Arbitrage and Equilibrium with Portfolio Constraints

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Abstract

We consider a multiperiod financial exchange economy with nominal assets and restricted participation, where each agent's portfolio choice is restricted to a closed, convex set containing zero, as in Siconolfi (1989). Using an approach that dates back to Cass (1984, 2006) in the unconstrained case, we seek to isolate arbitrage-free asset prices that are also quasi-equilibrium or equilibrium asset prices. In the presence of such portfolio restrictions, we need to confine our attention to *aggregate arbitrage-free* asset prices, i.e., for which there is no arbitrage in the space of marketed portfolios. Our main result states that such asset prices are quasi-equilibrium prices under standard assumptions and then deduce that they are equilibrium prices under a suitable condition on the accessibility of payoffs by agents, i.e., every payoff that is attainable in the aggregate can be marketed through some agent's portfolio set. This latter result extends previous work by Martins-da-Rocha and Triki (2005).

Keywords: Stochastic Financial exchange economies; Incomplete markets; Financial equilibrium; Constrained portfolios; Multiperiod models; Arbitrage-free asset prices

JEL Classification C62; D52; D53; G11; G12

1 Introduction

In financial markets, agents face several restrictions on what assets they can trade and the extent to which they can trade in these assets. Such constraints on agents' portfolios are not exceptional cases and can also explain why markets are incomplete. Some of the well known institutional restrictions are transactions costs, short sales constraints, margin requirements, frictions due to bid-ask spreads and taxes, collateral requirements, capital adequacy ratios and target ratios. Elsinger and Summer (2001) give an extensive discussion of these institutional constraints and how to model them in a general financial model. On the other hand, agents may be restricted due to some behavioral reasons. For instance, following Radner and Rothschild (1975), we can suppose that agents have limits on how much information they can process. This may cause each investor to concentrate on only a subset of assets to begin with.

Given that agents can face such restrictions on their portfolio choices, there are two ways in which such restrictions can be incorporated into a general financial model. The first is to assume that these restrictions are institutional, hence exogenously given, and we can take them as primitives of the model. In this paper we adopt this approach and consider very general restrictions on portfolio sets which are assumed to be closed, convex and contain zero, as in Siconolfi (1989). Such general portfolio sets are able to capture all the institutional restrictions listed earlier (see Elsinger and Summer (2001)). Alternatively, we can model these restrictions as arising endogenously, as in Cass et al. (2001) and more recently Carosi et al. (2009), and in a truly general model, this is what we would expect. Villanacci et al. (2002) summarize some earlier work in this direction.

In such models, the existence issue has been extensively studied since the seminal paper by Radner (1972). Duffie and Shafer (1985) showed a generic existence result with real assets, hence answering the issue raised by Hart's counterexample (Hart (1975)), with the drop of rank in the

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payoff matrix. An extensive body of literature is built upon their argument, see for example Geanakoplos and Shafer (1990), Hirsch et al. (1990), Husseini et al. (1990) and Bich and Cornet (2004, 2009). Another approach was to consider the cases of nominal or numéraire assets for which there is no drop of rank in the payoff matrix, hence no need for generic existence. Cass (1984, 2006), Duffie (1987), and Werner (1985) showed existence with nominal assets. Existence with numéraire assets was provided subsequently by Geanakoplos and Polemarchakis (1986). However, the presence of nominal assets will result in real indeterminacy at equilibrium. See (Balasko and Cass (1989), Geanakoplos and Mas-Colell (1989), Cass (1992), Dubey and Geanakoplos (2006). Polemarchakis and Siconolfi (1993), show that at noninformative prices restricted information can be modeled as restricted asset market participation and with nominal assets the same indeterminacy issue arises. Alternatively, Cornet and De Boisdeffre (2002, 2009) provide a model of sequential elimination of arbitrage states under asymmetric information. Magill and Shafer (1991) provide an extensive survey on the existence of financial markets equilibria and contingent markets equilibria.

In the case of purely financial securities, Cass (1984, 2006) was able to characterize equilibrium asset prices as arbitrage-free assets prices when some agent in the economy has no portfolio restriction. In a trick initiated by Cass (1984, 2006) and used subsequently by Werner (1985), Duffie (1987), and also by Duffie and Shafer (1985) and Bich and Cornet (2004) for pseudo-equilibria, the treatment of the agents is asymmetric with this (unconstrained) agent behaving competitively as in an Arrow-Debreu economy. A symmetric approach to the existence problem was considered by Radner (1972), Siconolfi (1989), Martins-da-Rocha and Triki (2005), Florig and Meddeb (2007), and Aouani and Cornet (2009), in the sense that no agent plays a particular role, hence the Cass trick is ruled out. Moreover, Martins-da-Rocha and Triki (2005) generalize Cass' characterization of equilibrium asset prices as arbitrage-free asset prices to the case where agents have portfolio restrictions. Although, in a strategic markets framework the market clearing asset prices themselves are determined by arbitrage activity and hence there may exist arbitrage activity even at equilibrium (see Koutsougeras and Papadoupoulos (2004)).

In this paper, we will consider a multiperiod model with nominal assets and restricted participation, where each agent's portfolio choice is restricted to a closed, convex set containing zero, as in Siconolfi (1989). The multiperiod model is better equipped to capture the evolution of time and uncertainty and is a first step before studying infinite horizon models. Following the pioneering model of Debreu (1959) we consider an event-tree to represent the evolution of time and uncertainty, and we will follow the presentation by Angeloni and Cornet (2006), which extends the standard model presented in Magill and Quinzii (1996). In the presence of portfolio restrictions, we need to confine our attention to *aggregate arbitrage-free* asset prices, i.e., for which there is no arbitrage opportunity in the space of marketed portfolios. Our main result states that aggregate arbitrage-free asset prices are accounts clearing quasi-equilibrium prices under standard assumptions. The notion of quasi-equilibrium is closely related to the one introduced by Gottardi and Hens (1996) in a two-date incomplete markets model without consumption in the first date, and then suitably modified by Seghir et al. (2004) to include consumption in the first date. However, we differ from these notions by assuming that only the financial accounts are cleared at quasi-equilibrium, instead of the standard portfolio clearing condition as in Gottardi and Hens (1996).

The paper is organized as follows. In Section 2, we describe the multiperiod model with restricted participation, we define the notions of equilibria with portfolio clearing and accounts clearing in the financial markets and we introduce the two notions of individual and aggregate arbitrage-free asset prices. In Section 3, we present our notion of accounts clearing quasi-equilibrium and discuss the relationship with the one introduced by Gottardi and Hens (1996) and Seghir et al. (2004). We state our main existence result of accounts clearing quasi-equilibria (Theorem 2) and present the way to go from quasi-equilibria to equilibria (Section 3.5). This allows us to deduce the existence of accounts clearing equilibria (Theorem 1) under a suitable condition on the accessibility of payoffs by agents, i.e., every payoff that is attainable in the aggregate can be marketed through some agent's portfolio set, a weaker version of the locally collectively frictionless condition by Martins-da-Rocha and Triki (2005). We also discuss the relationship with other results on the subject by Cass (1984, 2006), Werner (1985), Duffie (1987), in the unrestricted portfolio case and by Martins-da-Rocha and Triki (2005), and Angeloni and Cornet (2006) in the restricted portfolio case. The proof of our main result (Theorem 2) is given in Section 4.

2 The multiperiod financial exchange economy

2.1 Time and uncertainty in a multiperiod model

We [1] consider a multiperiod exchange economy with $(T+1)$ dates, $t \in \mathcal{T} := \{0, \dots, T\}$, and a finite set of agents $\mathcal{I} = \{1, ..., I\}$. The stochastic structure of the model is described by a finite event-tree $\mathcal{D} = \{0, 1, 2, ..., D\}$ of length T and we refer to Magill and Quinzii (1996) for a more detailed presentation together with an equivalent formulation with information partitions. The set $\mathcal{D}_t \subset \mathcal{D}$ denotes the nodes (also called date-events) that could occur at date t and the family $(\mathcal{D}_t)_{t \in \mathcal{T}}$ defines a partition of the set $\mathcal{D}$; for each $\xi \in \mathcal{D}$ we denote by $t(\xi)$ the unique time $t \in \mathcal{T}$ at which ξ can occur, i.e., such that $\xi \in \mathcal{D}_t$.

At each nonterminal date, $t \neq T$, there is an a priori uncertainty about which node will prevail in the next date. There is a unique non-stochastic event occurring at date $t = 0$, which is denoted by 0 so $\mathcal{D}_0 = \{0\}$. Finally, every $\xi \in \mathcal{D}_t, t \neq 0$ has a unique immediate predecessor in $\mathcal{D}_{t-1}$, denoted ξ^- or $pr^1(\xi)$, and the mapping $pr^1 : \mathcal{D} \setminus \{0\} \longrightarrow \mathcal{D}$ is assumed to satisfy $pr^1(\mathcal{D}_t) = \mathcal{D}_{t-1}$, for every $t \neq 0$. For each $\xi \in \mathcal{D}$, we let $\xi^+ = \{\bar{\xi} \in \mathcal{D} : \xi = \bar{\xi}^-\}$ be the set of immediate successors of ξ; we notice that the set ξ^+ is nonempty if and only if $\xi \notin \mathcal{D}_T$.

Moreover, for $\tau \geq 2$ and $\xi \in \mathcal{D} \setminus \bigcup_{t=0}^{\tau-1} \mathcal{D}_t$ we define, by induction, $pr^\tau(\xi) = pr^1(pr^{\tau-1}(\xi))$ and we let the set of (not necessarily immediate) successors and the set of predecessors of ξ be respectively defined by

$$\mathcal{D}^+(\xi) = \{\xi' \in \mathcal{D} : \exists \tau \in \mathcal{T} \setminus \{0\} \ : \ \xi = pr^\tau(\xi')\},$$

$$\mathcal{D}^-(\xi) = \{\xi' \in \mathcal{D} : \exists \tau \in \mathcal{T} \setminus \{0\} \ : \ \xi' = pr^\tau(\xi)\}.$$

If $\xi' \in \mathcal{D}^+(\xi)$ [resp. $\xi' \in \mathcal{D}^+(\xi) \cup \{\xi\}$], we shall also use the notation $\xi' > \xi$ [resp. $\xi' \geq \xi$]. We notice that $\mathcal{D}^+(\xi)$ is nonempty if and only if $\xi \notin \mathcal{D}_T$ and $\mathcal{D}^-(\xi)$ is nonempty if and only if $\xi \neq 0$. Moreover, one has $\xi' \in \mathcal{D}^+(\xi)$ if and only if $\xi \in \mathcal{D}^-(\xi')$ and similarly $\xi' \in \xi^+$ if and only if $\xi = (\xi')^-$.

2.2 The stochastic exchange economy

At each node $\xi \in \mathcal{D}$, there is a spot market where ℓ divisible physical goods are available. We assume that each physical good does not last for more than one period. In this model, a commodity is a couple (h, ξ) of a physical good $h \in \mathcal{H} := \{1, \dots, \ell\}$ and a node $\xi \in \mathcal{D}$ at which it will be available, so the commodity space is $\mathbb{R}^L$, where $L := \ell D$ and we let $\mathcal{L} = \mathcal{H} \times \mathcal{D}$.

