,

$$R_i(N,y,T) + R_j(N,y,T) \leq R_i(N\setminus\{j\}, y', T).$$

We will investigate logical relations between the three axioms, invoking in the process some of the following standard axioms.[4]

The next axiom requires that a rule should give the same tax profile when it is applied for any subset of agents as when it is applied for the whole population.

Consistency. For each $(N,y,T) \in \mathscr{D}$, each $M \subset N$, and each $i \in M$,

$$R_i(M, y_M, \sum_{i\in M} x_i) = x_i,$$

where $(x_i)_{i\in N} \equiv R(N,y,T)$ and $y_M \equiv (y_i)_{i\in M}$.

The next two axioms require that tax contributions and post-tax incomes be in the order of pre-tax income (Aumann and Maschler 1985).

Tax order preservation. For each $(N,y,T) \in \mathscr{D}$ and each pair $i, j \in N$, if $y_i \geq y_j$, $R_i(N,y,T) \geq R_j(N,y,T)$.

Income order preservation. For each $(N,y,T) \in \mathscr{D}$ and each pair $i, j \in N$, if $y_i \geq y_j$, $y_i - R_i(N,y,T) \geq y_j - R_j(N,y,T)$.

Note that *progressivity* implies *tax order preservation*.

Finally, the next axiom says that small changes in incomes or revenue do not produce a jump in tax schedules.

[4]We refer readers to Thomson (2003, 2005) for a detailed discussion on these axioms.

Continuity. For each $N \in \mathcal{N}$, each sequence $\{(N, y^n, T^n) : n \in \mathbb{N}\}$ in $\mathcal{D}^N$, and each $(N, y, T) \in \mathcal{D}^N$, if (y^n, T^n) converges to (y, T), then $R(N, y^n, T^n)$ converges to $R(N, y, T)$.

4 Characterizations and logical relation among axioms

4.1 Progressivity and merging-proofness

Lemma 1 gives a necessary and sufficient condition for a parametric rule to satisfy *progressivity*. A parametric representation $f\colon [a,b] \times \mathbb{R}_+ \to \mathbb{R}$ is *superhomogeneous in income* if for each $\lambda \in [a,b]$, each $y_0 \in \mathbb{R}_+$ and each $\alpha \geq 1$, $f(\lambda, \alpha y_0) \geq \alpha f(\lambda, y_0)$.

Lemma 1. *A parametric rule satisfies progressivity if and only if it has a parametric representation that is superhomogeneous in income.*

Proof. Let R be a parametric rule and $f\colon [a,b] \times \mathbb{R}_+ \to \mathbb{R}$ a parametric representation of R. Assume that R is *progressive*. Let $\lambda \in [a,b]$, $y_0 > 0$ and $\alpha \geq 1$. Let $T^\lambda \equiv f(\lambda, y_0) + f(\lambda, \alpha y_0)$ and $N \equiv \{1,2\}$. Then, $R(N, (y_0, \alpha y_0), T^\lambda) = (f(\lambda, y_0), f(\lambda, \alpha y_0))$. By *progressivity*, $f(\lambda, y_0)/y_0 \leq f(\lambda, \alpha y_0)/(\alpha y_0)$. Thus $\alpha f(\lambda, y_0) \leq f(\lambda, \alpha y_0)$, which shows that f is superhomogeneous in income.

Conversely, assume that f is superhomogeneous in income. Let $(N, y, T) \in \mathcal{D}$ and $i, j \in N$ be such that $0 < y_i \leq y_j$. Let $\lambda \in [a,b]$ be such that $R(N, y, T) = (f(\lambda, y_i))_{i \in N}$. Then, by superhomogeneity, $f(\lambda, y_j) = f(\lambda, \frac{y_j}{y_i} \cdot y_i) \geq \frac{y_j}{y_i} \cdot f(\lambda, y_i)$.Thus

$$\frac{R_j(N, y, T)}{y_j} = \frac{f(\lambda, y_j)}{y_j} \geq \frac{f(\lambda, y_i)}{y_i} = \frac{R_i(N, y, T)}{y_i},$$

which shows the *progressivity* of R. ∎

It is evident that *progressivity* implies the following axiom, which says that any two agents with the same income should pay the same tax.

Equal treatment of equals. For each $(N, y, T) \in \mathcal{D}$ and each pair $i, j \in N$ with $y_i = y_j$, $R_i(N, y, T) = R_j(N, y, T)$.

Young (1987, Theorem 1) shows that the parametric rules are the only rules satisfying *consistency*, *equal treatment of equals*, and *continuity*. Therefore, using his result and Lemma 1 we obtain:

Proposition 1. *A rule satisfies progressivity, consistency, and continuity if and only if it has a parametric representation that is superhomogeneous in income.*[5]

[5] It is worth noting that, although there might be different representations of a parametric rule, superhomogeneity in income is invariant; that is, either every representation is superhomogeneous in income or none of them is superhomogeneous in income.

Remark 1. Marshall and Olkin (1979, p.453) and Bruckner and Ostrow (1962, Lemma 3) offer similar results for tax functions $\xi : \mathbb{R}_+ \to \mathbb{R}$.[6] The main difference between their model and ours is that our rules are *multivariate vector valued* functions with the two constraints of (income) *boundedness* or *balancedness*. Despite the differences, Proposition 1 shows that, thanks to Young's (1987) characterization of parametric rules, the earlier results can be extended in our model without much difficulty.

Lemma 2 gives a necessary and sufficient condition for a parametric rule to satisfy *merging-proofness*. A parametric representation $f\colon [a,b] \times \mathbb{R}_+ \to \mathbb{R}$ is *superadditive in income* if for each $\lambda \in [a,b]$ and each pair $y_0, y_0' \in \mathbb{R}_+$, $f\left(\lambda, y_0 + y_0'\right) \geq f\left(\lambda, y_0\right) + f\left(\lambda, y_0'\right)$.[7] Ju (2003, Proposition 1) offers the following result:

Lemma 2 (Ju 2003)**.** *A parametric rule satisfies merging-proofness if and only if it has a parametric representation that is superadditive in income.*

The proof of the lemma is given in the appendix.

The next lemma says that *consistency* and *merging-proofness* together imply *equal treatment of equals*.

Lemma 3. *Merging-proofness and consistency together imply equal treatment of equals.*[8]

Combining Lemmas 2 and 3 and Young's (1987) characterization of parametric rules, we obtain:

Proposition 2. *A rule satisfies merging-proofness, consistency, and continuity if and only if it has a parametric representation that is superadditive in income.*[9]

Now, due to Propositions 1 and 2, the logical relation between *progressivity* and *merging-proofness* can be established from the following relation between superhomogeneity and superadditivity.

