Operational identification of the complete class of superlative index numbers: an application of Galois theory[1]

William A. Barnett[a] and Ki-Hong Choi[b]

[a]University of Kansas
Department of Economics
Lawrence, KS 66045-7585
e-mail: barnett@ku.edu
Tel: 785-864-2844
Fax: 785-832-1527

[b]National Pension Research Institute
Nonhyeon-dong 4-15, Gangnam-gu
Seoul 135-811, Korea
e-mail: khchoi@nps4u.or.kr
Tel: 82-2-3218-8631
Fax: 82-2-541-2901

February 10, 2006

Abstract

We provide an operational identification of the complete class of superlative index numbers to track the exact aggregator functions of economic aggregation theory. If an index number is linearly homogeneous and a second order approximation in a formal manner that we define, we prove the index to be in the superlative index number class of nonparametric functions. Our definition is mathematically equivalent to Diewert's most general definition. But when operationalized in practice, our definition permits use of the full class, while Diewert's definition, in practice, spans only a strict subset of the general class. The relationship between the general class and that strict subset is a consequence of Galois theory. Only a very small number of elements of the general class have been found by Diewert's method, despite the fact that the general class contains an infinite number of functions. We illustrate our operational, general approach by proving for the first time that a particular family of nonparametric functions, including the Sato-Vartia index, is within the superlative index number class.

JEL Classification Codes: C8, E01, D
Keywords: Exact index numbers, superlative index number class, Divisia line integrals, aggregator function space, Galois theory.

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[1] The corresponding author is William A. Barnett.

1. Introduction

Diewert (1976, p. 117; 1978, p. 884) defined an index number as *superlative*, if it is exact for a linearly homogeneous flexible functional form aggregator function or its dual function, where the class of flexible functional forms is defined to be the class of second order functions. Along with this definition of superlativeness, he provided the example of the quadratic mean of order r≠0 price index,

$$P_r = P_r(\mathbf{p}_0,\mathbf{p}_1,\mathbf{x}_0,\mathbf{x}_1) = \sqrt{\left\{\sum_{i=1}^{n} w_{i,0}\left(p_{i,1}/p_{i,0}\right)^{r/2}\right\}^{r/2}\left\{\sum_{i=1}^{n} w_{i,1}\left(p_{i,1}/p_{i,0}\right)^{-r/2}\right\}^{-r/2}}\ , \qquad (1.1)$$

which he proved to be in the superlative class, where the quantity of good i consumed in period s = 0, 1 is $x_{i,s}$ and its price is $p_{i,s}$, with the expenditure share of good i in period s being $w_{i,s} = \dfrac{p_{i,s}x_{i,s}}{\mathbf{p}_s{}'\mathbf{x}_s}$ for strictly positive prices $\mathbf{p}_s = (p_{1,s}, \ldots , p_{n,s})'$ and quantities $\mathbf{x}_s = (x_{1,s}, \ldots , x_{n,s})'$.

The superlative class of nonparametric functions thereby also includes the corresponding quantity index for $r \neq 0$,

$$X_r = X_r(\mathbf{p}_0,\mathbf{p}_1,\mathbf{x}_0,\mathbf{x}_1) = \sqrt{\left\{\sum_{i=1}^{n} w_{i,0}\left(x_{i,1}/x_{i,0}\right)^{r/2}\right\}^{r/2}\left\{\sum_{i=1}^{n} w_{i,1}\left(x_{i,1}/x_{i,0}\right)^{-r/2}\right\}^{-r/2}}\ .$$

The quadratic-mean-of-order-r subset of the superlative index number class includes some of the best known index numbers, such as the Fisher ideal (for r = 2) and the Walsh (for r=1). Although not within the space of functions defined by the quadratic mean of order r, the famous Törnqvist index is on the open boundary of that function set, since that index can be reached in the limit as r→0. Subsequent to the publication of Diewert's (1976, 1978) original papers, no further elements of the superlative class have been found.

But the class of superlative index numbers contains an infinite number of nonparametric functions. As a result, it might seem surprising that there has been no further progress in finding index numbers in the class. There is a reason, well understood by mathematicians, but less well incorporated into applied economics. Existence of a function does not alone imply existence of an algebraic closed form representation of that function. Additional assumptions are required to get from a class of functions that exists, to the strict subset of those functions that can be expressed in algebraic closed form. In fact many of the best known functions can only be tabulated by numerical methods.[2]

The additional assumptions needed to assure existence of closed form algebraic representations can be found in Galois theory (see, e.g., Artin, 1998). Yet the search for superlative index numbers by Diewert's approach has been limited to the search for the subset of those superlative index numbers that are exact for flexible functional forms expressible in algebraic closed form. In practice, Diewert's approach to locating superlative index numbers thereby becomes limited to the identification of index numbers that are exact for aggregator functions in that subset.[3] The potential use of implicit function representations or nonalgebraic representations of aggregator functions renders Diewert's approach unreasonably difficult to apply.[4]

A notable example is the Sato-Vartia index, which is on a par with the Fisher ideal index in terms of its ability to satisfy statistical index number tests, as has been shown by Balk (1995) and Reinsdorf and Dorfman (1999, p. 45). However, the superlativeness of the Sato-Vartia index has remained undetermined. No flexible functional form that it can track exactly has yet been found, despite such efforts as those of Sato (1976) and Lau (1979).