An element x in $\mathbb{R}^L$ is called a *consumption* and we will use the notation $x = (x(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^L$, where $x(\xi) = (x_1(\xi), \dots, x_\ell(\xi)) \in \mathbb{R}^\ell$, denotes the spot consumption at node $\xi \in \mathcal{D}$. Similarly we denote by $p = (p(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^L$ the vector of spot prices and $p(\xi) = (p_1(\xi), \dots, p_\ell(\xi)) \in \mathbb{R}^\ell$ is called the spot price at node $\xi \in \mathcal{D}$. The spot price $p(h, \xi)$ is the price paid at time $t(\xi)$ for one unit of the physical good h at node ξ if this node prevails. Thus the value of the spot consumption $x(\xi)$ at node $\xi \in \mathcal{D}$ (evaluated in unit of account of node ξ) is

$$p(\xi) \bullet_\ell x(\xi) = \sum_{h=1}^{\ell} p_h(\xi) x_h(\xi).$$

[1] In this paper, we shall use the following notations. Let x, y be in $\mathbb{R}^n$, we denote by $x \cdot y := \sum_{i=1}^n x_i y_i$ the scalar product of $\mathbb{R}^n$, also denoted $x \bullet_n y$ when we use scalar products on different Euclidean spaces. We denote by $\|x\| := \sqrt{x \cdot x}$ the Euclidean norm of $\mathbb{R}^n$ and the closed ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$ is denoted $B_n(x, r) := \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$. We shall use the notation $x \geq y$ (resp. $x \gg y$) if $x_h \geq y_h$ (resp. $x_h \gg y_h$) for every $h = 1, \dots, n$ and $x > y$ means that $x \geq y$ and $x \neq y$. We let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$ and $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x \gg 0\}$.

Consider a $(D \times J)-$matrix A with D rows and J columns, with entries $A^j_\xi (\xi \in D, j \in J)$, we denote by A_ξ the ξ-th row of A (hence a row vector, i.e., a $(1 \times J)$-matrix, often identified to a vector in $\mathbb{R}^J$ when there is no risk of confusion) and A^j denotes the $j-$th column of A (hence a column vector, i.e., a $(D \times 1)$-matrix, similarly often identified to a vector in $\mathbb{R}^D$). Again if there is no risk of confusion, we will use the same notation for the $(D \times J)-$matrix A and the associated linear mapping $A : \mathbb{R}^J \to \mathbb{R}^D$. We recall that the transpose of A is the unique $(J \times D)-$matrix denoted by A^T satisfying $(A^T)^\xi_j = A^j_\xi (\xi \in D, j \in J)$, which in terms of linear mapping can be formulated as $(Ax) \bullet_D y = x \bullet_J (A^T y)$, for every $x \in \mathbb{R}^J$, $y \in \mathbb{R}^D$. We shall denote by rank A the rank of the matrix A, by ker A the kernel of A, that is, the set $\{x \in \mathbb{R}^J : Ax = 0\}$, and by Im A the image of A, that is, the set $\{Ax : x \in \mathbb{R}^J\}$.

Each agent $i \in \mathcal{I}$ is endowed with a *consumption set* $X_i \subset \mathbb{R}^L$, which is the set of her possible consumptions. The tastes of each consumer $i \in \mathcal{I}$ are represented by a *strict preference correspondence* P_i from $\prod_{j \in \mathcal{I}} X_j$ to X_i, where $P_i(x)$ defines the set of consumptions that are strictly preferred by i to x_i, given the consumptions x_j for the other consumers $j \neq i$. Thus P_i represents the tastes of consumer i but also her behavior under time and uncertainty, in particular her impatience and her attitude towards risk. If consumers' preferences are represented by utility functions $u_i : X_i \longrightarrow \mathbb{R}$, the strict preference correspondence is defined by $P_i(x) = \{\bar{x}_i \in X_i : u_i(\bar{x}_i) > u_i(x_i)\}$. Finally, at each node $\xi \in \mathcal{D}$,

every consumer $i \in \mathcal{I}$ has a *node-endowment* $e_i(\xi) \in \mathbb{R}^\ell$ (contingent to the fact that ξ prevails) and we denote by $e_i = (e_i(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^L$ her *endowment vector* across the different nodes. The exchange economy $\mathcal{E}$ can thus be summarized by

$$\mathcal{E} = (\mathcal{D}, \ell, \mathcal{I}, (X_i, P_i, e_i)_{i \in \mathcal{I}}).$$

2.3 The financial structure

We consider finitely many financial assets and we denote by $\mathcal{J} = \{1, ..., J\}$ the set of assets. An asset $j \in \mathcal{J}$ is a contract, which is issued at a given and unique node in $\mathcal{D}$, denoted by $\xi(j)$ and called the *emission node* of j. Each asset j is bought (or sold) at its emission node $\xi(j)$ and only yields payoffs at the successor nodes ξ' of $\xi(j)$, that is, for $\xi' > \xi(j)$. In this basic financial structure and in the statement of our main existence result, we only consider nominal assets, that is the payoffs do not depend on the spot price vector p (and only in the existence proof of Section 4 will we need to consider an additional real asset).

For the sake of convenient notations, we shall in fact consider payoffs of asset j at every node $\xi \in \mathcal{D}$ and assume that it is zero if ξ is not a successor of the emission node $\xi(j)$. Formally, we denote by V_ξ^j the payoff of asset j at node $\xi \in D$, by $V^j = (V_\xi^j)_{\xi \in D} \in \mathbb{R}^D$ its payoff across all nodes and we assume that $V_\xi^j = 0$ if $\xi \notin \mathcal{D}^+(\xi(j))$. With the above convention, we notice that every asset has a zero payoff at the initial node, that is, $V_0^j = 0$ for every $j \in \mathcal{J}$. Furthermore, every asset j which is emitted at the terminal date T has a zero payoff, that is, if $\xi(j) \in \mathcal{D}_T$, $V_\xi^j = 0$ for every $\xi \in \mathcal{D}$. The price of asset j is denoted by q_j and we recall that it is paid at its emission node $\xi(j)$. We let $q = (q_j)_{j \in \mathcal{J}} \in \mathbb{R}^J$ be the asset price (vector).

For every consumer $i \in \mathcal{I}$, we denote by $z_i = (z_i^j)_{j \in \mathcal{J}} \in \mathbb{R}^J$ the *portfolio* of agent i and we make the following convention: if $z_i^j > 0$ [resp. $z_i^j < 0$], then $|z_i^j|$ denotes the quantity of asset $j \in \mathcal{J}$ bought [resp. sold] by agent i at the emission node $\xi(j)$. We assume that each consumer $i \in \mathcal{I}$ is endowed with a *portfolio set* $Z_i \subset \mathbb{R}^J$, which represents the set of portfolios that are admissible for agent i. If some agent $i \in \mathcal{I}$ has no constraints on her portfolio choices then $Z_i = \mathbb{R}^J$. Throughout this paper we consider portfolio sets that are closed, convex and contain zero for every agent, a framework general enough to cover most of the constraints considered in the literature (see Elsinger and Summer (2001)).

To summarize, the financial asset structure $\mathcal{F} = (\mathcal{J}, (\xi(j), V^j)_{j \in \mathcal{J}}, (Z_i)_{i \in \mathcal{I}})$ consists of

- A finite set of assets $\mathcal{J}$ and each asset $j \in \mathcal{J}$ is characterized by its node of issue $\xi(j) \in \mathcal{D}$ and its payoff vector $V^j = (V_\xi^j)_{\xi \in D} \in \mathbb{R}^D$, with $V_\xi^j = 0$ if $\xi \notin \mathcal{D}^+(\xi(j))$.
- The portfolio set $Z_i \subset \mathbb{R}^J$ for every agent $i \in \mathcal{I}$.
- The *space of marketed portfolios* $\mathcal{Z}_\mathcal{F} := \left\langle \bigcup_{i \in \mathcal{I}} Z_i \right\rangle$, that is, the linear space in which portfolio activity of the economy takes place[2].
- The payoff matrix of $\mathcal{F}$ is the $D \times J$ matrix having V^j ($j = 1, \ldots, J$) for column vectors.
- The full payoff matrix $W(q)$ is the $(D \times J)-$matrix with entries $W_\xi^j(q) := V_\xi^j - \delta_{\xi, \xi(j)} q_j$, where $\delta_{\xi,\xi'} = 1$ if $\xi = \xi'$ and $\delta_{\xi,\xi'} = 0$ otherwise.

[2]Given a subset $A \subset \mathbb{R}^n$ we denote by $\langle A \rangle :=$ span A, the linear space spanned by A.

So for a given portfolio $z \in \mathbb{R}^J$ (and asset price q) the full flow of payoffs across all nodes is $W(q)z$, a vector in $\mathbb{R}^D$ whose ξ-th component is the (full) financial payoff at node ξ, that is

$$[W(q)z](\xi) = W(q,\xi) \bullet_J z = \sum_{j \in \mathcal{J}} V_\xi^j z_j - \sum_{j \in \mathcal{J}} \delta_{\xi,\xi(j)} q_j z_j = \sum_{\{j \in \mathcal{J}\ :\ \xi(j) < \xi\}} V_\xi^j z_j - \sum_{\{j \in \mathcal{J}:\ \xi(j) = \xi\}} q_j z_j.$$

2.4 Equilibria of the financial economy

We now consider a financial exchange economy $(\mathcal{E}, \mathcal{F})$, which is defined as the couple of an exchange economy $\mathcal{E}$ and a financial structure $\mathcal{F}$ as described above. Given the price $(p,q) \in \mathbb{R}^L \times \mathbb{R}^J$, the *budget set* of consumer $i \in \mathcal{I}$ is[3]

$$\begin{aligned} B_i(p,q) &= \{(x_i, z_i) \in X_i \times Z_i :\ \forall \xi \in \mathcal{D},\ \ p(\xi) \bullet_\ell (x_i(\xi) - e_i(\xi)) \le [W(q)z_i](\xi)\} \\ &= \{(x_i, z_i) \in X_i \times Z_i :\ p \Box (x_i - e_i) \le W(q) z_i\}. \end{aligned}$$

We now introduce the standard notion of portfolio clearing equilibrium and an alternative definition, called accounts clearing equilibrium that we will adhere to in this paper. The latter equilibrium notion is called *weak equilibrium* by Martins-da-Rocha and Triki (2005). Note that the following definitions are slightly more general than the standard ones in a multiperiod model and we refer to Angeloni and Cornet (2006) for the relationship with the standard definitions as defined in Magill and Quinzii (1996).

Definition 2.1 *A* portfolio clearing equilibrium *(resp.* accounts clearing equilibrium*) of $(\mathcal{E}, \mathcal{F})$ is a list of strategies and prices $\left(\bar{x}, \bar{z}, \bar{p}, \bar{q}\right) \in (\mathbb{R}^L)^I \times (\mathbb{R}^J)^I \times \mathbb{R}^L \times \mathbb{R}^J$ such that $\bar{p} \neq 0$ and*

(a) every consumer $i \in \mathcal{I}$ maximizes her preference under the budget constraint $B_i(\bar{p}, \bar{q})$, that is,

$(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q})$ and $B_i(\bar{p}, \bar{q}) \cap [P_i(\bar{x}) \times Z_i] = \emptyset$;

(b) $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e_i$; [Commodity market clearing condition]

(c) $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$ (resp. $\sum_{i \in \mathcal{I}} W(\bar{q}) \bar{z}_i = 0$). [Portfolio market (resp. accounts) clearing condition]

In the above definition, an accounts clearing equilibrium only requires that the payoffs (or accounts) of the financial markets are cleared, that is $\sum_{i \in \mathcal{I}} W(\bar{q}) \bar{z}_i = 0$, which is weaker than the portfolio clearing condition: $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$. The relationship between the two equilibrium notions is given in the next proposition.