Lemma 4. *Superhomogeneity in income implies superadditivity in income.*

Proof. Let y_0 and y_0' be such that $0 < y_0 \leq y_0'$. Let $\alpha \equiv (y_0 + y_0')/y_0'$. Then, by *superhomogeneity*, $f\left(\lambda, \alpha y_0'\right) \geq \alpha f\left(\lambda, y_0'\right)$, that is, $f\left(\lambda, y_0 + y_0'\right) / \left(y_0 + y_0'\right) \geq f(\lambda, y_0')/y_0'$. Thus, $f(\lambda, y_0 + y_0') \geq f(\lambda, y_0') + \frac{y_0}{y_0'} f(\lambda, y_0')$. By *superhomogeneity*, $\frac{y_0}{y_0'} f(\lambda, y_0') \geq f\left(\lambda, y_0\right)$. Hence $f(\lambda, y_0 + y_0') \geq f(\lambda, y_0') + f\left(\lambda, y_0\right)$, which shows that f is *superadditive* in income. ∎

It follows from Propositions 1 and 2 and Lemma 4 that:

[6] See also Proposition 2 in Thon (1987).

[7] Like superhomogeneity, superadditivity in income is also invariant with respect to the choice of the representation.

[8] Chambers and Thomson (2002, Lemma 3) show that *consistency* and *equal treatment of equals* together imply *anonymity*, which says that the chosen tax profile should not depend on the names of agents. Combining this with our lemma, *merging-proofness* and *consistency* imply *anonymity*.

[9] This strengthens Theorem 2 in Ju (2003) by dropping *equal treatment of equals*.

Corollary 1. *Let R be a rule satisfying consistency and continuity. If R is progressive, then R is merging-proof. But the converse does not hold.*[10]

Remark 2. Without *consistency* and *continuity*, the logical relation between *progressivity* and *merging-proofness* in Corollary 1 does not hold, as shown by Example 1 in Section 5.

Remark 3. Since rules take only non-negative values, if a parametric representation is *superadditive in income* (or *superhomogeneous*, by Lemma 4), then it is *non-decreasing in income*. Thus the corresponding parametric rule satisfies *tax order preservation*. Therefore, among parametric rules, *merging-proofness* (or *progressivity*) implies *tax order preservation*.

Note that any convex function that crosses the origin is superhomogeneous. This, together with Proposition 1 and Corollary 1, gives the following:

Corollary 2. *Any rule with a parametric representation that is convex in income is progressive and merging-proof.*

Both the leveling tax and the flat tax have parametric representations that are convex in income. Thus, they are both *progressive* and *merging-proof*. The same argument applies to show that two other classical tax rules, such as the proposals by Cohen-Stuart and Cassel (and formulated as rules by Young, 1988), are *progressive* and *merging-proof*.

4.2 Progressivity and inequality reduction

We now investigate the logical relation between *progressivity* and *inequality reduction*. The following additional axioms are also considered.

Revenue continuity. For each $N \in \mathscr{N}$, each $y \in \mathbb{R}^N_+$, each sequence $\{T^n : n \in \mathbb{N}\}$ in $\mathbb{R}_+$ and each $T \in \mathbb{R}_+$, if T^n converges to T, then $R(N,y,T^n)$ converges to $R(N,y,T)$.

Revenue monotonicity. For each $(N,y,T) \in \mathscr{D}$ and each $T' \geq T$, $R(N,y,T') \geqq R(N,y,T)$.

Young (1987) offers the following useful lemma:

Lemma 5 (Young 1987)**.** *Equal treatment of equals, revenue continuity, and consistency together imply revenue monotonicity.*

Now we are ready to prove the following result.

Proposition 3. *The following statements hold*:
(i) *Progressivity and income order preservation together imply inequality reduction.*
(ii) *Inequality reduction and consistency together imply progressivity.*
(iii) *Inequality reduction, together with consistency and revenue continuity* (*or revenue monotonicity*)*, implies income order preservation.*

[10] An example of a rule satisfying *merging-proofness* but violating *progressivity* can be provided upon request.

Proof. The proof of parts (i) and (ii) will be provided in the appendix. Here we prove part (iii). Let R be a rule satisfying *consistency*, *revenue continuity* and *inequality reduction* (the same argument applies when *revenue continuity* is replaced with *revenue monotonicity*). Then by the second statement, R satisfies *progressivity* and also *equal treatment of equals*. By Lemma 5, R also satisfies *revenue monotonicity*. Suppose, by contradiction, that R violates *income order preservation*. Then, there exist $(N,y,T) \in \mathscr{D}$ and $i,j \in N$ such that $y_i < y_j$ and $y_i - x_i > y_j - x_j$, where $x \equiv R(N,y,T)$. By *consistency*, $R\left(\{i,j\},(y_i,y_j),x_i+x_j\right) = (x_i,x_j)$. Let $n \in \mathbb{N}$ be such that

$$n-1 > \frac{(y_j-x_j)(y_j-y_i)}{y_i(y_i-x_i-y_j+x_j)}. \tag{1}$$

Consider the problem $(N',y',T') \in \mathscr{D}$ with $N' = \{i,j\} \cup M$ such that $|M| = n-1$, $M \cap N = \emptyset$, $y'_j = y_j$, $y'_k = y_i$ for each $k \in M \cup \{i\}$, and $T' = nx_i + x_j$. By *equal treatment of equals*, there exist $a,b \in \mathbb{R}_+$ such that for each $k \in M \cup \{i\}$, $R_k(N',y',T') = a$ and $R_j(N',y',T') = b$. If $a+b > x_i + x_j$, then by *consistency* and *revenue monotonicity*, $R\left(\{i,j\},(y_i,y_j),a+b\right) = R(\{i,j\},(y'_i,y'_j),a+b) = (a,b) \geq \left(x_i,x_j\right) = R\left(\{i,j\},(y_i,y_j),x_i+x_j\right)$. Then $na+b > nx_i + x_j = T'$, contradicting *balancedness*. A similar contradiction occurs if $a+b < x_i+x_j$. Therefore, $a+b = x_i+x_j$. This, together with $na+b = nx_i+x_j$, implies $a = x_i$ and $b = x_j$. Therefore, for each $k \in M \cup \{i,j\}$,

$$R_k\left(N',y',T'\right) = \begin{cases} x_i & \text{if } k \in M \cup \{i\} \\ x_j & \text{if } k = j \end{cases}$$

Thus, by *inequality reduction*,

$$\frac{y_i}{y_j+ny_i} = \frac{\min_{k\in N'}\{y'_k\}}{\sum_{k=1}^n y'_k} \leq \frac{\min_{k\in N'}\{y'_k - R_k(N',y',T')\}}{\sum_{k=1}^n (y'_k - R_k(N',y',T')} = \frac{y_j-x_j}{(y_j-x_j)+n(y_i-x_i)},$$

which implies that

$$n \leq \frac{(y_j-x_j)(y_j-y_i)}{y_i(y_i-x_i-y_j+x_j)},$$

contradicting (1). ∎

The next result follows directly from Proposition 3.