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[2] For example, trigonometric, hyperbolic, and Bessel functions usually can only be tabulated at predetermined precision from the partial sums of series expansions (Taylor, Laurent, or hypergeometric) with analytic continuation.

[3] In addition, if one conversely starts with a flexible functional form and then tries to find a statistical index number that is exact for it, no clear procedure exists. For example, the minflex Laurent flexible functional form of Barnett and Lee (1985) was originated two decades ago, but no statistical index number that is exact for it has yet been found.

2. Flexible aggregator functions and second order approximations

The source of the desirability of superlative indexes is their ability to attain second order approximations to the underlying theoretical aggregator functions, which under linear homogeneity assumptions are either weakly separable subfunctions of consumer utility or firm technology or are the corresponding dual unit cost functions. In discrete time, an aggregator function is evaluated at two periods of time, and the logarithm of the ratio of the aggregator function in those two periods defines the growth rate of the economic functional index number. A superlative index number seeks to approximate that ratio nonparametrically up to the second order. Barnett (1983) proved the equivalence of the mathematical definition of "second order local approximation" and the definition of "flexible" approximation, where the latter definition is the one used in economics to define the class of flexible functional forms in function space. The terminology and results are reviewed below.

Definition 1: Diewert defines a function f* having vector of parameters θ to be a flexible functional form approximation to any function f on the domain set of variables, $\mathbf{x}$, if for any f in the relevant function space and any point $\mathbf{x}_0$ in the functions' domain, there exists a value of $\theta = \theta\ (\mathbf{x}_0)$ such that

$$f^*(\mathbf{x}_0) = f(\mathbf{x}_0), \tag{2.1}$$

$$\partial f^* / \partial \mathbf{x}\big|_{\mathbf{x}=\mathbf{x}_0} = \partial f / \partial \mathbf{x}\big|_{\mathbf{x}=\mathbf{x}_0}, \tag{2.2}$$

$$\partial^2 f^* / \partial \mathbf{x} \partial \mathbf{x}'\big|_{\mathbf{x}=\mathbf{x}_0} = \partial^2 f / \partial \mathbf{x} \partial \mathbf{x}'\big|_{\mathbf{x}=\mathbf{x}_0}. \tag{2.3}$$

But in mathematics, the following is the common definition of local second order approximation.

[4] See, e.g., Blackorby et al. (1991) for some relevant theory applicable to the algebraic implicit function case.

Definition 2: A function $f^* = f^*_{\boldsymbol{\theta}}$ having vector of parameters $\boldsymbol{\theta}$ can provide a second order local approximation to any function f on the domain set of variables, **x**, if for any f in the relevant function space and any point $\mathbf{x}_0$ in the functions' domain, there exists a value of $\boldsymbol{\theta} = \boldsymbol{\theta}(\mathbf{x}_0)$ such that

$$\left(f^*(\mathbf{x}) - f(\mathbf{x})\right) \big/ \|\mathbf{x} - \mathbf{x}_0\|^2 \to 0 \qquad (2.4)$$

as $\mathbf{x} \to \mathbf{x}_0$.

Equivalently in a different common notation equation (2.4) can be written as

$$f^*(\mathbf{x}) - f(\mathbf{x}) = o(\|\mathbf{x} - \mathbf{x}_0\|^2),$$

which often is read to say that the remainder term is of smaller order than $\|\mathbf{x} - \mathbf{x}_0\|^2$. Another common notation, using "big O" order is:

$$f^*(\mathbf{x}) - f(\mathbf{x}) = O(\|\mathbf{x} - \mathbf{x}_0\|^3),$$

which often is read to say that the remainder term is at most of order $\|\mathbf{x} - \mathbf{x}_0\|^3$.[5]

Definition 2, although less common than Defintion 1 in econometric specifications of flexible functional forms, has long been used by Barnett and his coauthors.[6] The relationship between the two definitions is central to the results in this paper. In particular, we shall need the following lemma proved by Barnett (1983).

Lemma 1: Let f^* and f both be twice continuously differentiable functions of **x**. Then (2.1), (2.2) and (2.3) are necessary and sufficient for (2.4) in the limit, as $\mathbf{x} \to \mathbf{x}_0$.

Thus if, by Definition 1, we can identify a flexible functional form that is exactly tracked by an index number, we can conclude that the index can approximate any aggregator

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[5] If $f^*(\mathbf{x})$ is a polynomial, the big O notation implies that the remainder terms contain third order or higher terms of the polynomial. But for more general functions, the big O notation is defined to mean that $\left(f^*(\mathbf{x}) - f(\mathbf{x})\right) \big/ \|\mathbf{x} - \mathbf{x}_0\|^3$ is bounded as $\mathbf{x} \to \mathbf{x}_0$.