Proposition 2.1 (a) *Every portfolio clearing equilibrium of the economy $(\mathcal{E}, \mathcal{F})$ is an accounts clearing equilibrium of $(\mathcal{E}, \mathcal{F})$.*

(b) *Conversely, let $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ be an accounts clearing equilibrium of $(\mathcal{E}, \mathcal{F})$ satisfying one of the following conditions (i) $\bigcup_{i \in \mathcal{I}} Z_i$ is a vector space, or (ii) $\ker W(q) \subset \bigcup_{i \in \mathcal{I}} Z_i$, then there exists a portfolio clearing equilibrium $(\bar{x}, \hat{z}, \bar{p}, \bar{q})$ of $(\mathcal{E}, \mathcal{F})$ which differs only in terms of the portfolio profile.*

The proof of Proposition 2.1 is given in the Appendix (Section 5.1). We now give some examples in which the above Conditions (i) or (ii) are satisfied.

Example 2.1 (Cass Condition) Following Cass (1984, 2006), the set $\bigcup_{i \in \mathcal{I}} Z_i$ is a vector space if the three following conditions hold:

- for some $i_0 \in \mathcal{I}$, Z_{i_0} is a vector space,
- for every $i \in \mathcal{I}$, Z_i is closed, convex, contains zero, and $Z_i \subset Z_{i_0}$.

Example 2.2 *At least one of the above Conditions (i) or (ii) (in Proposition 2.1) is true when one of the following holds*[4]*:*

[3] For $x = (x(\xi))_{\xi \in \mathcal{D}}, p = (p(\xi))_{\xi \in \mathcal{D}}$ in $\mathbb{R}^L = \mathbb{R}^{\ell D}$ (with $x(\xi), p(\xi) \in \mathbb{R}^\ell$) we let $p \Box x = (p(\xi) \bullet_\ell x(\xi))_{\xi \in \mathcal{D}} \in \mathbb{R}^D$.

[4] For every closed convex subset $Z \subset \mathbb{R}^n$ we denote by $\boldsymbol{A}Z := \{\eta \in \mathbb{R}^n\ :\ Z + \eta \subset Z\}$ the asymptotic cone of Z.

- $\bigcup_{i\in\mathcal{I}} Z_i = \mathbb{R}^J$;
- $\ker W(q) = \{0\}$;
- $\ker W(q) \subset \bigcup_{i\in\mathcal{I}} \boldsymbol{A}Z_i$ *(Martins-da-Rocha and Triki (2005)).*[5]

2.5 Arbitrage and equilibrium

Each agent faces different constraints on portfolios, so the arbitrage opportunities that open up to individuals will be different from one another and different from what may be available to the market as a whole. Hens et al. (2006) show this distinction in a 2-date model with linear portfolio sets and Cornet and Gopalan (2006) extend this result to a multiperiod model. A portfolio $\bar{z}_i \in Z_i$ is said to *have no arbitrage opportunities* (or to be *arbitrage-free*) for agent $i \in \mathcal{I}$ at the price $\bar{q} \in \mathbb{R}^J$ if there is no portfolio $z_i \in Z_i$ such that $W(\bar{q})z_i > W(\bar{q})\bar{z}_i$. It is well known that, under a standard non-satiation assumption[6], at equilibrium there is no arbitrage at the individual level, that is, if $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an equilibrium of the economy $(\mathcal{E}, \mathcal{F})$, then $\bar{z}_i$ is arbitrage-free at $\bar{q}$ for every $i \in \mathcal{I}$ (see Angeloni and Cornet (2006)). However, in this paper, we will confine our attention to the stronger notion of *aggregate arbitrage-free* asset price, i.e., for which there is no arbitrage in the space of marketed portfolios.

Definition 2.2 *We say that the asset price $\bar{q} \in \mathbb{R}^J$ is aggregate arbitrage-free if one of the following equivalent conditions hold:*
(i) $W(\bar{q})\mathcal{Z}_{\mathcal{F}} \cap \mathbb{R}^D_+ = \{0\}$;
(ii) *There exists $\lambda \in \mathbb{R}^D_{++}$ such that $\lambda \bullet_D W(\bar{q})z = 0$ for all $z \in \mathcal{Z}_{\mathcal{F}}$.*

Notice that if $\bar{q}$ is aggregate arbitrage-free and $\sum_{i\in\mathcal{I}} \bar{z}_i = 0$, then $(\bar{q}, \bar{z}_i)$ is arbitrage-free for each agent $i \in \mathcal{I}$. The converse is true in particular if some agent is unconstrained, i.e., if $Z_i = \mathcal{Z}_{\mathcal{F}}$ for some agent $i \in \mathcal{I}$.

3 Existence of equilibria and quasi-equilibria

3.1 Existence of accounts clearing equilibria

We posit the main assumptions on the consumption side of the economy and we first recall the definition of the set $\widehat{X}$ of attainable consumptions

$$\widehat{X} = \{x \in \prod_{i\in\mathcal{I}} X_i \ : \ \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i\}.$$

Assumption C (**Consumption Side**) *For all $i \in \mathcal{I}$ and all $\bar{x} \in \prod_{i\in\mathcal{I}} X_i$,*

(i) *[Consumption Sets] $X_i \subset \mathbb{R}^L$ is closed, convex and bounded below by $\underline{x}_i$;*

(ii) *[Continuity] the preference correspondence P_i from $\prod_{i\in\mathcal{I}} X_i$ to X_i, is lower semicontinuous;*[7]

(iii) *[Openness-type] for every $x_i \in P_i(\bar{x})$ for every $x'_i \in X_i, x'_i \neq x_i$ then $[x'_i, x_i) \cap P_i(\bar{x}) \neq \emptyset$;*[8]

(iv) *[Convexity] $P_i(\bar{x})$ is convex;*

[5] A still weaker condition is provided in Martins-da-Rocha and Triki (2005).

[6] That is, Assumptions C (vi) and K (ii) defined below.

[7] A correspondence Φ from X to Y is said to be lower semicontinuous at $x_0 \in X$ if, for every open set $U \subset Y$ such that $\Phi(x_0) \cap U$ is not empty, there exists a neighborhood N of x_0 in X such that, for all $x \in N$, $\Phi(x) \cap U$ is nonempty. The correspondence Φ is said to be lower semicontinuous if it is lower semicontinuous at each point of X.

[8] This is satisfied, in particular, when $P_i(\bar{x})$ is open in X_i (for its relative topology). However, Proposition 2 (used in the proof of Theorem 2) is not true in general when one replaces the opennes-type assumption by the assumption that P_i has open values: see Bich and Cornet (2004) for a counter-example.

(v) *[Irreflexivity]* $\bar{x}_i \notin P_i(\bar{x})$;

(vi) *[Non-Satiation at Every Node]* $\forall \bar{x} \in \widehat{X}$, $\forall \xi \in \mathcal{D}$, $\exists x_i(\xi) \in \mathbb{R}^\ell$, $(x_i(\xi), \bar{x}_i(-\xi)) \in P_i(\bar{x})$; [9]

(vii) *[Survival Assumption] For all* $i \in \mathcal{I}$, $e_i \in X_i$.

Note that these assumptions on P_i are satisfied in particular when agents preferences are given by a utility function that is continuous, strongly monotonic, and quasi-concave.

Assumption S (Strong Survival Assumption) *For all* $i \in \mathcal{I}$, $e_i \in$ *int* X_i.

Let $\bar{q} \in \mathbb{R}^J$, we consider the following assumptions on the financial side of the economy . The first one need no additional comment and the second one is discussed in the next section.

Assumption F (Financial Side) *For all* $i \in \mathcal{I}$, Z_i *is closed, convex,* $0 \in Z_i$, *and* $W(\bar{q})Z_i$ *is closed.*[10]

Financial Accessibility FA: *The closed cone spanned by* $\bigcup_{i \in \mathcal{I}} W(\bar{q})(Z_i)$ *is a linear space.*

Our first result states that every aggregate arbitrage-free asset price will also be an equilibrium price under the previous assumptions.

Theorem 3.1 *Let* $\bar{q} \in \mathbb{R}^J$ *be aggregate arbitrage-free and suppose* $(\mathcal{E}, \mathcal{F})$ *satisfies Assumptions* **C**, **S**, **F**, *and* **FA**. *Then there exists* $(\bar{x}, \bar{z}, \bar{p})$ *such that* $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ *is an accounts clearing equilibrium.*

The proof of Theorem 3.1 is given in Section 3.5 as a consequence of a more general result on the existence of quasi-equilibria, stated in Section 3.4.

3.2 Financial and Portfolio Accessibility Conditions

The Financial Accessibility Condition **FA** requires that a fraction of any payoff in the aggregate is accessible by some agent. In the unrestricted case, the Strong Survival Assumption **S** guarantees sufficient wealth accessibility to agents in order to establish the existence of an equilibrium (see Cass (1984, 2006), Duffie (1987), and Werner (1985)). However, with restricted participation in asset markets, even with Assumption **S**, there is a need for an accessibility condition on the financial side as well: see Martins-da-Rocha and Triki (2005) and Angeloni and Cornet (2006) (and the Conditions **FA2'** and **FA3** below). It is worth pointing out that Assumption **FA** is satisfied under the following portfolio accessibility condition

Portfolio Accessibility PA: *The closed cone spanned by* $\bigcup_{i \in \mathcal{I}} Z_i$ *is a linear space.*

This condition is discussed in the next example and represented in Figure 1, together with two stronger conditions, also of interest.

Example 3.1 *: [Portfolio Accessibility]* Consider two assets and two agents. The second asset can be bought only by the first agent and sold only by the second agent. The first asset is unconstrained for both agents in Figure (a) and for the second agent in Figure (b); it is constrained in the other cases.

- The set $\bigcup_{i \in \mathcal{I}} Z_i$ *is a linear space*, as in Figure 1 (a).
- *The cone spanned by the set* $\bigcup_{i \in \mathcal{I}} Z_i$ *is a linear space*, as in Figure 1 (b),
- *The closed cone spanned by the set* $\bigcup_{i \in \mathcal{I}} Z_i$ *is a linear space*, as in Figure 1 (c),

[9] Given $\xi \in \mathcal{D}$, we denote $x_i(-\xi) := (x_i(\xi'))_{\xi' \neq \xi}$.

[10] Note that this last assumption is satisfied when each portfolio set Z_i is polyhedral, that is, it is defined by linear inequality and equality constraints (see Rockafellar (1970)).

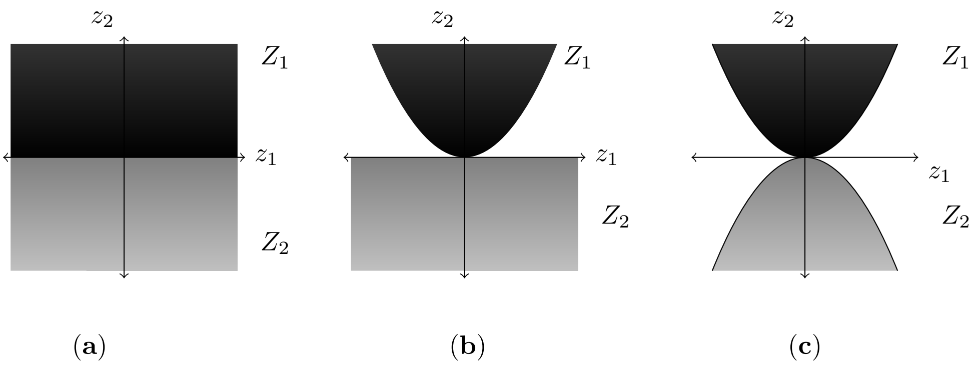

Figure 1: Portfolio accessibility

3.3 Some consequences of Theorem 3.1

We now give some consequences of Theorem 3.1 and show the relationship with the literature on this subject under different financial accessibility conditions.