Corollary 3. *For consistent and revenue continuous* (*or revenue monotonic*) *rules, the combination of progressivity and income order preservation is equivalent to inequality reduction.*

Remark 4. A similar result is established for tax functions by Eichhorn et al. (1984). In order to extend that result in our model, we need the two additional axioms, *consistency* and *revenue continuity* (or *revenue monotonicity*).

It follows from Proposition 3 that since the leveling tax and the flat tax satisfy both *progressivity* and *income order preservation*, they satisfy *inequality reduction*. After strengthening *revenue continuity* to (full) *continuity*, we obtain the following result.

Proposition 4. *A rule satisfies inequality reduction, consistency and continuity if and only if it has a parametric representation $f\colon [a,b]\times\mathbb{R}_+\to\mathbb{R}$ such that f is superhomogeneous in income and for each $\lambda\in[a,b]$, the function $g^\lambda(x)=x-f(\lambda,x)$ is non-decreasing.*[11]

Proof. Let R be a rule satisfying *inequality reduction*, *consistency,* and *continuity*. By Proposition 3, R satisfies *progressivity* and *income order preservation*. Then, by Proposition 1, R has a parametric representation $f\colon [a,b]\times\mathbb{R}_+\to\mathbb{R}$, where $a,b\in\mathbb{R}\cup\{\pm\infty\}$, which is superhomogeneous in income. Let $\lambda\in[a,b]$. Let $g^\lambda:\mathbb{R}_+\to\mathbb{R}$ be such that $g^\lambda(x)=x-f(\lambda,x)$ for all $x\in\mathbb{R}_+$. Suppose, by contradiction, that there exist $x,y\in\mathbb{R}_+$ such that $x<y$ and $g^\lambda(x)>g^\lambda(y)$. Let $T\equiv f(\lambda,x)+f(\lambda,y)$. Consider the problem $(\{1,2\},(x,y),T)$. Then, $R(\{1,2\},(x,y),T)=(f(\lambda,x),f(\lambda,y))$. Thus,

$$x-R_1(\{1,2\},(x,y),T))=g^\lambda(x)>g^\lambda(y)=y-R_2(\{1,2\},(x,y),T)),$$

contradicting *income order preservation*.

Conversely, let R be a rule with parametric representation $f\colon [a,b]\times\mathbb{R}_+\to\mathbb{R}$ such that f is superhomogeneous in income and for each $\lambda\in[a,b]$, $g^\lambda(x)=x-f(\lambda,x)$ is non-decreasing. By Proposition 1, R satisfies *progressivity*, *continuity* and *consistency*. Then by Proposition 3, we only have to show *income order preservation*. Suppose, by contradiction, that there exist $(N,y,T)\in\mathscr{D}$ and $i,j\in N$ such that $y_i<y_j$ and $y_i-R_i(N,y,T)>y_j-R_j(N,y,T)$. Let $\lambda\in[a,b]$ be such that $R(N,y,T)=(f(\lambda,y_i))_{i\in N}$. Then

$$y_i-f(\lambda,y_i)=y_i-R_i(N,y,T)>y_j-R_j(N,y,T)=y_j-f\left(\lambda,y_j\right),$$

contradicting the non-decreasing property of $g^\lambda(\cdot)$. ∎

5 Operators: what axioms are preserved?

An *operator* is a function that maps a rule into another, possibly the same, rule. An axiom is said to be *preserved* under an operator if any rule that satisfies the axiom is mapped by the operator into a rule that also satisfies the axiom. We consider two operators introduced by Thomson and Yeh (2001) and study preservation of our three main axioms.

5.1 Convexity operators

When two rules compete, a natural compromise is to mix the two rules by a convex combination as suggested by *convexity operators*. Formally, given a "reference rule" $\bar{R}(\cdot)$ and a weight $\alpha\in[0,1]$, the *convexity operator* associated with $\bar{R}$ and α maps each rule $R(\cdot)$ into the convex combination $(1-\alpha)R(\cdot)+\alpha\bar{R}(\cdot)$.[12] The idea of mixing two rules is also useful

[11] This property is also invariant.

[12] This definition is slightly different from the definition in Thomson and Yeh (2001). The convexity operator in Thomson and Yeh (2001) maps an ordered list of a finite number of rules into the weighted average rule. Our results can easily be adapted to establish the same results for their convexity operator.

for a smooth transition from one rule to another when such a transition is required.

Mixing two rules may lose its appeal if such an operation does not preserve some basic axioms of taxation. Fortunately, all of our three main axioms are preserved:

Proposition 5. *Consider convexity operators associated with a reference rule $\bar{R}(\cdot)$. If $\bar{R}(\cdot)$ satisfies progressivity, then each of these convexity operators preserves progressivity. And the same results hold for inequality reduction and merging-proofness.*

Proof. We skip the proof of preservations of *progressivity* and *merging-proofness*, which is straightforward. Suppose that $R(\cdot)$ and $\bar{R}(\cdot)$ satisfy *inequality reduction*. Let $\alpha \in [0,1]$. Let $(N,y,T) \in \mathscr{D}$, $\bar{x} \equiv \bar{R}(N,y,T)$, $x \equiv R(N,y,T)$ and $x^{\alpha} \equiv R^{\alpha}(N,y,T)$. Without loss of generality, assume that $N \equiv \{1,\ldots,n\}$ and that $y_1 \leq y_2 \leq \cdots \leq y_n$. Let $\bar{\sigma}$, σ and $\pi\colon N \to N$ be permutations on N such that for each $i \in \{1,\ldots,n-1\}$, $y_{\bar{\sigma}(i)} - \bar{x}_{\bar{\sigma}(i)} \leq y_{\bar{\sigma}(i+1)} - \bar{x}_{\bar{\sigma}(i+1)}$, $y_{\sigma(i)} - x_{\sigma(i)} \leq y_{\sigma(i+1)} - x_{\sigma(i+1)}$, and $y_{\pi(i)} - x^{\alpha}_{\pi(i)} \leq y_{\pi(i+1)} - x^{\alpha}_{\pi(i+1)}$. Let $i \in \{1,\ldots,n-1\}$. Note that $\sum_{j=1}^{i}\left(y_{\pi(j)} - \bar{x}_{\pi(j)}\right) \geq \sum_{j=1}^{i}\left(y_{\bar{\sigma}(j)} - \bar{x}_{\bar{\sigma}(j)}\right)$ because, by definition of $\bar{\sigma}$, the right-hand side is the sum of the i lowest post-tax incomes associated with the tax profile $\bar{x}$. Similarly, $\sum_{j=1}^{i}\left(y_{\pi(j)} - x_{\pi(j)}\right) \geq \sum_{j=1}^{i}\left(y_{\sigma(j)} - x_{\sigma(j)}\right)$. Therefore,

$$\begin{aligned}\sum_{j=1}^{i}\left(y_{\pi(j)} - x^{\alpha}_{\pi(j)}\right) &= (1-\alpha)\sum_{j=1}^{i}\left(y_{\pi(j)} - x_{\pi(j)}\right) + \alpha\sum_{j=1}^{i}\left(y_{\pi(j)} - \bar{x}_{\pi(j)}\right)\\ &\geq (1-\alpha)\sum_{j=1}^{i}\left(y_{\sigma(j)} - x_{\sigma(j)}\right) + \alpha\sum_{j=1}^{i}\left(y_{\bar{\sigma}(j)} - \bar{x}_{\bar{\sigma}(j)}\right).\end{aligned}$$