[6] See, e.g., Barnett (1977, 1979a, 1979b) and Barnett and Lee (1985).

function up to the second order, by Definition 2. The converse also can be shown. We formally provide that result below asTheorem 1, after we have completed defining the relevant terminology.

But Diewert's approach to proving superlativeness of an index number depends upon the possibility of finding a flexible function form that is exactly tracked by the index number. While it is true that such a flexible functional form, $f^*(\mathbf{x}) = f(\mathbf{x}) + O(\| \mathbf{x} - \mathbf{x}_0 \|^3)$, exists for any superlative index number, existence is not enough to render this approach generally operational in applications. This dependence upon the possibility of locating $f^*(\mathbf{x})$ introduces the complications of Galois theory into the proof of superlativeness of an index number. We propose bypassing the intermediate search for such a flexible functional form, as defined in Definition 1, and instead advocate direct proof of second order approximation of an index number to an arbitrary aggregator function, as in Definition 2.

Under our assumptions of linearly-homogeneous weak separability of aggregator functions, it has been shown by Hulten (1973) that the Divisia (1925) line integral, defining the Divisia index in continuous time, exactly tracks any aggregator function for a rational optimizing economic agent. Hence the ability of a statistical index number to track the Divisia line integral is equivalent to the index's ability to track the underlying aggregator function. To find discrete time index numbers that are equivalent to the Divisia line integral in continuous time, Barnett et al. (2003) proposed proving convergence of the discrete time indexes to the Divisia index as the discrete time intervals converge to zero. This approach has revealing consequences, by requiring that candidate index numbers be put into log change form and treated as approximations to the Divisia in continuous time. This transformation to log change form can be very informative about the properties of the index. But we find the requirement to produce convergence to continuous time to be an unnecessary complication. We propose and apply a simpler and more direct approach.

3. Index number functions space and Galois theory

To make our point more formally, we define three sets of index number formulas as follows.[7] Let f be an "*aggregator function*," defined to concave and monotonically (isotone) increasing in the vector of strictly positive goods quantities, $\mathbf{x}$, having strictly positive prices, $\mathbf{p}$. Suppose $\mathbf{x}_t$ is the solution to $\max_{\mathbf{x}}\{f(\mathbf{x})\text{: } \mathrm{p}_t{}'\mathbf{x} \leq \mathrm{p}_t{}'\mathbf{x}_t,\ \mathbf{x} \geq \mathbf{0}\}$, for discrete time periods $t = 0, 1, \ldots, T$. In the analogous continuous time case, $\mathbf{x}(t)$ is the instantaneous solution to $\max_{\mathbf{x}}\{f(\mathbf{x})\text{: } \mathrm{p}(t)'\mathbf{x} \leq \mathrm{p}(t)'\mathbf{x}(t),\ \mathbf{x} \geq \mathbf{0}\}$, at instant of time $t \in [0, T]$. This conditional instantaneous decision is nested within the implied intertemporal decision that optimizes the subjectively discounted integral of $\mathrm{f}(\mathbf{x}(\mathrm{t}))$ over time.

In discrete time, define $Q(\mathbf{p}_0,\mathbf{p}_t,\mathbf{x}_0,\mathbf{x}_t)$ by

$$Q(\mathbf{p}_0,\mathbf{p}_t,\mathbf{x}_0,\mathbf{x}_t) = f(\mathbf{x}_t)/f(\mathbf{x}_0) \tag{3.1}$$

for $t = 1, 2, \ldots, T$, while in continuous time, define $\mathrm{Q}(\mathbf{p}(0),\mathbf{p}(t),\mathbf{x}(0),\mathbf{x}(t))$ by

$$Q(\mathbf{p}(0),\mathbf{p}(t),\mathbf{x}(0),\mathbf{x}(t)) = f(\mathbf{x}(t))/f(\mathbf{x}(0)) \tag{3.2}$$

for $t \in [0, T]$. In either case, $\log Q$ defines the growth rate of the index from 0 to t.

Following Diewert (1976), we define a nonparametric index number function, Q_{I}, to be "*exact*" for an aggregator function, f, in discrete time, if (Q_{I},f) satisfies (3.1) for all strictly positive $(\mathbf{p}_0,\mathbf{p}_t)$ or "exact" for an aggregator function, f, in continuous time, if $(\mathrm{Q}_{\mathrm{I}},f)$ satisfy (3.2) for all strictly positive $(\mathbf{p}(0),\mathbf{p}(t))$. An aggregator function f that is exact in the discrete time case can be used in the continuous time case, and visa versa, since f maps from n dimensional space to 1 dimensional space in either case. The class of functions, f, is a subset of the same function space, regardless of whether or not used in continuous time or discrete time applications. We now have the necessary terminology to provide the following

[7] In discrete time, we assume that time intervals are closed on the left and open on the right. Prices are announced at the start of each period, and purchases for that period are made at the start of the period. Hence all activity takes place at the instant of time at the left hand boundary of each interval. As time interval lengths decline to zero, discrete time converges to continuous time, with purchases per period becoming instantaneous rates of change of purchases.

Theorem 1.