Corollary 3.1 (Cass (1984, 2006), Werner (1985), Duffie (1987)) *Let $\bar{q} \in \mathbb{R}^J$ be aggregate arbitrage-free. Suppose $(\mathcal{E}, \mathcal{F})$ satisfies* **C**, **F** *and* **S**, *together with*
FA1: *there exists $i_0 \in \mathcal{I}$ such that $Z_{i_0} = \mathbb{R}^J$.*
Then there exists $(\bar{x}, \bar{z}, \bar{p})$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a portfolio clearing equilibrium.

Corollary 3.2 (Cass (1984, 2006)) *Let $\bar{q} \in \mathbb{R}^J$ be aggregate arbitrage-free. Suppose $(\mathcal{E}, \mathcal{F})$ satisfies* **C**, **F** *and* **S**, *together with*
FA2: *for some $i_0 \in \mathcal{I}$, Z_{i_0} is a linear space and for all $i \in \mathcal{I}$, $Z_i \subset Z_{i_0}$.*
Then there exists $(\bar{x}, \bar{z}, \bar{p})$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is a portfolio clearing equilibrium.

Corollary 3.3 (Angeloni and Cornet (2006)) *Let $\bar{q} \in \mathbb{R}^J$ be aggregate arbitrage-free. Suppose $(\mathcal{E}, \mathcal{F})$ satisfies* **C**, **F** *and* **S**, *together with*
FA2′: *there exists $i \in \mathcal{I}$ such that $0 \in$ int Z_i.*
Then there exists $(\bar{x}, \bar{z}, \bar{p})$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an accounts clearing equilibrium.

Corollary 3.4 (Martins-da-Rocha and Triki (2005)) *Let $\bar{q} \in \mathbb{R}^J$ be aggregate arbitrage-free. Suppose $(\mathcal{E}, \mathcal{F})$ satisfies* **C**, **F** *and* **S**, *together with*
FA3: *the cone spanned by $\left(\bigcup_{i \in \mathcal{I}} W(\bar{q}) Z_i\right)$ is equal to Im $W(\bar{q})$.*
Then there exists $(\bar{x}, \bar{z}, \bar{p})$ such that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an accounts clearing equilibrium.

The proofs of the above corollaries are direct consequences of Theorem 3.1 and one only needs to check the following two sets of implications: [**FA1** $\Rightarrow$ **FA2** $\Rightarrow$ **FA3** $\Rightarrow$ **FA**], and [**FA2′** $\Rightarrow$ **FA**].

3.4 Existence of accounts clearing quasi-equilibria

We now consider a mapping $\gamma : \mathbb{R}^L \to \mathbb{R}^D$ and define the associated *γ-budget sets* and *γ-quasi-budget sets* defined for $(p, q) \in \mathbb{R}^L \times \mathbb{R}^J$. The most simple choice of such a function is to take $\gamma = 0$ (see Remark 3.1 below) but it is not general enough for our purpose (see the next Section 3.5). The choice of the function γ to enlarge the budget sets, plays the same role as the mapping α to define the modified budget sets of Martins-da-Rocha and Triki (2005).

$$B_i^\gamma(p,q) = \{(x_i, z_i) \in X_i \times Z_i : \exists \tau_i \in [0,1], p \square (x_i - e_i) \leq W(q) z_i + \tau_i \gamma(p)\},$$
$$\breve{B}_i^\gamma(p,q) = \{(x_i, z_i) \in X_i \times Z_i : \exists \tau_i \in [0,1], p \square (x_i - e_i) \ll W(q) z_i + \tau_i \gamma(p)\}.$$

Definition 3.1 *An* accounts clearing γ-quasi-equilibrium *of* $(\mathcal{E},\mathcal{F})$ *is a list* $\big(\bar{x},\bar{z},\bar{p},\bar{q}\big)\in(\mathbb{R}^L)^I\times(\mathbb{R}^J)^I\times\mathbb{R}^L\times\mathbb{R}^J$ *such that* $\bar{p}\neq 0$ *and*

(a-i) for every $i\in\mathcal{I}$, $(\bar{x}_i,\bar{z}_i)\in B_i(\bar{p},\bar{q})$;

(a-ii) $\breve{B}_i^{\gamma}(\bar{p},\bar{q})\neq\emptyset\Rightarrow B_i^{\gamma}(\bar{p},\bar{q})\cap[P_i(\bar{x})\times Z_i]=\emptyset$;

(b) $\sum_{i\in\mathcal{I}}\bar{x}_i=\sum_{i\in\mathcal{I}}e_i$ *and* $\sum_{i\in\mathcal{I}}W(\bar{q})\bar{z}_i=0$.

Our main result states.

Theorem 3.2 *Let* $\bar{q}\in\mathbb{R}^J$ *be aggregate arbitrage-free, let* $\lambda\in\mathbb{R}^D_{++}$ *be an associated state price (as in Definition 2.2), assume that* $(\mathcal{E},\mathcal{F})$ *satisfies* **C**, **F** *and that* $\gamma:\mathbb{R}^L\to\mathbb{R}^D$ *is a continuous mapping satisfying*

Assumption $\boldsymbol{\Gamma_1(\bar{q})}$ **:** $\forall p\in\mathbb{R}^L,\ \gamma(p)\cdot\lambda=0$ *and* $\gamma(p)\cdot[W(\bar{q})z_i]=0$ *for all* $z_i\in Z_i$ *and all* $i\in\mathcal{I}$.

Then there exists $(\bar{x},\bar{z},\bar{p})$ *such that* $(\bar{x},\bar{z},\bar{p},\bar{q})$ *is an accounts clearing* γ*-quasi-equilibrium.*

The proof of Theorem 3.2 is given in Section 4.

Remark 3.1 *When* $\gamma=0$, $B_i^{\gamma}(p,q)=B_i(p,q)$ *and the* γ*-quasi-budget set* $\breve{B}_i^{\gamma}(p,q)$ *is simply denoted* $\breve{B}_i(p,q)$*, that is*

$$\breve{B}_i(p,q)=\{(x_i,z_i)\in X_i\times Z_i:\ p\Box(x_i-e_i)\ll W(q)z_i\}.$$

Thus, when $\gamma=0$*, an accounts clearing* γ*-quasi-equilibrium is simply called an accounts clearing quasi-equilibrium, and Condition (a-ii) states that*

$$\breve{B}_i(\bar{p},\bar{q})\neq\emptyset\Rightarrow B_i(\bar{p},\bar{q})\cap[P_i(\bar{x})\times Z_i]=\emptyset. \tag{1}$$

Remark 3.2 *For every mapping* γ*, we have* $B_i(p,q)\subset B_i^{\gamma}(p,q)$ *and* $\breve{B}_i(p,q)\subset\breve{B}_i^{\gamma}(p,q)$ *(taking* $\tau_i=0$*) and equality holds when* $\gamma=0$*. Consequently, every accounts clearing* γ*-quasi-equilibrium is an accounts clearing quasi-equilibrium*[11] *and it is an accounts clearing equilibrium whenever we know that* $\breve{B}_i^{\gamma}(\bar{p},\bar{q})\neq\emptyset$ *for all* i.

Thus, the choice of γ *gives some flexibility to parametrize accounts clearing quasi-equilibria in a way that will be fully exploited hereafter to deduce Theorem 3.1 from Theorem 3.2.*

3.5 From quasi-equilibria to equilibria: proof of Theorem 3.1

In this section, we provide a proof of Theorem 3.1, as a direct consequence of Theorem 3.2. As explained previously in Remark 3.2, we only need to choose a suitable mapping γ so that the accounts clearing γ-quasi-equilibrium $(\bar{x},\bar{z},\bar{p},\bar{q})$ that we get from Theorem 3.2, satisfies the property that $\breve{B}_i^{\gamma}(\bar{p},\bar{q})\neq\emptyset$ for every $i\in\mathcal{I}$. This is possible by choosing γ as in the following lemma, in a way similar to Martins-da-Rocha and Triki (2005). The proof of the lemma is given in the appendix.

Lemma 3.1 *Let* $\bar{q}$ *be aggregate arbitrage-free, let* $\lambda\in\mathbb{R}^D_{++}$ *be an associated state price vector. Under the assumptions of Theorem 3.1, there exists a continuous mapping* $\gamma:\mathbb{R}^L\to\mathbb{R}^D$ *satisfying* $\boldsymbol{\Gamma_1(\bar{q})}$ *and*

$\boldsymbol{\Gamma_2(\bar{q})}$ **:** $\forall\,\bar{p}\in\mathbb{R}^L,\bar{p}\neq 0,\exists\, i_0\in\mathcal{I},\ \textit{such that}\ \breve{B}_{i_0}^{\gamma}(\bar{p},\bar{q})\neq\emptyset.$

The end of the proof of Theorem 3.1 is a consequence of the following claim.

Claim 1 *(i) For all* $\xi\in\mathcal{D},\bar{p}(\xi)\neq 0$;
(ii) For all $i\in\mathcal{I},\breve{B}_i^{\gamma}(\bar{p},\bar{q})\neq\emptyset$.

[11] Note first that Condition $(a\text{-}i)$ in Definition 3.1 [$(\bar{x}_i,\bar{z}_i)\in B_i(\bar{p},\bar{q})$ for every $i\in\mathcal{I}$] is defined for the standard budget set (and not the γ-budget set) and second that the quasi-equilibrium Condition (1) is a consequence of the γ-quasi-equilibrium Condition $(a\text{-}ii)$, using the above inclusions, which imply that:

$[\breve{B}_i(\bar{p},\bar{q})\neq\emptyset\Rightarrow\breve{B}_i^{\gamma}(\bar{p},\bar{q})\neq\emptyset]$ and $[B_i^{\gamma}(\bar{p},\bar{q})\cap[P_i(\bar{x})\times Z_i]=\emptyset\Rightarrow B_i(\bar{p},\bar{q})\cap[P_i(\bar{x})\times Z_i]=\emptyset]$.

Proof. *Part (i).* From Lemma 3.1, recalling that $\bar{p} \neq 0$, there exists $i_0 \in \mathcal{I}$ such that $\breve{B}^{\gamma}_{i_0}(\bar{p}, \bar{q}) \neq \emptyset$. Hence $[P_{i_0}(\bar{x}) \times Z_{i_0}] \cap B^{\gamma}_{i_0}(\bar{p}, \bar{q}) = \emptyset$, (from the γ-quasi-equilibrium condition of the i_0-th agent). Suppose there exists $\xi \in \mathcal{D}$ such that $\bar{p}(\xi) = 0$. Since $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e_i$ (from the Market Clearing Condition), from the Non-Satiation Assumption **C (v)**, there exists $x_{i_0} \in P_{i_0}(\bar{x})$ such that $x_{i_0}(\xi') = \bar{x}_{i_0}(\xi')$ for every $\xi' \neq \xi$. Hence $\bar{p} \square (x_{i_0} - e_{i_0}) = \bar{p} \square (\bar{x}_{i_0} - e_{i_0})$, which together with $(\bar{x}_{i_0}, \bar{z}_{i_0}) \in B^{\gamma}_{i_0}(\bar{p}, \bar{q})$, implies that $(x_{i_0}, \bar{z}_{i_0}) \in B^{\gamma}_{i_0}(\bar{p}, \bar{q})$. Consequently $(x_{i_0}, \bar{z}_{i_0}) \in B^{\gamma}_{i_0}(\bar{p}, \bar{q}) \cap [P_{i_0}(\bar{x}) \times Z_{i_0}]$, which contradicts the fact that it is empty from above.