By *inequality reduction* of $R(\cdot)$ and $\bar{R}(\cdot)$,

$$\frac{\sum_{j=1}^{i}\left(y_{\sigma(j)} - x_{\sigma(j)}\right)}{Y-T} \geq \frac{\sum_{j=1}^{i} y_j}{Y} \text{ and } \frac{\sum_{j=1}^{i}\left(y_{\bar{\sigma}(j)} - \bar{x}_{\bar{\sigma}(j)}\right)}{Y-T} \geq \frac{\sum_{j=1}^{i} y_j}{Y}.$$

Therefore,

$$\frac{\sum_{j=1}^{i}\left(y_{\pi(j)} - x^{\alpha}_{\pi(j)}\right)}{Y-T} \geq \frac{\sum_{j=1}^{i} y_j}{Y}.$$

showing *inequality reduction* of R. ∎

5.2 Minimal-burden operator

At each problem (N,y,T), if $T - \sum_{j\in N\setminus\{i\}} y_j > 0$ for an agent $i \in N$, this part of the revenue cannot be covered even if everyone other than i pays his full income. Thus this part can be viewed as the minimal burden imposed on agent i. For each $i \in N$, let $m_i(N,y,T) \equiv \min\{0, T - \sum_{j\neq i} y_j\}$ be i's *minimal burden*. Let $m(N,y,T) \equiv (m_i(N,y,T))_{i\in N}$ and $M(N,y,T) \equiv \sum_N m_i(N,y,T)$. The *minimal-burden operator* associates with each rule R the rule R^m defined by the following two-step payment procedure. For each problem, first each agent pays his minimal burden; second, each agent pays his tax according to R at the revised problem obtained by reducing agents' incomes by the amounts of their minimal burdens and the tax revenue by the total minimal burdens. That is, for each $(N,y,T)\in\mathscr{D}$,

$$R^m(N,y,T)\equiv m(N,y,T)+R(N,y-m(N,y,T),T-M(N,y,T)).$$

The next proposition shows what axioms are preserved under the minimal-burden operator.

Proposition 6. *The minimal burden operator preserves progressivity and inequality reduction. However, it does not preserve merging-proofness.*

The proof is provided in the appendix.

Example 1 below shows that the minimal-burden operator does not preserve *merging-proofness*.

Example 1. For each $(N,y,T)\in\mathscr{D}$, let

$$R(N,y,T)\equiv\begin{cases}R^L(N,y,T) & \text{if } T\geq 10\\ R^F(N,y,T) & \text{if } T<10\end{cases},$$

where R^L denotes the leveling tax and R^F the flat tax. Since both R^L and R^F are *merging-proof*, R is *merging-proof*. However, R^m is not *merging-proof*. To show this, consider the problem $(N,y,T)=(\{1,2,3\},(5,55,100),70)$. Then,

$$R^m(N,y,T)=(0,0,10)+R^L(\{1,2,3\},(5,55,90),60)=\left(0,\frac{25}{2},\frac{115}{2}\right).$$

Consider now the resulting problem in which agents 2 and 3 merge their incomes and are represented by agent 3, i.e., $(N\backslash\{2\},y',T)=(\{1,3\},(5,155),70)$. Then,

$$R^m\left(N\backslash\{2\},y',T\right)=(0,65)+R^F(\{1,3\},(5,90),5)=\left(\frac{5}{19},\frac{1325}{19}\right).$$

Consequently,

$$R^m_3(N\backslash\{2\},y',T)<R^m_2(N,y,T)+R^m_3(N,y,T),$$

which shows that R^m is not *merging-proof*. Note that R is *progressive*. By Proposition 5, so is R^m. Therefore, R^m is an example showing that *progressivity* does not imply *merging-proofness*, as claimed in Remark 2.

For rules satisfying the following very mild axiom, we show that the minimal-burden operator preserves *merging-proofness*.

Suppose that an agent donates part of his income and that the donation is used to finance tax revenue. Then both the donor's income and the tax revenue go down by the amount of the donation. The next axiom says that the donor's total payment (tax plus donation) should not be lower than his total payment without donation.

No Donation Paradox. For all $(N,y,T) \in \mathscr{D}$, all $i \in N$ and all $t \in [0, \min\{T, y_i\}]$,

$$R_i(N,y,T) \leq t + R_i(N, (y_i - t, y_{-i}), T - t).^{13}$$

Ju and Moreno-Ternero (2005) characterize a large family of rules satisfying *no donation paradox* and some other axioms. The family includes most of the well-known parametric rules, which shows *no donation paradox* is a very mild condition.

Note that the rule in Example 1 violates *no donation paradox*. To show this, consider the problem $(N,y,T) = (\{1,2\},(3,15),11)$. Then, $R(N,y,T) = (0,11)$ and $R(N,(3,13),9) = (27/16, 117/16)$. Thus, $R_2(N,y,T) = 11 > 2 + 117/16 = 2 + R_2(N,(3,13),9)$.

Proposition 7. *On the family of rules satisfying no donation paradox, the minimal-burden operator preserves merging-proofness.*

The proof is provided in the appendix.

6 Concluding remarks

We conclude with some remarks associated with two other operators in Thomson and Yeh (2001) and the axioms that are dual to our main axioms.

Truncation and Duality Operators

Truncation Operator maps each rule $R(\cdot)$ into $R^t(\cdot)$ defined as follows: for each $(N,y,T) \in \mathscr{D}$ and each $i \in N$,

$$R_i^t(N,y,T) \equiv R_i(N, (\min\{y_j, T\})_{j \in N}, T).$$

Progressivity is not preserved under truncation operator. To show this, we can use the flat tax (Thomson 2005, Table 3.2, p.205). Let us call the image of the flat tax under truncation operator truncated flat tax. It is easy to show that the truncated flat tax satisfies *regressivity* and differs from the flat tax. Thus it violates *progressivity* because the flat tax is the only rule satisfying both *progressivity* and *regressivity*.

Inequality reduction is not preserved under truncation operator. This is shown in Example 2.

Merging-proofness is not preserved under truncation operator. This is shown in Example 2. We can also use the flat tax and a similar argument to the above one provided for *progressivity*.

Example 2. Consider the leveling tax L. It is easy to show that, in the two-agent case, L^t (the image of L under the truncation operator) coincides with the so-called *concede-and-divide* (Thomson 2003). This rule has the following expression, for the problems with

[13] In bankruptcy problems, this axiom is introduced by Thomson and Yeh (2001). It is the dual of "claims monotonicity" (see p.100 and p.161 in Thomson 2005).