Theorem 1: If a nonparametric index number function, Q_I, is "exact" for an aggregator function, f^*, in discrete time, and f^* is a flexible functional form approximation to f, by Definition 1, the index number can approximate f up to the second order, by Definition 2. Conversely, if Q_I approximates f up to the second order by Definition 2, there exists a flexible functional form, f*, such that Q_I is "exact" for f^*.

Proof: The first part is immediate from Lemma 1. The converse follows, since Barnett's (1983) equivalence (bijective) results, illustrated by Lemma 1 above, prove isomorphism. In particular, if Definition 2 applies, then $f^*(\mathbf{x}) = f(\mathbf{x}) + O(\| \mathbf{x} - \mathbf{x}_0 \|^3)$ exists and is a flexible functional form in the sense defined by Definition 1. *Q. E. D.*

Define the expenditure shares in continuous time by $w_i(t) = \dfrac{p_i(t)x_i(t)}{\mathbf{p}(t)'\mathbf{x}(t)}$, and then define the differential equation for the growth rates of the quantity aggregate, $f(\mathbf{x})$, and the dual price (unit cost function) aggregates, $c(\mathbf{p})$, by

$$\frac{d \log f(\mathbf{x}(t))}{dt} = \sum_{i=1}^{n} w_i(t) \frac{d \log x_i(t)}{dt}, \tag{3.3a}$$

and

$$\frac{d \log c(\mathbf{p}(t))}{dt} = \sum_{i=1}^{n} w_i(t) \frac{d \log p_i(t)}{dt}. \tag{3.3b}$$

Equation (3.3a) can be shown to be derivable directly from the first order conditions from constrained optimization of f, under our assumption of linear homogeneity of f, with (3.3b) being the immediate dual. See, e.g., Divisia (1925). The solution to the differential equation, (3.3a), is the famous Divisia line integral:

$$f(\mathbf{x}(t)) = \oint_{\tau\in[0,\,t]} \left[\sum_{i=1}^{n} w_i(\tau) \frac{d \log x_i(\tau)}{d\tau} \right] d\tau \,,$$

which has been shown to be path independent by Hulten (1973) under our assumptions.

While Barnett et al. (2003) proposed proving convergence of index numbers in log change discrete time form to (3.3a,b) as $\Delta t \to 0$, we use a comparative statics differential form of (3.3a,b) in discrete time, as follows:

$$d \log f(\mathbf{x_t}) = \sum_{i=1}^{n} w_{i,t} d \log x_{i,t} \,, \tag{3.4a}$$

and

$$d \log c(\mathrm{p_t}) = \sum_{i=1}^{n} w_{i,t} d \log \mathrm{p}_{i,t} \,, \tag{3.4b}$$

where the expenditure shares in discrete time are $w_{i,t} = \dfrac{p_{i,t} x_{i,t}}{\mathbf{p_t}'\mathbf{x_t}'}$ for i =1, ..., n. The deriviation of (3.4a,b) is analogous to that of (3.3a,b). While (3.3a,b) is exactly correct in continuous time, equation (3.4a,b) is similarly exactly correct in discrete time. But it is important to recognize that (3.4a,b) is a comparative statics total differential regarding the effects of changes in variables during a single period of time, t. A mathematically equivalent representation of (3.4a,b) is:

$$\frac{\partial \log f(\mathbf{x_t})}{\partial \log x_{i,t}} = w_{i,t} \,, \tag{3.5a}$$

and

$$\frac{\partial \log c(\mathbf{p_t})}{\partial \log p_{i,t}} = w_{i,t} \,, \tag{3.5b}$$

for i = 1, ... ,n. While the continuous time form, (3.3a,b), is used in Barnett et al. (2003), we instead use the comparative statics discrete time form, (3.4a,b), particularly its Shephard's lemma implication, (3.5b).

We now have the following three definitions.

Definition 3: The class, S_1, of superlative index numbers is the set of functions Q, such that Q is exact for any linearly homogeneous flexible functional form.

Definition 4: The class, S_2, of index numbers is the set of functions Q, such that Q is exact for any linearly homogeneous flexible functional form *that can be expressed as an algebraic function in closed form*.

Definition 5: The class, S_3, of index numbers is the set of functions Q, such that Q is a second order discrete time approximation to any linearly homogeneous function.

Theorem 2: $S_1 = S_3$ and $S_2 \subset S_1$.

Proof: $S_1 = S_3$ follows immediately from Lemma 1, while $S_2 \subset S_1$ follows from Galois theory. *Q. E. D.*

As discussed above, Definition 3 is the definition of the class of superlative index numbers that is used by Diewert (1976,1978) in his search for superlative index numbers and is the definition that has been used in the literature since the appearance of Diewert's two seminal papers on superlative index numbers. While mathematically equivalent to our Definition 5, the operational version of S_1 that Diewert and others have applied is S_2, which is a strict subset of S_1. Our operational definition, S_3, spans all of the index numbers in the theoretical class S_1 and is not constrained by Galois theory to the strict subset, S_2, in applications.