Part (ii). From Part (i) of this Claim, $\bar{p}(\xi) \neq 0$, for all $\xi \in \mathcal{D}$ and we notice that $\bar{p} \square \bar{p} \gg 0$. Taking $x_i = e_i - t\bar{p}$, for $t > 0$ small enough, we deduce that $x_i \in \operatorname{int} X_i$ (since $e_i \in \operatorname{int} X_i$ from the Strong Survival Assumption **S**). Thus $\bar{p} \square (x_i - e_i) = -t(\bar{p} \square \bar{p}) \ll 0 + 0\gamma(\bar{p})$ (taking $\bar{z}_i = 0 \in Z_i, \bar{\tau}_i = 0$). This shows that $(x_i, 0) \in \breve{B}^{\gamma}_i(\bar{p}, \bar{q}) \neq \emptyset$. □

3.6 Other definitions of quasi-equilibrium

Gottardi and Hens (1996) consider a two-date incomplete markets model without consumption in the first date $t = 0$. Their definition of a quasi-equilibrium, suitably modified by Seghir et al. (2004) to include consumption in first date is presented below. In this section we consider a two-date model with S states at the second date $t = 1$, that is, $\mathcal{D} = \{0, 1, \ldots S\}$. Hereafter we denote by $p \square_1 x$ the vector $(p(1) \bullet_{\ell} x(1), \ldots, p(S) \bullet_{\ell} x(S)) \in \mathbb{R}^S$.

Definition 3.2 *A list of strategies and prices* $(\bar{x}, \bar{z}, \bar{p}, \bar{q}) \in (\mathbb{R}^L)^I \times (\mathbb{R}^J)^I \times \mathbb{R}^L \times \mathbb{R}^J$ *is a* quasi-equilibrium *of the financial exchange economy* $(\mathcal{E}, \mathcal{F})$ *if* $\bar{p} \neq 0$ *and, for every* $i \in \mathcal{I}$,

$(a'\text{- }i)$ $\bar{x}_i \in X_i$, $\bar{z}_i \in Z_i$, *and* $\bar{p} \square (\bar{x}_i - e_i) = W(\bar{q})\bar{z}_i$;

$(a'\text{- }ii)$ $x_i \in P_i(\bar{x})$ *and* $\bar{p}(s) \cdot (x_i(s) - e_i(s)) \leq V_s \cdot z_i (s = 1, \ldots, S) \Rightarrow \bar{p}(0) \cdot (x_i(0) - e_i(0)) + \bar{q} \cdot z_i \geq 0$;

$(a'\text{- }iii)$ *for every* $s \in \mathcal{D}$, $(x_i(s), \bar{x}_i(-s)) \in P_i(\bar{x})$ *implies* $\bar{p}(s) \cdot x_i(s) \geq \bar{p}(s) \cdot \bar{x}_i(s)$;

(b) $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e_i$ *and* $\sum_{i \in \mathcal{I}} \bar{z}_i = 0$.

We can now compare the two notions of quasi-equilibria given in Definition 3.1 and 3.2, under the assumption (made by Gottardi and Hens (1996) and Seghir et al. (2004)) that there is no redundant asset, that is, rank $V = J$. In this case, the notions of portfolio clearing and accounts clearing quasi-equilibria of Definition 3.1 coincide; moreover both Market Clearing Conditions of Definition 3.1 and 3.2 coincide. Finally, Conditions $(a\text{ - }i)$ and $(a'\text{- }i)$ are equivalent as a direct consequence of the Market Clearing Conditions. As shown in the following remark, when $\gamma = 0$, Condition $(a'\text{- }ii)$ is stronger than $(a\text{ - }ii)$ and both are equivalent under the assumption (made by Gottardi and Hens (1996) and Seghir et al. (2004)) that there is a riskless asset, that is, for all i there exists $\zeta_i \in \boldsymbol{A}(Z_i)$ such that $V\zeta_i \gg 0$. Finally, the two notions cannot be further compared because Definition 3.1 introduces the mapping γ and Definition 3.2 introduces the additional Condition $(a'\text{- }iii)$, none of which being considered by the other.

Remark 3.3 *Condition* $(a'\text{- }ii)$ *of Definition 3.2 is satisfied if and only if* $\tilde{B}_i(\bar{p}, \bar{q}) \cap [P_i(\bar{x}) \times Z_i] = \emptyset$, *where*

$$\begin{aligned} \tilde{B}_i(p, q) = \{(x_i, z_i) \in X_i \times Z_i : & \text{for } s = 0,\ p(0) \cdot (x_i(0) - e_i(0)) < -q \cdot z_i, \\ & \text{for } s \neq 0,\ p(s) \cdot (x_i(s) - e_i(s)) \leq V_s \cdot z_i\}. \end{aligned}$$

Furthermore, we always have $\breve{B}_i(p, q) \subset \tilde{B}_i(p, q)$ *and the equality* $\breve{B}_i(p, q) = \tilde{B}_i(p, q)$ *holds*[12] *for agent* i *under the assumption (made by Gottardi and Hens (1996) and Seghir et al. (2004)) that the financial structure has a riskless asset. In this case, the two quasi-equilibrium conditions* $(a\text{- }ii)$ *and* $(a'\text{- }ii)$ *are equivalent.*

[12]Indeed, let $(x_i, z_i) \in \tilde{B}_i(\bar{p}, \bar{q})$. Then $\bar{p}(0) \cdot (x_i(0) - e_i(0)) < -\bar{q} \cdot z_i$ and for $s \neq 0$, $p(s) \cdot (x_i(s) - e_i(s)) \leq V_s \cdot z_i$. Let $\zeta_i \in \boldsymbol{A}Z_i$ such that $V\zeta_i \gg 0$, then, for all $t > 0$, $z_i + t\zeta_i \in Z_i$ (since $\zeta_i \in \boldsymbol{A}Z_i$) and for $t > 0$ small enough, $(x_i, z_i + t\zeta_i) \in \breve{B}_i(\bar{p}, \bar{q})$ since $\bar{p}(0) \cdot (x_i(0) - e_i(0)) < -\bar{q} \cdot z_i - \bar{q} \cdot (t\zeta_i) = -\bar{q}(z + t\zeta_i)$ and for $s \neq 0$, $p(s) \cdot (x_i(s) - e_i(s)) < V_s \cdot (z_i + t\zeta_i))$.

4 Proof of Theorem 3.2

The proof of Theorem 3.2 consists of two main steps. In Section 4.1 we provide a proof of Theorem 3.2 under the following additional assumptions (together with those already made in Theorem 3.2).

Assumption K: For every $i \in \mathcal{I}$,[13]

(i) *[Boundedness]* The sets X_i and $W(\bar{q})Z_i$, are bounded;

(ii) *[Local Non-Satiation]* for every $\bar{x} \in \widehat{X}$, for every $x_i \in P_i(\bar{x})$, $[x_i, \bar{x}_i) \subset P_i(\bar{x})$.

Then, in Section 4.2, we will give the proof of Theorem 3.2 in the general case, that is, without Assumption **K**. We will use a standard argument by modifying the original economy $(\mathcal{E}, \mathcal{F})$ into a new economy $(\mathcal{E}', \mathcal{F}')$, which satisfies Assumption **K**. Then we will check that accounts clearing γ-quasi-equilibria of $(\mathcal{E}', \mathcal{F}')$ correspond to accounts clearing γ-quasi-equilibria of the original economy $(\mathcal{E}, \mathcal{F})$.

4.1 Proof of Theorem 3.2 under additional assumptions

In the following, let $\bar{q} \in \mathbb{R}^J$ be aggregate arbitrage-free, let $\lambda \in \mathbb{R}^D_{++}$ be an associated state price vector (as in Definition 2.2) and let $\gamma : \mathbb{R}^L \to \mathbb{R}^D$ be a continuous mapping satisfying Assumption $\boldsymbol{\Gamma_1(\bar{q})}$ (but we do not assume Assumption $\boldsymbol{\Gamma_2(\bar{q})}$). We let[14]

$$B_L = \{p \in \mathbb{R}^L \;:\; ||\lambda \Box p|| \leq 1\},$$

$$\rho(\bar{p}) = (1 - \|\lambda \Box \bar{p}\|)\mathbb{1}_D,$$

$$\mathcal{B}_i^{\gamma\rho}(p) = \Big\{x_i \in X_i : \; \exists z_i \in Z_i, \exists \tau_i \in [0,1], p\Box(x_i - e_i) \leq W z_i + \tau_i \gamma(p) + \rho(p)\Big\},$$

$$\breve{\mathcal{B}}_i^{\gamma\rho}(p) = \Big\{x_i \in X_i : \; \exists z_i \in Z_i, \exists \tau_i \in [0,1], p\Box(x_i - e_i) \ll W z_i + \tau_i \gamma(p) + \rho(p)\Big\}.$$

4.1.1 The fixed point argument

For $(p, x) \in B_L \times \prod_{i \in \mathcal{I}} X_i$, we define $\Phi_i(p, x)$, for $i \in \mathcal{I}_0 := \{0\} \cup \mathcal{I}$, as follows:

$$\Phi_0(p, x) = \Big\{p' \in B_L \; : (\lambda \Box (p' - p)) \bullet_L \sum_{i \in \mathcal{I}} (x_i - e_i) > 0\Big\},$$

and for every $i \in \mathcal{I}$,

$$\Phi_i(p, x) = \begin{cases} \{e_i\} & \text{if } x_i \notin \mathcal{B}_i^{\gamma\rho}(p) \text{ and } \breve{\mathcal{B}}_i^{\gamma\rho}(p) = \emptyset, \\ \mathcal{B}_i^{\gamma\rho}(p) & \text{if } x_i \notin \mathcal{B}_i^{\gamma\rho}(p) \text{ and } \breve{\mathcal{B}}_i^{\gamma\rho}(p) \neq \emptyset, \\ \breve{\mathcal{B}}_i^{\gamma\rho}(p) \cap P_i(x) & \qquad \text{if } x_i \in \mathcal{B}_i^{\gamma\rho}(p). \end{cases}$$

The existence proof relies on the following fixed-point-type theorem.

Theorem 4.1 *[Gale and Mas-Colell (1975)] Let $\mathcal{I}_0$ be a finite set, let C_i ($i \in \mathcal{I}_0$) be a nonempty, compact, convex subset of some Euclidean space, let $C = \prod_{i \in \mathcal{I}_0} C_i$ and let Φ_i ($i \in \mathcal{I}_0$) be a correspondence from C to C_i, which is lower semicontinuous and convex-valued. Then, there exists $\bar{c} = (\bar{c}_i)_i \in C$ such that, for every $i \in \mathcal{I}_0$, either $\bar{c}_i \in \Phi_i(\bar{c})$ or $\Phi_i(\bar{c}) = \emptyset$.*

We now show that the sets $C_0 = B_L$, $C_i = X_i$ ($i \in \mathcal{I}$) and the above defined correspondences Φ_i ($i \in \mathcal{I}_0$) satisfy the assumptions of Theorem 4.1. This is a consequence of the following claim and the fact that B_L and X_i ($i \in \mathcal{I}$) are nonempty, convex and compact (by Assumptions **C** and **K**).