$(\{1,2\},(y_1,y_2),T)$ such that $y_1 \leq y_2$:

$$CD(\{1,2\},(y_1,y_2),T) = \begin{cases} \left(\frac{T}{2},\frac{T}{2}\right) & \text{if } T \leq y_1 \\ \left(\frac{y_1}{2},T-\frac{y_1}{2}\right) & \text{if } y_1 \leq T \leq y_2 \\ \left(y_1-\frac{Y-T}{2},y_2-\frac{Y-T}{2}\right) & \text{if } y_2 \leq T \end{cases}.$$

If $T = 1$ and $(y_1,y_2) = (1,3)$, we have

$$\frac{CD_1(\{1,2\},(y_1,y_2),T)}{y_1} = \frac{1}{2} > \frac{1}{6} = \frac{CD_2(\{1,2\},(y_1,y_2),T)}{y_2},$$

which shows that concede-and-divide (and therefore L^t) violates *progressivity*. Similarly,

$$\frac{CD_1(\{1,2\},(y_1,y_2),T)}{T} = \frac{1}{2} > \frac{1}{4} = \frac{y_1}{Y},$$

which shows that concede-and-divide (and therefore L^t) violates *inequality reduction*. Finally, consider the problem $P \equiv (\{1,2,3\},(1,2,3),2) \in \mathscr{D}$ and the resulting problem $P' \equiv (\{1,2\},(1,5),2) \in \mathscr{D}$ in which agents 2 and 3 merge their incomes. Then, it is straightforward to show that $L^t(P) = (0,1,1)$ and $L^t(P') = CD(P') = (\frac{1}{2},\frac{3}{2})$. Thus, $L^t_2(P) + L^t_3(P) > L^t_2(P')$, which shows that L^t is not *merging-proof*.

Duality Operator maps each rule $R(\cdot)$ into $R^d(\cdot)$ defined as follows: for each $(N,y,T) \in \mathscr{D}$ and each $i \in N$,

$$R^d_i(N,y,T) \equiv y_i - R_i(N,y,\sum_{j\in N} y_j - T).$$

Progressivity is not preserved under duality operator. This is because *regressivity* is the dual property of *progressivity* and so for any *progressive* rule $R(\cdot)$ that differs from the flat tax, its dual $R^d(\cdot)$ satisfies *regressivity* but not *progressivity*.

Inequality reduction is not preserved under duality operator. To show this, consider the leveling tax, of which the dual is the head tax. Note that the leveling tax satisfies *progressivity* and *income order preservation*. Thus by Proposition 3-(i), it also satisfies *inequality reduction*. On the other hand, the head tax satisfies *regressivity* and *consistency*. Thus by Proposition 3-(ii), it must violate *inequality reduction.*

Merging-proofness is not preserved under duality operator. This is because *merging-proofness* is the dual property of *splitting-proofness* and so for any *merging-proof* rule $R(\cdot)$ that differs from the flat tax, its dual $R^d(\cdot)$ satisfies *splitting-proofness*. Since the flat tax is the only rule satisfying both *merging-proofness* and *splitting-proofness*, then $R^d(\cdot)$ must violate *merging-proofness*.

	Minimal-burden	Truncation	Duality
Progressivity	Y	N	N
Inequality reduction	Y	N	N
Merging-proofness	N (Y under no donation paradox)	N	N

1	2	$3,\dots,n$	$n+1$	$n+2$
a	a	$y_{-\{1,2\}}$		
a	a	$y_{-\{1,2\}}$	0	0
	a	$y_{-\{1,2\}}$	a	0
	a	$y_{-\{1,2\}}$	a	
0	a	$y_{-\{1,2\}}$	a	0
a	a	$y_{-\{1,2\}}$		0
a	a	$y_{-\{1,2\}}$		

(a) Income profiles

1	2	$3,\dots,n$	$n+1$	$n+2$
x_1	x_2	$x_{-1,2}$		
x_1	x_2	$x_{-\{1,2\}}$	0	0
	x'_2	$x'_{-\{1,2\}}$	x'_{n+1}	0
	x'_2	$x'_{-\{1,2\}}$	x'_{n+1}	
0	x'_2	$x'_{-\{1,2\}}$	x'_{n+1}	0
x_1	x_2	$x_{-\{1,2\}}$		0
x_1	x_2	$x_{-\{1,2\}}$		

(b) Tax profiles

Table 1: Proof of Lemma 3.

Dual Axioms

As shown in Thomson (2005), dual axioms of *progressivity* and *merging-proofness* are *regressivity* and *splitting-proofness* respectively. Proposition 3.9 in Thomson (2005) says that an axiom is preserved under truncation operator if and only if the dual axiom is preserved under minimal-burden operator. Therefore, from Proposition 6 we obtain: truncation operator preserves *regressivity* and the dual axiom of *inequality reduction*, but not *splitting-proofness*. Also from Proposition 7, we obtain: on the family of rules satisfying *income monotonicity* (which is the dual of *no donation paradox*), truncation operator preserves *splitting-proofness*.

A Proofs

Proof of Lemma 3. Let $(N,y,T) \in \mathscr{D}$ and $i,j \in N$ be such that $i \neq j$ and $y_i = y_j$. For simplicity, let $i=1$ and $j=2$ and $N \equiv \{1,\dots,n\}$ (this problem is illustrated in the second row of Table 1-(a)). Let $x \equiv R(N,y,T)$ and $a \equiv y_1 = y_2$ (x is illustrated in the second row of Table 1-(b)). Let $N' \equiv N \cup \{n+1, n+2\}$. Consider the problem $(N',(y,0,0),T)(= (N',(a,a,y_{-\{1,2\}},0,0),T))$ where $n+1$ and $n+2$ have zero income and all agents in N have the same incomes as in (N,y,T) (see the third row of Table 1-(a)). By *boundedness*, $R_{\{n+1,n+2\}}(N',(y,0,0),T) = (0,0)$. By *balancedness* and *consistency*, $R_N(N',(y,0,0),T) = R(N,y,T)$ (see the third row of Table 1-(b)). Now consider the problem $(N' \backslash \{1\},(a,y_{-\{1,2\}},a,0),T)$ obtained by merging the incomes of agents 1 and $n+1$ at $(N',(y,0,0),T)$ into the income of agent $n+1$ (see the fourth row of Table 1-(a)). Let $x' \equiv R(N' \backslash \{1\},(a,y_{-\{1,2\}},a,0),T)$ (see the fourth row of Table 1-(b)). Then $x'_{n+2} = 0$ and by *merging-proofness*, $x'_{n+1} \geq x_1$. By *consistency*, $(x'_2, x'_{-\{1,2\}}, x'_{n+1}) = R(\{2,\dots,n+1\},(a,y_{-\{1,2\}},a),T)$.