In the following sections, we define a log-change index number with normalized symmetric mean weights. We call this class of indexes the Theil-Sato class. This class of indexes contains such important index numbers as the Sato-Vartia index, the Walsh index[8], and the Törnqvist index. We prove three lemmas and the main theorem, which is that the Theil-Sato index is a superlative index by our Definition 5 and hence (because of Theorem 2) by Diewert's Definition 3. Nevertheless, we do not know whether this index is in S_2, since no closed form algebraic flexible functional form has been found, for which this index is exact. But we have no need to determine such an exactly tracked closed-form flexible functional form, since the existence of such an intermediate function has no relevancy to the desired result on the order of the remainder term.

4. Normalized symmetric-mean-weight log-change index

In general, log-change indexes are characterized by their weight functions. We define the class of log-change indexes, P^{T-S}, between periods 0 and 1, with normalized symmetric mean weights as follows:

$$\ln P^{T-S} = \sum_{i=1}^{n} \frac{m(w_{i,1}, w_{i,0})}{\sum_{j=1}^{n} m(w_{j,1}, w_{j,0})} \ln \frac{p_{i,1}}{p_{i,0}}, \tag{4.1}$$

where $m(x, y)$ is a *symmetric mean*. As shown by Samuelson and Swamy (1974, p.582) and Diewert (1978, p. 897), the class of symmetric mean functions includes functions that are linearly homogeneous with the properties $m(x, y) = m(y, x)$, $m(x, x) = x$, and $\min(x, y) \le m(x, y) \le \max(x, y)$. This class of functions includes most mathematical means of two positive numbers, including the arithmetic, geometric, logarithmic, and harmonic means. Since the sum of symmetric mean weights, $m(w_{i,1}, w_{i,0})$, is not necessarily unity, we normalize those weights by their sum to produce a linear homogeneous price index, P^{T-S}. We call the index number, P^{T-S}, the Theil-Sato index, since this index number was first formalized in Theil (1973) and advocated by Sato (1974, 1976).

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[8] See Theil (1973).

4.1. Special Cases

To emphasize the importance of the Theil-Sato class of indexes, we provide some of its special cases. Each member of the normalized symmetric-mean-weight log-change indexes is characterized by its weight function:

$$w_i^{T-S} = \frac{m(w_{i,1}, w_{i,0})}{\sum_{j=1}^{n} m(w_{j,1}, w_{j,0})} . \tag{4.2}$$

As the functional form of $m(w_{i,1}, w_{i,0})$ varies, the Theil-Vartia becomes Walsh, Törnqvist, Sato-Vartia, among infinitely more. The following are examples.

The Walsh index in Theil (1973) is produced by using as $m(w_{i,1}, w_{i,0})$ the geometric mean $(w_{i,1}w_{i,0})^{1/2}$. The Törnqvist discrete time approximation to the Divisia index is acquired by setting $m(w_{i,1}, w_{i,0})$ to be the arithmetic mean $(w_{i,1} + w_{i,0})/2$. The Sato-Vartia index is acquired by setting $m(w_{i,1}, w_{i,0})$ to be the logarithmic mean function defined by Sato (1976) and Vartia (1976) as $\dfrac{w_{i,1} - w_{i,0}}{\log w_{i,1} - \log w_{i,0}}$.

Theil (1973, p.499) defined a special case acquired by setting $m(w_{i,1}, w_{i,0})$ to be $\left(\dfrac{w_{i,1} + w_{i,0}}{2} w_{i,1} w_{i,0}\right)^{1/3}$. Sato (1975, p. 551) defined another special case by setting $m(w_{i,1}, w_{i,0})$ to be $\dfrac{1}{3}\left(\dfrac{w_{i,1} + w_{i,0}}{2} + 2\sqrt{w_{i,1} w_{i,0}}\right)$.

5. Approximation order to the economic index numbers

The following quadratic approximation lemma was popularized in index number theory by Theil (1971; 1975, pp. 37-38):

$$f\left(\mathbf{x}+\mathbf{h}\right)-f\left(\mathbf{x}\right)=\sum_{i=1}^{n}\frac{1}{2}\left(\frac{\partial f\left(\mathbf{x}+\mathbf{h}\right)}{\partial x_i}+\frac{\partial f\left(\mathbf{x}\right)}{\partial x_i}\right)h_i+O\left(\left\|\mathbf{h}\right\|^3\right), \tag{5.1}$$

for $\mathbf{h} = (h_1, \ldots ,h_2)'$. This lemma is a variant of the first order Taylor expansion. Its usefulness results from the fact that the approximation order is increased by one by averaging the first-order derivatives evaluated at the two points **x** + **h** and **x**. Consequently, although (5.1) contains only first order derivatives, the remainder term is third order. Our Lemma 3 below shows that ability to raise the order of the approximation is retained, if we replace the arithmetic mean in (5.1) with the more general symmetric mean, to produce an extended version of the quadratic approximation lemma.

But first we need Lemma 2, providing the approximation property of symmetric means with respect to the arithmetic mean.

Lemma 2: The symmetric mean, $m(x_1,x_0)$, has the following approximation relationship with the arithmetic mean, $(x_1+x_0)/2$:

$$m(x_1,x_0)=\frac{x_1+x_0}{2}+O\left(\Delta x^2\right), \tag{5.2}$$

where $\Delta x = x_1 - x_0$.