[13]Note that we only assume the $W(\bar{q})Z_i$ to be bounded and not the Z_i. This allows us to have redundant assets, and we do not assume any rank condition on the matrix $W(\bar{q})$.

[14]We let $\mathbb{1}_D$ denote the vector in $\mathbb{R}^D$, whose coordinates are all equal to 1.

Claim 2 *(i) For every $\bar{c} := (\bar{p}, \bar{x}) \in B_L \times \prod_{i \in \mathcal{I}} X_i$, $\Phi_i(\bar{c})$ is convex (possibly empty);*

(ii) For every $i \in \mathcal{I}_0$, the correspondence Φ_i is lower semicontinuous on $B_L \times \prod_{i \in \mathcal{I}} X_i$.

Proof. *Part (i).* Clearly $\Phi_0(\bar{c})$ is convex and for all $i \in \mathcal{I}$, $\Phi_i(\bar{c})$ is convex (by Assumptions **C** and **F**). *Part (ii).* From the definition of Φ_0 it is clearly lower semicontinuous. The proof of the lower semicontinuity of Φ_i $(i \in \mathcal{I})$ is given in the appendix. □

In view of Claim 2, we can now apply the fixed-point Theorem 4.1. Thus there exists $\bar{c} := (\bar{p}, \bar{x}) \in B_L \times \prod_{i \in \mathcal{I}} X_i$ such that, for every $i \in \mathcal{I}_0$, either $\Phi_i(\bar{p}, \bar{x}) = \emptyset$ or $\bar{c}_i \in \Phi_i(\bar{p}, \bar{x})$. We now check that the second condition can never hold. Indeed, for $i = 0$, $\bar{c}_0 = \bar{p} \notin \Phi_0(\bar{c})$, from the way it is defined; for every $i \in \mathcal{I}, \bar{c}_i = \bar{x}_i \notin \Phi_i(\bar{c})$ since $\bar{x}_i \notin P_i(\bar{x})$ (from the Irreflexivity Assumption **C (v)**) and $e_i \in \mathcal{B}_i^{\gamma\rho}(\bar{p})$. Thus we have shown that $\Phi_i(\bar{p}, \bar{x}) = \emptyset$ for all i. Written coordinatewise (and noticing that $\mathcal{B}_i^{\gamma\rho}(\bar{p}) \neq \emptyset$) we get:

$$\forall\, p \in B_L,\ (\lambda \Box p) \bullet_D \sum_{i \in \mathcal{I}} (\bar{x}_i - e_i) \leq (\lambda \Box \bar{p}) \bullet_D \sum_{i \in \mathcal{I}} (\bar{x}_i - e_i) \tag{2}$$

$$\forall i \in \mathcal{I}, \bar{x}_i \in \mathcal{B}_i^{\gamma\rho}(\bar{p}) \text{ and } \breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \cap P_i(\bar{x}) = \emptyset. \tag{3}$$

Thus, for all $i \in \mathcal{I}$, there exists $\bar{z}_i \in Z_i$ and $\bar{\tau}_i \in [0, 1]$, such that

$$\bar{p} \Box (\bar{x}_i - e_i) \leq W(\bar{q}) \bar{z}_i + \bar{\tau}_i \gamma(\bar{p}) + \rho(\bar{p}). \tag{4}$$

4.1.2 $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an accounts clearing γ-quasi-equilibrium of $(\mathcal{E}, \mathcal{F})$

We first prove that $\bar{x} = (\bar{x}_i)_{i \in \mathcal{I}}$ satisfies the commodity market clearing condition.

Claim 3 $\sum_{i \in \mathcal{I}} \bar{x}_i = \sum_{i \in \mathcal{I}} e_i$.

Proof. Suppose $\sum_{i \in \mathcal{I}} (\bar{x}_i - e_i) \neq 0$. From the Fixed-Point Assertion (2) we deduce that $(\lambda \Box \bar{p}) = \frac{\sum_{i \in \mathcal{I}} (\bar{x}_i - e_i)}{|| \sum_{i \in \mathcal{I}} (\bar{x}_i - e_i) ||}$, hence $||\lambda \Box \bar{p}|| = 1$. So

$$(\lambda \Box \bar{p}) \bullet_L \sum_{i \in \mathcal{I}} (\bar{x}_i - e_i) > 0. \tag{5}$$

Summing up over $i \in \mathcal{I}$, in the Inequalities (4) we get:

$$\bar{p} \Box \sum_{i \in \mathcal{I}} (\bar{x}_i - e_i) \leq \sum_{i \in \mathcal{I}} W(\bar{q}) \bar{z}_i + (\sum_{i \in \mathcal{I}} \bar{\tau}_i) \gamma(\bar{p}) + (\# \mathcal{I}) \rho(\bar{p}).$$

Taking the scalar product of both sides with $\lambda \gg 0$ we get,

$$(\lambda \Box \bar{p}) \bullet_L \sum_{i \in \mathcal{I}} (\bar{x}_i - e_i) \leq \lambda \bullet_D \sum_{i \in \mathcal{I}} W(\bar{q}) \bar{z}_i + (\sum_{i \in \mathcal{I}} \bar{\tau}_i) \lambda \bullet_D \gamma(\bar{p}) + (\# \mathcal{I}) \lambda \bullet_D \rho(\bar{p}).$$

On the right hand side, we have $\lambda \bullet_D \sum_{i \in \mathcal{I}} W(\bar{q}) \bar{z}_i = 0$ (by Definition 2.2), $\lambda \bullet_D \gamma(\bar{p}) = 0$ (by Assumption $\mathbf{\Gamma}_1(\bar{q})$), and $\rho(\bar{p}) = 0$ (since $||\lambda \Box \bar{p}|| = 1$). Thus $(\lambda \Box \bar{p}) \bullet_L \sum_{i \in \mathcal{I}} (\bar{x}_i - e_i) \leq 0$, which contradicts Inequality (5). □

Claim 4 *For $i \in \mathcal{I}$, such that $\breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \neq \emptyset$, one has:*
(i) $\mathcal{B}_i^{\gamma\rho}(\bar{p}) \cap P_i(\bar{x}) = \emptyset$,
(ii) $\bar{p} \Box (\bar{x}_i - e_i) = W(\bar{q}) \bar{z}_i + \bar{\tau}_i \gamma(\bar{p}) + \rho(\bar{p})$.

Proof. *Part (i).* Suppose that $\mathcal{B}_i^{\gamma\rho}(\bar{p}) \cap P_i(\bar{x})$ contains some element x_i. Since $\breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \neq \emptyset$, we let $\breve{x}_i \in \breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p})$. Clearly $x_i \neq \breve{x}_i$ (otherwise, $x_i \in \breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \cap P_i(\bar{x}) \neq \emptyset$, which contradicts the Fixed-Point Assertion (3)). For $\alpha \in (0, 1]$, we let $x_i(\alpha) := \alpha \breve{x}_i + (1 - \alpha) x_i$ and we check that $x_i(\alpha) \in \breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p})$ (since $x_i \in \mathcal{B}_i^{\gamma\rho}(\bar{p})$ and $\breve{x}_i \in \breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p})$). Moreover, for α small enough, $x_i(\alpha) \in P_i(\bar{x})$

(by the Openness-type Assumption **C (iii)** since $x_i \in P_i(\bar{x})$, $\breve{x}_i \in X_i$ and $x_i \neq \breve{x}_i$). Consequently, $x_i(\alpha) \in \breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \cap P_i(\bar{x}) \neq \emptyset$, which contradicts the Fixed-Point Assertion (3). $\square$

Part (ii). In view of the Budget Inequality (4), suppose that the equality does not hold, then

$$\bar{p}\square(\bar{x}_i - e_i) < W(\bar{q})\bar{z}_i + \bar{\tau}_i\gamma(\bar{p}) + \rho(\bar{p}),$$

that is, there exist $\xi \in \mathcal{D}$ such that

$$\bar{p}(\xi) \bullet_\ell (\bar{x}_i(\xi) - e_i(\xi)) < W(\bar{q})\bar{z}_i(\xi) + \bar{\tau}_i\gamma(\bar{p})(\xi) + \rho(\bar{p}).$$

From the Non-Satiation Assumption **C (vi)** for consumer i (recalling that $\bar{x}_i \in \widehat{X}_i$, by Claim 3), there exists $x_i \in P_i(\bar{x})$ such that $x_i(\xi') = \bar{x}_i(\xi')$ for every $\xi' \neq \xi$. Consequently, we can choose $x \in [x_i, \bar{x}_i)$ close enough to $\bar{x}_i$ so that $x \in \mathcal{B}_i^{\gamma\rho}(\bar{p})$. But, from the Local Non-Satiation (Assumption **K (ii)**), $[x_i, \bar{x}_i) \subset P_i(\bar{x})$. Consequently, $x \in \mathcal{B}_i^{\gamma\rho}(\bar{p}) \cap P_i(\bar{x}) \neq \emptyset$ which contradicts Part (i). $\square$

Claim 5 *(i)* $\|\lambda\square\bar{p}\| = 1$, *that is,* $\rho(\bar{p}) = 0$;
(ii) $\sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i = 0$ *and for all* $i \in \mathcal{I}, \bar{\tau}_i\gamma(\bar{p}) = 0$;

Proof. *Part (i).* Suppose that $\|\lambda\square\bar{p}\| < 1$, then $\rho(\bar{p}) \gg 0$. Then, for all $i \in \mathcal{I}$, $e_i \in \breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \neq \emptyset$, since $e_i \in X_i$ (by the Survival Assumption **C (vii)**) and $0 \in Z_i$, taking $\tau_i = 0$. Summing up over $i \in I$, the binding budget constraints (in Part (ii) of Claim 4) and using the commodity market clearing condition (Claim 3) we get:

$$0 = \bar{p}\square\sum_{i\in\mathcal{I}}(\bar{x}_i - e_i) = \sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i + (\sum_{i\in\mathcal{I}}\bar{\tau}_i)\gamma(\bar{p}) + (\#\mathcal{I})\rho(\bar{p}). \tag{6}$$

Taking above the scalar product with $\lambda \gg 0$ and recalling that $\rho(\bar{p}) = (1 - \|\lambda\square\bar{p}\|)\mathbb{1}_D$,

$$0 = \lambda \bullet_D (\sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i) + (\sum_{i\in\mathcal{I}}\bar{\tau}_i)\lambda \bullet_D \gamma(\bar{p}) + (\#\mathcal{I})(1 - \|\lambda\square\bar{p}\|)\sum_{\xi\in\mathcal{D}}\lambda(\xi).$$

Consequently, $\|\lambda\square\bar{p}\| = 1$, since $\lambda \bullet_D (\sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i) = 0$ (by Definition 2.2) and $0 = \lambda \bullet_D \gamma(\bar{p})$ (by Assumption $\mathbf{\Gamma_1(\bar{q})}$). A contradiction. $\square$

Part (ii). We first claim that

$$0 = \sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i + (\sum_{i\in\mathcal{I}}\bar{\tau}_i)\gamma(\bar{p}). \tag{7}$$

Indeed, summing over $i \in \mathcal{I}$, the budget inequalities (4), recalling that $\sum_{i\in\mathcal{I}}\bar{x}_i = \sum_{i\in\mathcal{I}} e_i$ (from Claim 3), and that $\rho(\bar{p}) = 0$, we get

$$0 = \bar{p}\square\sum_{i\in\mathcal{I}}(\bar{x}_i - e_i) \leq \sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i + (\sum_{i\in\mathcal{I}}\bar{\tau}_i)\gamma(\bar{p}) + 0.$$

Taking the scalar product of both sides with $\lambda \gg 0$, recalling that $\sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i \in \lambda^\perp := \{w \in \mathbb{R}^D : \lambda \cdot w = 0\}$ and $\gamma(\bar{p}) \in \lambda^\perp$ (by Assumption $\mathbf{\Gamma_1(\bar{q})}$), we get

$$0 = \lambda \bullet_D \sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i + \lambda \bullet_D (\sum_{i\in\mathcal{I}}\bar{\tau}_i)\gamma(\bar{p}).$$

Consequently, Equality (7) holds.