Consider the problem $(N',(0,a,y_{-\{1,2\}},a,0),T)$ where 1 and $n+2$ have zero income and all others in N' have the same incomes as in $(\{2,\dots,n+1\},(a,y_{-\{1,2\}},a),T)$ (see the sixth row of Table 1-(a)). Then, by *boundedness* and *consistency*, $R(N',(0,a,y_{-\{1,2\}},a,0),T) = (0,x'_2,x'_{-\{1,2\}},x'_{n+1},0)$. Now making the reverse argument but merging the incomes of 1 and $n+1$ at $(N',(0,a,y_{-\{1,2\}},a,0),T)$ into 1's income and applying *merging-proofness*, we can show $x_1 \geq x'_{n+1}$, as $x_1 = R_1(N' \backslash \{n+1\},(a,a,y_{-\{1,2\}},0),T)$.

Therefore, $x_1 = x'_{n+1}$. By *balancedness*, $x_2 + \cdots + x_n = x'_2 + \cdots + x'_n$. Thus, the two reduced problems of (N, y, T) and $(\{2,\ldots,n+1\},(a,y_{-\{1,2\}},a),T)$ for the coalition $\{2,\ldots,n\}$ are identical. By *consistency*, $(x_2,\ldots,x_n) = (x'_2,\ldots,x'_n)$.

To summarize, by replacing agent 1's income at $(N,(a,a,y_{-\{1,2\}}),T)$ with agent $(n+1)$'s income, we transformed the problem into $(\{2,\ldots,n,n+1\},(a,y_{-\{1,2\}},a),T)$ and showed that 1's tax at the original problem is equal to $(n+1)$'s tax in the new problem and the taxes of all others do not change.

Now, transforming $(\{2,\ldots,n,n+1\},(a,y_{-\{1,2\}},a),T)$ into $(\{3,\ldots,n,n+1,n+2\},(y_{-\{1,2\}},a,a),T)$ and letting $\bar{x} \equiv R(\{3,\ldots,n,n+1,n+2\},(y_{-\{1,2\}},a,a),T)$, we can show that $\bar{x}_{\{3,\ldots,n+1\}} = x'_{\{3,\ldots,n+1\}} = (x_{\{3,\ldots,n\}},x'_{n+1}) = (x_{\{3,\ldots,n\}},x_1)$ and $x_2 = \bar{x}_{n+2}$. Therefore, $x_1 = \bar{x}_{n+1}$ and $x_2 = \bar{x}_{n+2}$.

Applying the symmetric argument (the whole argument above) switching the role of $n+1$ and the role of $n+2$, we can show that $x_2 = \bar{x}_{n+1}$ and $x_1 = \bar{x}_{n+2}$. Therefore, $x_1 = x_2$. ∎

Proof of Proposition 3, parts (i) and (ii). The proofs of parts (i) and (ii) below are similar to Eichhorn et al. (1984).

(i) Let R be a rule satisfying *progressivity* and *income order preservation*. Let $(N,y,T) \in \mathscr{D}$. Assume, without loss of generality, that $0 < y_1 \le y_2 \le \cdots \le y_n$. Let $x \equiv R(N,y,T)$. Then, by *progressivity*,

$$\frac{x_1}{y_1} \le \frac{x_2}{y_2} \le \cdots \le \frac{x_n}{y_n}. \tag{2}$$

Let $k \in \{1,...,n-1\}$. By (2), $x_i y_j \le x_j y_i$, for all $i = 1,...,k$ and $j = k+1,...,n$. Thus, $\sum_{i=1}^{k} x_i \sum_{j=k+1}^{n} y_j \le \sum_{j=k+1}^{n} x_j \sum_{i=1}^{k} y_i$. Equivalently, $\sum_{i=1}^{k} x_i \sum_{j=1}^{n} y_j \le \sum_{j=1}^{n} x_j \sum_{i=1}^{k} y_i$, which says that

$$\sum_{i=1}^{n} y_i \sum_{i=1}^{k} (y_i - x_i) \ge \sum_{i=1}^{k} y_i \sum_{i=1}^{n} (y_i - x_i). \tag{3}$$

By *income order preservation*, the post-tax income profile $(y_i - x_i)_{i\in N}$ preserves the order of the pre-tax income profile y. Thus, (3) shows that the post-tax income profile Lorenz dominates the pre-tax income profile.

(ii) Let R be a rule satisfying *inequality reduction*. Suppose, by contradiction, that R is not *progressive*. Then, there exist $(N,y,T) \in \mathscr{D}$ and $i, j \in N$, such that $0 < y_i \le y_j$ and $R_i(N,y,T)/y_i > R_j(N,y,T)/y_j$. Let $a_i \equiv 1 - \frac{R_i(N,y,T)}{y_i}$ and $a_j \equiv 1 - \frac{R_j(N,y,T)}{y_j}$. Then, $a_i < a_j$, and therefore,

$$\frac{y_i}{y_i + y_j} > \frac{a_i y_i}{a_i y_i + a_j y_j} \ge \frac{\min\{a_i y_i, a_j y_j\}}{a_i y_i + a_j y_j}. \tag{4}$$

Now, let $T' \equiv R_i(N,y,T) + R_j(N,y,T)$. Consider $(\{i,j\},(y_i,y_j),T') \in \mathscr{D}$. By consistency, $R_k(\{i,j\},(y_i,y_j),T') = R_k(N,y,T)$ for each $k = i,j$, and therefore, $y_k - R_k(\{i,j\},(y_i,y_j),T') = a_k y_k$ for each $k = i,j$. Thus, (4) contradicts *inequality reduction*. ∎

Proof of Proposition 6. *Progressivity*: Let R be a rule satisfying *progressivity*. Let $(N,y,T) \in \mathscr{D}$ and $x^m \equiv R^m(N,y,T)$. Assume, without loss of generality, that $N = \{1,2,...,n\}$ and

$y_1 \leq y_2 \leq \cdots \leq y_n$. Let $k \in N$ be the first agent whose minimal burden is strictly positive, i.e., $y_{k-1} \leq Y - T < y_k$. Then, $m_1(N,y,T) = \cdots = m_{k-1}(N,y,T) = 0 < m_k(N,y,T) \leq m_{k+1}(N,y,T) \leq \cdots \leq m_n(N,y,T)$. For each $i \geq k$, $m_i(N,y,T) = y_i - Y + T$. Let $y' \equiv y - m(N,y,T) = (y_1, \ldots, y_{k-1}, Y-T, \ldots, Y-T)$ and $T' \equiv T - \sum_{i=1}^n m_i(N,y,T) = T - \sum_{i=k}^n (y_i - Y + T)$. Let $x' \equiv R(N,y',T')$. Then

$$x_i^m = \begin{cases} x_i' \text{ if } i \leq k-1; \\ y_i - Y + T + x_i' \text{ if } i \geq k. \end{cases} \tag{5}$$

Let $i, j \in N$ be such that $y_i \leq y_j$. There are three cases.