Proof: The following relationship among means and growth rates is shown in Theil (1973, p.501):

$$x_1 = \bar{x}\left(1+\frac{g}{2}\right),\ x_0 = \bar{x}\left(1-\frac{g}{2}\right), \tag{5.3}$$

where $\bar{x} = (x_1 + x_0)/2$ is the arithmetic average and $g = 2(x_1 - x_0)/(x_1 + x_0) = \Delta x/\bar{x}$ is the growth rate of two numbers.

By using (5.3) and the linear homogeneity property of the mean function, the symmetric mean (5.2) can be written as

$$m(x_1, x_0) = \frac{x_1 + x_0}{2} m\left(1+\frac{g}{2}, 1-\frac{g}{2}\right). \tag{5.4}$$

But by the second order Taylor expansion at the point (1,1), we can write $m(1+g/2, 1-g/2)$ as

$$\begin{aligned} m\left(1+\frac{g}{2}, 1-\frac{g}{2}\right) &= m(1,1) + \begin{bmatrix} \frac{\partial m}{\partial x_1}(1,1) & \frac{\partial m}{\partial x_2}(1,1) \end{bmatrix} \begin{bmatrix} g/2 \\ -g/2 \end{bmatrix} + \\ &\frac{1}{2}\begin{bmatrix} g/2 & -g/2 \end{bmatrix} \begin{bmatrix} \frac{\partial^2 m}{\partial x_1{}^2}(1,1) & \frac{\partial m^2}{\partial x_1 \partial x_2}(1,1) \\ \frac{\partial m^2}{\partial x_1 \partial x_2}(1,1) & \frac{\partial^2 m}{\partial x_2{}^2}(1,1) \end{bmatrix} \begin{bmatrix} g/2 \\ -g/2 \end{bmatrix} + O\left(g^3\right) \\ &= 1 + \frac{1}{2}\frac{\partial^2 m}{\partial x_1{}^2}(1,1) g^2 = 1 + O\left(g^2\right) = 1 + O\left(\Delta x^2\right). \end{aligned} \tag{5.5}$$

In this result, we have used the following properties of symmetric means:[9] $\partial m/\partial x_1(1,1) = \partial m/\partial x_2(1,1) = 1/2$, $\partial m^2/\partial x_1{}^2(1,1) = \partial m^2/\partial x_2{}^2(1,1) = -\partial m^2/\partial x_1 \partial x_2(1,1)$, and $m(1,1) = 1$. *Q. E. D.*

[9] See, e.g., Diewert (1978, pp. 897-898).

Using the symmetric logarithmic mean, $L(x_1, x_0) = (x_1 - x_0)/(\log x_1 - \log x_0)$, defined by Vartia (1976) and Sato(1976), we provide a useful special case of Lemma 2. The following is the resulting relationship between the symmetric logarithmic mean and the arithmetic mean:

$$\begin{aligned}
L(x_1, x_0) &= \frac{x_1 + x_0}{2} L\left(1 + \frac{g}{2}, 1 - \frac{g}{2}\right) \\
&= \frac{x_1 + x_0}{2} \frac{\left[\left(1 + \frac{g}{2}\right) - \left(1 - \frac{g}{2}\right)\right]}{\left[ln\left(1 + \frac{g}{2}\right) - ln\left(1 - \frac{g}{2}\right)\right]} \\
&= \frac{x_1 + x_0}{2}\left(1 - \frac{1}{12} g^2 - \frac{1}{180} g^4 \ldots\right) = \frac{x_1 + x_0}{2} + O\left(\Delta x^2\right). (5.6)
\end{aligned}$$

We now are able to prove our extended quadratic approximation lemma.

Lemma 3:

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \sum_{i=1}^{n} m\left(\frac{\partial f(\mathbf{x} + \mathbf{h})}{\partial x_i} + \frac{\partial f(\mathbf{x})}{\partial x_i}\right) h_i + O\left(\|\mathbf{h}\|^3\right). \tag{5.7}$$

Proof: The following equation (5.8) is a direct application of lemma 2.

$$\begin{aligned}
\frac{1}{2}\left(\frac{\partial f(\mathbf{x} + \mathbf{h})}{\partial x_i} + \frac{\partial f(\mathbf{x})}{\partial x_i}\right) &= m\left(\frac{\partial f(\mathbf{x} + \mathbf{h})}{\partial x_i} + \frac{\partial f(\mathbf{x})}{\partial x_i}\right) + O\left[\left(\frac{\partial f(\mathbf{x} + \mathbf{h})}{\partial x_i} - \frac{\partial f(\mathbf{x})}{\partial x_i}\right)^2\right] \\
&= m\left(\frac{\partial f(\mathbf{x} + \mathbf{h})}{\partial x_i} + \frac{\partial f(\mathbf{x})}{\partial x_i}\right) + O\left(\|\mathbf{h}\|^2\right),
\end{aligned} \tag{5.8}$$

since

$$\frac{\partial f(\mathbf{x}+\mathbf{h})}{\partial x_i} - \frac{\partial f(\mathbf{x})}{\partial x_i} = \sum_{j=1}^{n} \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} h_j + O\left(\|\mathbf{h}\|^2\right) = O\left(\|\mathbf{h}\|\right). \tag{5.9}$$