Taking the scalar product of both sides of Equality (7) with $\gamma(\bar{p})$, we get

$$0 = \gamma(\bar{p}) \bullet_D \sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i + (\sum_{i\in\mathcal{I}}\bar{\tau}_i)\|\gamma(\bar{p})\|^2.$$

But $\gamma(\bar{p}) \bullet_D W(\bar{q})\bar{z}_i = 0$ for all $i \in \mathcal{I}$ (from Assumption $\mathbf{\Gamma_1(\bar{q})}$), hence $(\sum_{i\in\mathcal{I}}\bar{\tau}_i)\|\gamma(\bar{p})\| = 0$. Since $\bar{\tau}_i \geq 0$ for every i, we deduce that, for all $i \in \mathcal{I}, \bar{\tau}_i\|\gamma(\bar{p})\| = 0$, hence $\bar{\tau}_i\gamma(\bar{p}) = 0$. Consequently, from Equality (7), we get $\sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i = 0$. $\square$

To conclude the proof that the list $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an accounts clearing γ-quasi-equilibrium, firstly note that $\sum_{i\in\mathcal{I}}\bar{x}_i = \sum_{i\in\mathcal{I}} e_i$ (Claim 3), $\sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i = 0$ (Claim 5), for all $i \in \mathcal{I}$, $(\bar{x}_i, \bar{z}_i) \in B_i(\bar{p}, \bar{q})$, (from the budget inequalities (4) and the fact that $\rho(\bar{p}) = 0$ and $\bar{\tau}_i\gamma(\bar{p}) = 0$ for all $i \in \mathcal{I}$, by Claim 5). Finally, for all $i \in \mathcal{I}$ such that $\breve{B}_i^\gamma(\bar{p}, \bar{q}) \neq \emptyset$, then $\breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \neq \emptyset$ and $B_i^\gamma(\bar{p}, \bar{q}) \cap (P_i(\bar{x}) \times Z_i) = \emptyset$ (by Claim 4 and the fact that $\rho(\bar{p}) = 0$, from Claim 5). $\square$

4.2 Proof of Theorem 3.2 in the general case

We now give the proof of Theorem 3.2, without considering the additional Assumption **K**, as in the previous section. We will first enlarge the strictly preferred sets of the agents in $\mathcal{E}$, as in Gale and Mas-Colell (1975), to get a new economy $\hat{\mathcal{E}}$. Then we truncate the financial economy $(\hat{\mathcal{E}}, \mathcal{F})$ by a standard argument to define a new financial economy $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$, which satisfies all the assumptions of $(\mathcal{E}, \mathcal{F})$, together with the additional Assumption **K**. From the previous section, we will get an accounts clearing γ-quasi-equilibrium of $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$ and we will then check that it is also an accounts clearing γ-quasi-equilibrium of $(\mathcal{E}, \mathcal{F})$.

4.2.1 Enlarging the preferences as in Gale and Mas-Colell (1975)

The original preferences P_i are replaced by the "enlarged" preferences $\hat{P}_i$ defined as follows. For every $i \in \mathcal{I}$, $\bar{x} \in \prod_{i\in\mathcal{I}} X_i$ we let

$$\hat{P}_i(\bar{x}) := \bigcup_{x_i \in P_i(\bar{x})} (\bar{x}_i, x_i] = \{\bar{x}_i + t(x_i - \bar{x}_i) \mid t \in (0,1],\ x_i \in P_i(\bar{x})\}.$$

This allows us to consider the new economy $\hat{\mathcal{E}} = (\mathcal{D}, \ell, \mathcal{I}, (X_i, \hat{P}_i, e_i)_{i\in\mathcal{I}})$. The next proposition shows that $\hat{P}_i$ satisfies the same properties as P_i, for every $i \in \mathcal{I}$, together with the additional Local Non-satiation Assumption **K (ii)** (Condition (vii) hereafter).

Proposition 4.1 *Under Assumption* **C**, *for every $i \in \mathcal{I}$ and every $\bar{x} \in \prod_{i\in\mathcal{I}} X_i$ one has:*

(i) $P_i(\bar{x}) \subset \hat{P}_i(\bar{x}) \subset X_i$;

(ii) [Continuity] the correspondence $\hat{P}_i$ is lower semicontinuous at $\bar{x}$;

(iii) [Openness-type] for every $x_i \in \hat{P}_i(\bar{x})$ for every $x'_i \in X_i, x'_i \neq x_i$ then $[x'_i, x_i) \cap \hat{P}_i(\bar{x}) \neq \emptyset$;

(iv) [Convexity] $\hat{P}_i(\bar{x})$ is convex;

(v) [Irreflexivity] $\bar{x}_i \notin \hat{P}_i(\bar{x})$;

(vi) [Non-Satiation at Every Node] if $\bar{x} \in \widehat{X}$, for every $\xi \in \mathcal{D}$, there exists $x_i \in \hat{P}_i(\bar{x})$ that may differ from $\bar{x}_i$ only at the node ξ, i.e., for each $\xi' \neq \xi$, $x_i(\xi') = \bar{x}_i(\xi')$;

(vii) [Local Non-Satiation] if $\bar{x}_i \in \widehat{X}_i$ for every $x_i \in \hat{P}_i(\bar{x})$, then $[x_i, \bar{x}_i) \subset \hat{P}_i(\bar{x})$.

The proof of this result can be found in Gale and Mas-Colell (1975) and a detailed argument is given in Angeloni and Cornet (2006). Note that the enlarged preferred set $\hat{P}_i$ may not have open values when P_i has open values (see Bich and Cornet (2004) for a counter-example), a property that holds for the weaker opennes-type assumption (Part (iii)).

4.2.2 Truncating the economy

The set $\widehat{X}_i$ of admissible consumptions and the set $\widehat{W}_i$ of admissible income transfers are defined by:

$$\widehat{X}_i := \{x_i \in X_i \ :\ \exists (x_j)_{j\neq i} \in \prod_{j\neq i} X_j, \sum_{i\in\mathcal{I}} x_i = \sum_{i\in\mathcal{I}} e_i\},$$

$$\begin{aligned}\widehat{W}_i := \{w_i \in \mathbb{R}^D \ : \quad & \exists z_i \in Z_i, w_i = W(\bar{q})z_i, \exists p \in B_L(0,1), \exists x_i \in \widehat{X}_i, p \square (x_i - e_i) \leq w_i,\\ & \exists (w_j)_{j\neq i} \in \prod_{j\neq i} W(q) Z_i, \sum_{i\in\mathcal{I}} w_i = 0\}.\end{aligned}$$

Lemma 4.1 *The sets $\widehat{X}_i$ and $\widehat{W}_i$ are bounded.*

Proof. The set $\widehat{X}_i$ is clearly bounded since X_i is bounded below (by Assumption **C (i)**). To show that $\widehat{W}_i$ is bounded, let $w_i \in \widehat{W}_i$ then there exist $x_i \in \widehat{X}_i$ and $p \in B_L(0,1)$ such that $p\Box(x_i - e_i) \leq w_i$. Since $\widehat{X}_i$ and $B_L(0,1)$ are compact sets, there exists $\alpha_i \in \mathbb{R}^D$ such that $\alpha_i \leq p\Box(x_i - e_i) \leq w_i$. Using the fact that $\sum_{i\in\mathcal{I}} w_i = 0$ we also have $w_i = -\sum_{j\neq i} w_j \leq -\sum_{j\neq i}\alpha_j$. Thus $\widehat{W}_i$ is bounded for every $i \in \mathcal{I}$. □

We now define the "truncated economy" as follows. Since $\widehat{X}_i$ and $\widehat{W}_i$ are bounded (by Lemma 4.1), there exists a real number $r > 0$ such that, for every agent $i \in \mathcal{I}$, $\widehat{X}_i \subset \text{int}\, B_L(0,r)$ and $\widehat{W}_i \subset \text{int}\, B_D(0,r)$. The truncated economy $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$ is then defined as follows

$$(\hat{\mathcal{E}}^r, \mathcal{F}^r) = [\mathcal{D}, \ell, \mathcal{I}, (X_i^r, \hat{P}_i^r, e_i)_{i\in\mathcal{I}}, \mathcal{J}, (\xi(j), V^j)_{j\in\mathcal{J}}, (Z_i^r)_{i\in\mathcal{I}}], \text{ where}$$

$$X_i^r = X_i \cap B_L(0,r),\ Z_i^r = \{z \in Z_i \ :\ W(\bar{q})z \in B_D(0,r)\} \text{ and } \hat{P}_i^r(x) = \hat{P}_i(x) \cap \text{int}B_L(0,r).$$

4.2.3 Existence of an accounts clearing γ-quasi-equilibrium of $(\mathcal{E}, \mathcal{F})$

The existence of an accounts clearing γ-quasi-equilibrium $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ of $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$ is then a consequence of Section 4.1, that is, Theorem 3.2 under the additional Assumption **K**. Indeed, we just have to check that Assumption **K** and Assumption **C**, **F**, made in Theorem 3.2 are satisfied by $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$. Clearly, this is the case for the financial Assumption **F** and the Boundedness Assumption **K (i)** (by Lemma 4.1); in view of Proposition 4.1, this is also the case for the Local Non-Satiation Assumption **K (ii)** and the Consumption Assumption **C**, but the Survival Assumption **C (vii)** that is proved via a standard argument that we recall hereafter. Indeed, we first notice that for every $i \in \mathcal{I}$, $e_i \in \hat{X}_i \subset \text{int}B_L(0,r)$, since $e_i \in X_i$ (from the Survival Assumption **C (vii)**). Consequently, $e_i \in X_i \cap \text{int}B_L(0,r) \subset [X_i \cap B_L(0,r)] = X_i^r$.

We end the proof by checking that $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is also an accounts clearing γ-quasi-equilibrium of $(\mathcal{E}, \mathcal{F})$.

Proposition 4.2 *If $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ is an accounts clearing γ-quasi-equilibrium of $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$, then it is also an accounts clearing γ-quasi-equilibrium of $(\mathcal{E}, \mathcal{F})$.*

Proof. Let $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ be an accounts clearing γ-quasi-equilibrium of the economy $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$. To prove that it is also an accounts clearing γ-quasi-equilibrium of $(\mathcal{E}, \mathcal{F})$ we only have to check that, for every $i \in \mathcal{I}$, $[P_i(\bar{x}) \times Z_i] \cap B_i^\gamma(\bar{p}, \bar{q}) = \emptyset$, whenever $\breve{B}_i^\gamma(\bar{p}, \bar{q}) \neq \emptyset$.