Case 1: $y_i \leq y_j < y_k$. By *progressivity* of R at (N,y',T'), $x_i^m/y_i = x_i'/y_i' \leq x_j'/y_j' = x_j^m/y_j$.
Case 2: $y_k \leq y_i \leq y_j$. By *equal treatment of equals* of R at (N,y',T') (implied by the *progressivity* of R), $x_i' = x_j' = a$. By *boundedness*, $x_i' = x_j' = a \leq Y - T$ and so $Y - T - x_i' = Y - T - x_j' = Y - T - a \geq 0$. Therefore, since $y_i \leq y_j$,

$$\frac{x_i^m}{y_i} = \frac{y_i - Y + T + x_i'}{y_i} = 1 - \frac{Y-T-a}{y_i} \leq 1 - \frac{Y-T-a}{y_j} = \frac{y_j - Y + T + x_j'}{y_j} = \frac{x_j^m}{y_j}.$$

Case 3: $y_i < y_k \leq y_j$. By *progressivity* of R at (N,y',T'),

$$\frac{x_i'}{y_i} \leq \frac{x_j'}{Y-T}. \tag{6}$$

Now, since $Y - T < y_j$ and, by *boundedness*, $x_j' \leq Y - T$, then $x_j' y_j \leq (Y-T)\left(y_j - Y + T + x_j'\right)$. Hence,

$$\frac{x_j'}{Y-T} \leq \frac{y_j - Y + T + x_j'}{y_j}. \tag{7}$$

Therefore, combining (6) and (7),

$$\frac{x_i^m}{y_i} = \frac{x_i'}{y_i} \leq \frac{x_j'}{Y-T} \leq \frac{y_j - Y + T + x_j'}{y_j} = \frac{x_j^m}{y_j}.$$

Inequality Reduction: Let R be a rule satisfying *inequality reduction*. Let $(N,y,T) \in \mathscr{D}$, (N,y',T'), x^m and x' be given as in the above proof. Note that $y - x^m = y' - x'$. By the *inequality reduction* of R at (N,y',T'), $y' - x'$ Lorenz dominates y'. Thus, we only have to show that y' Lorenz dominates y.[14] It is clear that for each $l \leq k-1$, $\sum_{i=1}^l y_i'/Y' \geq \sum_{i=1}^l y_i/Y$. Assume $l \geq k$. Note that $\sum_{i=1}^l y_i'/Y' \geq \sum_{i=1}^l y_i/Y$ is equivalent to

$$\sum_{i=1}^k y_i \left(\sum_{i=l+1}^n (y_i - Y + T) \right) \geq (Y-T)\left((n-l) \sum_{i=k+1}^l y_i - (l-k) \sum_{i=l+1}^n y_i \right),$$

[14]Note that if y is increasingly ordered, so is y'.

which is true because the left-hand side is non-negative and the right-hand side is non-positive.[15] ∎

To prove Proposition 7, we need the following additional axiom and lemma.

No donation paradox and *merging-proofness* together imply the following useful property, as shown in the next lemma. Suppose that two agents i and j merge their income into j's income and agent j donates i's income. The property says that the total payment by the two agents should not be lowered by such a donation.

Donation-Proofness. For all $(N,y,T) \in \mathscr{D}$ and all $i, j \in N$, such that $T \geq y_i$

$$R_i(N,y,T) + R_j(N,y,T) \leq y_i + R_j\left(N\backslash\{i\}, y_{N\backslash\{i\}}, T - y_i\right).$$

Lemma 6. *Merging-proofness and no donation paradox together imply donation-proofness.*

Proof. Let R be a rule satisfying *merging-proofness* and *no donation paradox*. Let $(N,y,T) \in \mathscr{D}$ and $i, j \in N$ such that $T \geq y_i$. By *merging-proofness*,

$$R_i(N,y,T) + R_j(N,y,T) \leq R_j(N\backslash\{i\}, \left(y_i + y_j, y_{N\backslash\{i,j\}}\right), T).$$

By *no donation paradox*, applied to agent j with donation y_i at $\left(N\backslash\{i\}, \left(y_i + y_j, y_{N\backslash\{i,j\}}\right), T\right)$,

$$R_j(N\backslash\{i\}, \left(y_i + y_j, y_{N\backslash\{i,j\}}\right), T) \leq y_i + R_j(N\backslash\{i\}, \left(y_j, y_{N\backslash\{i,j\}}\right), T - y_i).$$

Combining the two inequalities, we obtain

$$R_i(N,y,T) + R_j(N,y,T) \leq y_i + R_j\left(N\backslash\{i\}, y_{N\backslash\{i\}}, T - y_i\right),$$

which shows *donation-proofness*. ∎

Now we are ready to prove Proposition 7.

Proof of Proposition 7. Let R be a rule satisfying *no donation paradox* and *merging-proofness*. By Lemma 6, R satisfies *donation-proofness*. Let $(N,y,T) \in \mathscr{D}$. Assume, without loss of generality, that $N = \{1,2,...,n\}$ and $y_1 \leq y_2 \leq \cdots \leq y_n$. Let $k \in N$ be the first agent whose minimal burden is strictly positive, i.e., k is such that $y_{k-1} \leq Y - T < y_k$. Let $i, j \in N$ and $\hat{y} \in \mathbb{R}_+^{N\backslash\{i\}}$ be such that $\hat{y}_j = y_i + y_j$ and $\hat{y}_{N\backslash\{i,j\}} = y_{N\backslash\{i,j\}}$. Let $x \equiv R(N,y,T)$ and $\hat{x} \equiv R(N\backslash\{i\}, \hat{y}, T)$. Let $x^m \equiv R^m(N,y,T)$ and $\hat{x}^m \equiv R^m(N\backslash\{i\}, \hat{y}, T)$. We show $x_i^m + x_j^m \leq \hat{x}_j^m$ below.

Let $M \equiv M(N,y,T)$ and $\hat{M} \equiv M(N\backslash\{i\}, \hat{y}, T)$. Let $y' \equiv (y_1, \ldots, y_{k-1}, Y - T, \ldots, Y - T)$ and $x' \equiv R(N, y', T - M)$.

[15] Note that $(n-l)\sum_{i=k+1}^{l} y_i \leq (n-l)\sum_{i=k+1}^{l} y_l = (n-l)(l-k)y_l$ and $(l-k)\sum_{i=l+1}^{n} y_i \geq (l-k)\sum_{i=l+1}^{n} y_{l+1} = (l-k)(n-l)y_{l+1}$. The two inequalities imply $(n-l)\sum_{i=k+1}^{l} y_i - (l-k)\sum_{i=l+1}^{n} y_i \leq (n-l)(l-k)(y_l - y_{l+1}) \leq 0$.