Inserting the right hand side of (5.8) into (5.1), we obtain the following:

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \sum_{i=1}^{n} \left[m\left(\frac{\partial f(\mathbf{x}+\mathbf{h})}{\partial x_i} + \frac{\partial f(\mathbf{x})}{\partial x_i} \right) + O\left(\|\mathbf{h}\|^2\right) \right] h_i + O\left(\|\mathbf{h}\|^3\right)$$

$$= \sum_{i=1}^{n} m\left(\frac{\partial f(\mathbf{x}+\mathbf{h})}{\partial x_i} + \frac{\partial f(\mathbf{x})}{\partial x_i} \right) h_i + O\left(\|\mathbf{h}\|^3\right). \qquad \textit{Q. E. D.} \quad (5.10)$$

Prior to providing our main theorem, we need one more useful lemma about economic index numbers under optimizing behavior. The result provides the approximation order of the weight changes. In this result, as well as in our main theorem, we adopt the notation, $O_n \equiv O\left[\left\|\log \mathbf{p}_1 - \log \mathbf{p}_0\right\|^n\right]$, where we define the log of a vector as log $\mathbf{p}_i$ = (log $p_{1,i}$, ... ,log $p_{n,i}$)′.

Lemma 4: $\Delta w_i = w_{i,1} - w_{i,0} = O\left[\left\|\log \mathbf{p}_1 - \log \mathbf{p}_0\right\|\right] = O_1$.

Proof:

$$\Delta w_i = w_{i,1} - w_{i,0} = \frac{\partial \log c(\mathbf{p}_1)}{\partial \log p_i} - \frac{\partial \log c(\mathbf{p}_0)}{\partial \log p_i}$$

$$= \sum_{j=1}^{n} \frac{\partial^2 \log c(\mathbf{p}_0)}{\partial \log p_i \partial \log p_j} \left(\log p_{j,1} - \log p_{j,0} \right) + O_2 = O_1, \tag{5.11}$$

where we have used Shephard's lemma, (3.5b), to acquire the second equality. *Q. E. D.*

We now can provide our main theorem. Although it is produced for an economic price index, the same result applies to quantity aggregation, by symmetrically interchanging prices and quantities and replacing the unit cost function, *c*, with a quantity aggregator function, *f*.

Theorem 3: The Theil-Sato price index, $P^{T\text{-}S}$, defined by equation (4.1) can provide a second order approximation to any arbitrary unit cost function, where that "second order approximation" is in the following Definition 2 sense:

$$\log P^{T-S} = \log \frac{c(\mathbf{p}_1)}{c(\mathbf{p}_0)} + O_3, \tag{5.12}$$

where O_3 is the remainder denoted by $O_n \equiv O\left[\left\|\log \mathbf{p}_1 - \log \mathbf{p}_0\right\|^n\right]$.

Proof: Applying the extended quadratic approximation lemma, (5.7), to the logarithm change of the unit cost function between periods 0 and 1 and using Shephard's lemma, (3.5b), we obtain:

$$\begin{aligned} \log \frac{c(\mathbf{p}_1)}{c(\mathbf{p}_0)} &= \sum_{i=1}^{n} m\left(\frac{\partial \log c(\mathbf{p}_1)}{\partial \log p_i}, \frac{\partial \log c(\mathbf{p}_0)}{\partial \log p_i}\right)(\log p_{i,1} - \log p_{i,0}) + O_3 \\ &= \sum_{i=1}^{n} m(w_{i,1}, w_{i,0})(\ln p_{i,1} - \ln p_{i,0}) + O_3 . \end{aligned} \tag{5.13}$$

By Lemmas 2 and 3, we obtain the following:

$$\sum_{i=1}^{n} m(w_{i,1}, w_{i,0}) = \sum_{i=1}^{n} \frac{w_{i,1} + w_{i,0}}{2}\left[1 + O\left(\Delta w_i^{\,2}\right)\right]$$

$$= 1 + \sum_{i=1}^{n} \frac{w_{i,1} + w_{i,0}}{2} O\left(\Delta w_i^{\,2}\right)$$

$$= 1 + \sum_{i=1}^{n} \frac{w_{i,1} + w_{i,0}}{2} O\left[\left(O_1\right)^2\right]$$

$$= 1 + O_2 . \tag{5.14}$$

Multiplying the right hand side of (5.14) by $\sum_{j=1}^{n} m\left(w_{i,1}, w_{i,0}\right) \Big/ \sum_{j=1}^{n} m\left(w_{i,1}, w_{i,0}\right)$, we obtain the following:

$$\ln \frac{c\left(p^1\right)}{c\left(p^0\right)} = \sum_{i=1}^{n} \frac{m\left(w_{i,1}, w_{i,0}\right)}{\sum_{j=1}^{n} m\left(w_{j,1}, w_{j,0}\right)} \sum_{j=1}^{n} m\left(w_{j,1}, w_{j,0}\right) \ln \frac{p_{i,1}}{p_{i,0}} + O_3$$

$$= \sum_{i=1}^{n} \frac{m\left(w_{i,1}, w_{i,0}\right)}{\sum_{j=1}^{n} m\left(w_{j,1}, w_{j,0}\right)} (1 + O_2) \ln \frac{p_{i,1}}{p_{i,0}} + O_3$$

$$= \sum_{i=1}^{n} \frac{m\left(w_{i,1}, w_{i,0}\right)}{\sum_{j=1}^{n} m\left(w_{j,1}, w_{j,0}\right)} \ln \frac{p_{i,1}}{p_{i,0}} + O_3 .$$

$$= \ln P^{T-S} + O_3 .$$

Hence $\log P^{T-S} = \log \dfrac{c\left(\mathbf{p}_1\right)}{c\left(\mathbf{p}_0\right)} + O_3 .$ *Q. E. D.*

6. Concluding remarks

We provide an operational identification of the complete class of superlative index numbers. By this approach, we prove that an important family of log-change index numbers are superlative indexes. We call the class the Theil-Sato class or equivalently the normalized symmetric-mean-weight log-change index number family. As special cases, the class includes the Sato-Vartia, Törnqvist, and Walsh indexes.

Diewert (1976) showed that the quadratic mean of order r index number family is in the superlative index number class, but his approach to locating superlative index numbers is less general than ours, since his approach is subject to additional restrictions of Galois theory in practice. By our fully operational approach, we have successfully added the Theil-Sato index number family to the superlative class.

References.

Artin E, 1998, Galois Theory. Dover Publications: Mineola, New York; 1998. Reprinted from 1944 edition, University of Notre Dame Press.

Balk B M. Axiomatic price theory: a survey. International Statistical Review 1995; 63; 63-93

Barnett W A. Recursive subaggregation and a generalized hypocycloidal demand model. Econometrica 1977; 45; 1117-1136. Reprinted in: Barnett W A; Binner J M (Eds.), Functional structure and approximation in econometrics, North Holland: Amsterdam; 2004. p. 233-255.

Barnett W A. Theoretical foundations for the Rotterdam model. Review of Economic Studies 1979a; 46; 109-130. Reprinted in: Barnett W A; Binner J M (Eds.), Functional structure and approximation in econometrics, North Holland: Amsterdam; 2004. p. 9-39.

Barnett W A. The joint allocation of leisure and goods expenditure. Econometrica 1979b; 47; 539-563. Reprinted in: Barnett W A; Binner J M (Eds.), Functional structure and approximation in econometrics, North Holland: Amsterdam; 2004. p. 41-73.

Barnett W.A. Definitions of ‘second order approximations’ and of ‘flexible functional form.’ Economics Letters 1983; 12; 31-35. Reprinted in: Barnett W A; Binner J M (Eds.), Functional structure and approximation in econometrics, North Holland: Amsterdam; 2004. p. 75-78.

Barnett W A, Lee Y W. The global properties of the minflex Laurent, generalized Leontief, and translog flexible functional forms. Econometrica 1985; 53; 1421-1437. Reprinted in: Barnett W A; Binner J M (Eds.), Functional structure and approximation in econometrics, North Holland: Amsterdam; 2004. p. 79-97.

Barnett W A, Choi K H, Sinclair T M. The differential approach to superlative index number theory. Journal of Agricultural and Applied Economics (Henri Theil Memorial Issue) 2003; 35; 59-64.

Blackorby C, Davidson R, and Schworm W. Implicit separability: characterisation and implications for consumer demands. Journal of Economic Theory; 1991; 55; 364-399.

Diewert W E. Exact and superlative index numbers, Journal of Econometrics; 1976; 4; 115-145.

Diewert W E. Superlative index numbers and consistency in aggregation, Econometrica; 46; 1978; 883-900.

Divisia F. L'indice monétaire et la théorie de la monnaie. Revue d'Economie Politique 1925; 39; 980-1008.

Hulten C R. Divisia index numbers. Econometrica 1973; 63; 1017-1026.

Lau L J. On exact index numbers. Review of Economics and Statistics 1979; 61; 73-82.

Reinsdorf M B; Dorfman A H. The Sato-Vartia index and the monotonicity axiom. Journal of Econometrics 1999; 90; 45-61.

Sato K. Ideal index numbers that almost satisfy the factor reversal tests. The Review of Economics and Statistics 1974; 56; 549-552.

Sato K. The ideal log-change index number. The Review of Economics and Statistics 1976; 58; 223-228.

Theil H. Principles of econometrics. Wiley: New York; 1971.

Theil H. A new index number formula. The Review of Economics and Statistics 1973; 55; 498-502.

Theil H. Theory and measurement of consumer demand, vol 1. North-Holland: Amsterdam; 1975.

Vartia Y O. Ideal log-change index numbers. Scandinavian Journal of Statistics 1976; 3; 271-295.