Case 1: $y_i + y_j \leq Y - T$. Then $y_i, y_j \leq Y - T$ and so $x_i^m = x_i'$ and $x_j^m = x_j'$. Note that $M = \hat{M}$. Then, $R_j^m(N\backslash\{i\}, \hat{y}, T)$ equals j's award under $R(\cdot)$ at the problem obtained from y' after merging i and j's incomes. Therefore, *merging-proofness* of R at $(N, y', T - M)$ implies $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 2: $y_i, y_j > Y - T$. Without loss of generality, suppose $y_i \leq y_j$. In this case,

$$x_i^m + x_j^m = \begin{pmatrix} y_i - (Y-T) + R_i(N, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T - M) \\ +y_j - (Y-T) + R_j(N, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T - M) \end{pmatrix},$$
$$\hat{x}_j^m = y_i + y_j - (Y-T) + R_j(N\backslash\{i\}, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T - \hat{M}).$$

Since $\hat{M} = M + Y - T$, then by *donation-proofness*,

$$R_i(N, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T - M) + R_j(N, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T - M)$$
$$\leq (Y-T) + R_j(N\backslash\{i\}, y_1, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T - \hat{M}).$$

Therefore, $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 3: $y_i \leq Y - T < y_j$. Note that $\hat{M} = M + y_i$. We have

$$x_i^m + x_j^m = \begin{pmatrix} R_i(N, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T - M) + y_j - (Y-T) \\ +R_j(N, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T - M) \end{pmatrix},$$
$$\hat{x}_j^m = y_i + y_j - (Y-T) + R_j(N\backslash\{i\}, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T - \hat{M}).$$

Since $\hat{M} = M + y_i$, then by *donation-proofness*,

$$\begin{pmatrix} R_i(N, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T - M) \\ +R_j(N, y_1, \ldots, y_{i-1}, y_i, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T - M) \end{pmatrix}$$
$$\leq y_i + R_j(N\backslash\{i\}, y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T - \hat{M}).$$

Therefore, $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 4: $y_j \leq Y - T < y_i$. Note that $\hat{M} = M + y_j$. We have

$$x_i^m + x_j^m = \left(\begin{array}{l} y_i - (Y-T) + R_i(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ + R_j(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \end{array} \right),$$
$$\hat{x}_j^m = y_i + y_j - (Y-T) + R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{Y-T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-\hat{M}).$$

By *merging-proofness,*

$$\left(\begin{array}{l} R_i(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ + R_j(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \end{array} \right)$$
$$\leq R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{y_j + Y-T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-M).$$

By *no donation paradox* applied to agent j with donation y_j,

$$R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{y_j + Y-T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-M)$$
$$\leq y_j + R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{Y-T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-\hat{M}).$$

Combining the two inequalities, we obtain

$$R_i(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) + R_j(N, y_1, \ldots, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M)$$
$$\leq y_j + R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{Y-T}, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k}, T-\hat{M}),$$

which implies $x_i^m + x_j^m \leq \hat{x}_j^m$.

Case 5: $y_i, y_j \leq Y - T$ and $y_i + y_j > Y - T$. Then $\hat{M} = M + T - (Y - (y_i + y_j))$. We have

$$x_i^m + x_j^m = \left(\begin{array}{l} R_i(N, y_1, \ldots, y_i, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ + R_j(N, y_1, \ldots, y_i, y_j, \ldots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M), \end{array} \right)$$
$$\hat{x}_j^m = T - (Y - (y_i + y_j)) + R_j(N \backslash \{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{Y-T}, \cdots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-\hat{M}).$$

By *merging-proofness,*

$$x_i^m + x_j^m \leq R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{y_i + y_j}, \cdots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M).$$

Since $T - \hat{M} = T - M - (T - (Y - (y_i + y_j)))$, then applying *no donation paradox for* j with donation $T - (Y - (y_i + y_j))$,

$$\begin{aligned} & R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{y_i + y_j}, \cdots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-M) \\ & \leq T - (Y - (y_i + y_j)) + R_j(N\backslash\{i\}, y_1, \ldots, \underset{\substack{\uparrow \\ j^{\text{th}} \text{ income}}}{Y - T}, \cdots, y_{k-1}, \underbrace{Y-T, \ldots, Y-T}_{n-k+1}, T-\hat{M}). \end{aligned}$$

Therefore, $x_i^m + x_j^m \leq \hat{x}_j^m$. ∎

References

[1] Aumann, R., M. Maschler (1985), "Game theoretic analysis of a bankruptcy problem from the Talmud," J. Econ. Theory 36:195–213.

[2] Bruckner, A.M., E. Ostrow (1962), "Some function classes related to the class of convex functions," Pacific J. Math. 12:1203–1215.

[3] Chambers, C.P., W. Thomson (2002), "Group order preservation and the proportional rule for the adjudication of conflicting claims," Mathematical Social Sciences 44:235–252.

[4] Chun, Y. (1988), "The proportional solution for rights problems," Mathematical Social Sciences 15:231–246.

[5] Eichhorn, W., H. Funke, W.F. Richter (1984), "Tax progression and inequality of income distribution," Journal of Mathematical Economics 13:127-131.

[6] de Frutos, M.A. (1999), "Coalitional manipulations in a bankruptcy problem," Review of Economic Design 4:255-272.

[7] Ju, B.-G. (2003), "Manipulation via merging and splitting in claims problems," Review of Economic Design 8:205-215.

[8] Ju, B.-G., Miyagawa E., Sakai T. (2005), "Non-manipulable division rules in claim problems and generalizations," Journal of Economic Theory, forthcoming.

[9] Ju, B.-G., J. D. Moreno-Ternero (2005), "Avoiding donation paradox in taxation," Mimeo, The University of Kansas.

[10] Marshall, A., I. Olkin (1979), Inequalities: theory of majorization and its applications, Academic Press, New York.

[11] Moreno-Ternero, J.D. (2005), “Bankruptcy rules and coalitional manipulation,” International Game Theory Review, forthcoming.

[12] Moulin, H. (1987), “Equal or proportional division of a surplus, and other methods,” International Journal of Game Theory 16(3):161–186.

[13] Moulin, H. (2002), “Axiomatic cost and surplus-sharing,” in: K. Arrow, A. Sen, K. Suzumura, (Eds.), The Handbook of Social Choice and Welfare, Vol.1:289-357, North-Holland.

[14] Moulin, H. (2003), Fair division and collective welfare, MIT Press, Cambridge. MA.

[15] O’Neill, B. (1982), “A problem of rights arbitration from the Talmud,” Mathematical Social Sciences 2:345–371.

[16] Thomson, W. (2003), “Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey,” Mathematical Social Sciences 45:249-297.

[17] Thomson, W. (2004), Consistent allocation rules, Manuscript, University of Rochester.

[18] Thomson, W. (2005), How to divide when there isn’t enough: From the Talmud to game theory, Manuscript, University of Rochester

[19] Thomson, W., C.-H. Yeh (2001), “Minimal rights, maximal claims, and duality for division rules,” Mimeo, University of Rochester.

[20] Thon, D. (1987), “Redistributive properties of progressive taxation,” Mathematical Social Sciences 14:185-191.

[21] Young, P. (1987), “On dividing an amount according to individual claims or liabilities,” Mathematics of Operations Research 12:398-414.

[22] Young, P. (1988), “Distributive justice in taxation,” Journal of Economic Theory 44: 321–335.

[23] Young, P. (1994), Equity, Princeton University Press, Princeton.