Assume, on the contrary, that, for some $i \in \mathcal{I}$, $\breve{B}_i^\gamma(\bar{p}, \bar{q}) \neq \emptyset$ and $[P_i(\bar{x}) \times Z_i] \cap B_i^\gamma(\bar{p}, \bar{q}) \neq \emptyset$, hence contains a couple (x_i, z_i). Thus, for $t \in (0,1]$, let $x_i(t) := \bar{x}_i + t(x_i - \bar{x}_i) \in X_i$ and $z_i(t) := \bar{z}_i + t(z_i - \bar{z}_i) \in Z_i$, then $(x_i(t), z_i(t)) \in B_i^\gamma(\bar{p}, \bar{q})$, the budget set of agent i for the economy $(\mathcal{E}, \mathcal{F})$. From the Market Clearing Conditions, we deduce that, for every $i \in \mathcal{I}$, $\bar{x}_i \in \widehat{X}_i \subset \text{int}\, B_L(0,r)$ and $W(\bar{q})\bar{z}_i \in \widehat{W}_i \subset \text{int}\, B_D(0,r)$. Consequently, for $t > 0$ sufficiently small, $x_i(t) \in \text{int}\, B_L(0,r)$ and $W(\bar{q})z_i(t) \in \text{int}\, B_D(0,r)$, hence, $x_i(t) \in X_i^r := X_i \cap B_L(0,r)$, $z_i(t) \in Z_i^r := \{z \in Z_i \mid W(\bar{q})z \in B_D(0,r)\}$, and $(x_i(t), z_i(t))$ also belongs to the budget set $B_i^{\gamma r}(\bar{p}, \bar{q})$ of agent i (in the economy $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$). From the definition of $\hat{P}_i$, we deduce that $x_i(t) \in \hat{P}_i(\bar{x})$ (since from above $x_i(t) := \bar{x}_i + t(x_i - \bar{x}_i)$ and $x_i \in P_i(\bar{x})$). We have thus shown that, for $t \in (0,1]$ small enough, $(x_i(t), z_i(t)) \in [\hat{P}_i^r(\bar{x}) \times Z_i^r] \cap B_i^{\gamma r}(\bar{p}, \bar{q})$ (since $\hat{P}_i^r(\bar{x}) := \hat{P}_i(\bar{x}) \cap \text{int}\, B_L(0,r)$).

We now show that $\breve{B}_i^{\gamma r}(\bar{p}, \bar{q}) \neq \emptyset$ and the proof will be complete since this assertion implies that $[\hat{P}_i^r(\bar{x}) \times Z_i^r] \cap B_i^{\gamma r}(\bar{p}, \bar{q}) = \emptyset$ (from the γ-quasi-equilibrium condition of agent i in the economy $(\hat{\mathcal{E}}^r, \mathcal{F}^r)$) and contradicts the fact that we have shown above it is nonempty. Indeed, since $\breve{B}_i^\gamma(\bar{p}, \bar{q}) \neq \emptyset$, it contains a point $(\breve{x}_i, \breve{z}_i)$ and we notice that $(1-t)(\bar{x}_i, \bar{z}_i) + t(\breve{x}_i, \breve{z}_i) \in \breve{B}_i^{\gamma r}(\bar{p}, \bar{q})$ for $t \in (0,1]$ small enough, since $(\bar{x}_i, \bar{z}_i) \in B_i^{\gamma r}(\bar{p}, \bar{q})$. □

5 Appendix

5.1 Proof of Proposition 2.1

We prepare the proof by a lemma in which we will use the notion of the asymptotic cone $\boldsymbol{A}Z$ of a closed convex set $Z \subset \mathbb{R}^J$. We recall that $\boldsymbol{A}Z$ is the set of all $v \in \mathbb{R}^J$ such that, for all $z \in Z$,

$z + v \in Z$.[15]

Lemma 5.1 *Let Z_i ($i \in \mathcal{I}$) be finitely many nonempty closed convex subsets of $\mathbb{R}^J$, and let C be a cone such that $C \subset \cup_{i\in\mathcal{I}} Z_i$, then $C \subset \cup_{i\in\mathcal{I}} \boldsymbol{A}Z_i$.*

Proof. Let $v \in C$. Since C is a cone, for every $k \in \mathbb{N}$, $kv \in C \subset \cup_{i\in\mathcal{I}} Z_i$. Hence, $\mathbb{N} = \cup_{i\in\mathcal{I}}\{k \in \mathbb{N} : kv \in Z_i\}$ and since $\mathcal{I}$ is finite, one of the sets $\{k \in \mathbb{N} : kv \in Z_i\}$ is infinite, say for $i = 1$. In other words, there exists a sequence $k_n \to +\infty$ such that $k_n v \in Z_1$. Since Z_1 is closed and convex by assumption, we deduce that, for every $z_1 \in Z_1$ $z_1 + v = \lim_{n\to+\infty}(1 - \frac{1}{k_n})z_1 + \frac{1}{k_n}(k_n v) \in Z_1$. Consequently, from the definition of the asymptotic cone $\boldsymbol{A}Z_1$, we have $v \in \boldsymbol{A}Z_1 \subset \cup_{i\in\mathcal{I}} \boldsymbol{A}Z_i$. $\square$

We now come back to the proof of Proposition 2.1. The proof of Part (a) is straightforward and we now provide a proof of Part (b). Let $(\bar{x}, \bar{z}, \bar{p}, \bar{q})$ be an accounts clearing equilibrium of $(\mathcal{E}, \mathcal{F})$.

We claim that $-\sum_{i\in\mathcal{I}} \bar{z}_i \in \cup_{i\in\mathcal{I}} \boldsymbol{A}Z_i$. Under the first Condition (i), that is, the set $\cup_{i\in\mathcal{I}} Z_i$ is a vector space, we deduce that $-\sum_{i\in\mathcal{I}} \bar{z}_i \in \cup_{i\in\mathcal{I}} Z_i$ (since $\bar{z}_i \in Z_i$ for all i). From Lemma 5.1, we deduce that $\cup_{i\in\mathcal{I}} Z_i \subset \cup_{i\in\mathcal{I}} \boldsymbol{A}Z_i$ (since $C := \cup_{i\in\mathcal{I}} Z_i$ is a linear space, hence is a cone). Under the second Condition (ii), that is, ker $W(\bar{q}) \subset \cup_{i\in\mathcal{I}} Z_i$, from the equilibrium accounts clearing condition we deduce that $-\sum_{i\in\mathcal{I}} \bar{z}_i \in$ ker $W(\bar{q}) \subset \cup_{i\in\mathcal{I}} Z_i$. From Lemma 5.1, we deduce that ker $W(\bar{q}) \subset \cup_{i\in\mathcal{I}} \boldsymbol{A}Z_i$ (since $C :=$ker $W(\bar{q})$ is a cone). This ends the proof of the claim.

From the above claim, there exists $i_0 \in \mathcal{I}$ such that $-\sum_{i\in\mathcal{I}} \bar{z}_i \in \boldsymbol{A}(Z_{i_0})$. Consider the profile of portfolios $\hat{z} = (\hat{z}_i)_{i\in\mathcal{I}}$ defined by $\hat{z}_{i_0} = \bar{z}_{i_0} - \sum_{i\in\mathcal{I}} \bar{z}_i$ and $\hat{z}_i = \bar{z}_i$, for all $i \neq i_0$. Clearly, $\hat{z}_{i_0} \in Z_{i_0} + \boldsymbol{A}Z_{i_0} \subset Z_{i_0}$ from the definition of the asymptotic cone $\boldsymbol{A}Z_{i_0}$. Hence, $\sum_{i\in\mathcal{I}} \hat{z}_i = 0$, for all $i \in \mathcal{I}$, $\hat{z}_i \in Z_i$, and $W(\bar{q})\hat{z}_i = W(\bar{q})\bar{z}_i$ (since $\sum_{i\in\mathcal{I}} W(\bar{q})\bar{z}_i = 0$ from the equilibrium accounts clearing condition). Then one easily checks that $(\bar{x}, \hat{z}, \bar{p}, \bar{q})$ is a portfolio clearing equilibrium of $(\mathcal{E}, \mathcal{F})$. $\square$

5.2 Proof of the lower semicontinuity of Φ_i for $i \in \mathcal{I}$

In this section we provide a proof of the lower semicontinuity of Φ_i for all $i \in \mathcal{I}$, that is, Claim 2 (iii) (the proof for $i = 0$ having already been given).

We prepare the proof with the following claim.

Claim 6 *For every $i \in \mathcal{I}$: (a) The set $F_i := \{(p, x) \in B_L \times \prod_{i\in\mathcal{I}} X_i : x_i \in \mathcal{B}_i^{\gamma\rho}(p)\}$ is closed;*

(b) the set $\{(p, x_i) \in B_L \times X_i : x_i \in \breve{\mathcal{B}}_i^{\gamma\rho}(p)\}$ is open (in $B_L \times X_i$).

Proof. *Part (a).* Let $(p^n, x^n) \to (p, x)$ be such that $x_i^n \in \mathcal{B}_i^{\gamma\rho}(p^n)$. For all n, there exists z_i^n and τ_i^n such that

$$x_i^n \in X_i,\ z_i^n \in Z_i,\ \tau_i^n \in [0, 1],\ p^n \square (x_i^n - e_i) \leq W(\bar{q})z_i^n + \tau_i^n \gamma(p^n) + \rho(p^n).$$

Since the set $W(\bar{q})Z_i$ is bounded, without any loss of generality, we can assume that the sequence $(W(\bar{q})z_i^n, \tau_i^n)$ converges to some element $(w_i, \tau_i) \in \mathbb{R}^D \times [0, 1]$. Moreover $w_i \in W(\bar{q})Z_i$ since $W(\bar{q})Z_i$ is closed (by Assumption **F**), hence $w_i = W(\bar{q})z_i$ for some $z_i \in Z_i$. Thus, in the limit, since both mappings γ and ρ are continuous, we get $x_i \in X_i$, $z_i \in Z_i$, $\tau_i \in [0, 1]$, $p\square(x_i - e_i) \leq W(\bar{q})z_i + \tau_i\gamma(p) + \rho(p)$. Thus $x_i \in \mathcal{B}_i^{\gamma\rho}(p)$. $\square$

Part (b). Let $(\bar{p}, \bar{x}_i)$ such that $\bar{x}_i \in \breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p})$, that is, there exists $z_i \in Z_i$, and $\tau_i \in [0, 1]$ such that $\bar{p}\square(\bar{x}_i - e_i) \ll W(\bar{q})z_i + \tau_i\gamma(\bar{p}) + \rho(\bar{p})$. Clearly, this inequality still holds (for the same z_i and τ_i) when (p, x_i) belongs to some neighborhood N of $(\bar{p}, \bar{x}_i)$ small enough, recalling that the the mappings ρ and γ are both continuous. This shows that $x_i \in \breve{\mathcal{B}}_i^{\gamma\rho}(p)$ for every $(p, x_i) \in N$. $\square$

To show the lower semicontinuity of Φ_i for $i \in \mathcal{I}$ at $(\bar{p}, \bar{x})$, let U be an open subset of $\mathbb{R}^L$ such that $\Phi_i(\bar{p}, \bar{x}) \cap U \neq \emptyset$, we need to show that $\Phi_i(p, x) \cap U \neq \emptyset$ when (p, x) belongs to some open neighborhood N of $(\bar{p}, \bar{x})$. For the proof, we will distinguish the following three cases.

Case 1: $\bar{x}_i \notin \mathcal{B}_i^{\gamma\rho}(\bar{p})$ and $\breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) = \emptyset$. Recall that $\Phi_i(\bar{p}, \bar{x}) \cap U \neq \emptyset$. From the definition of Φ_i one has $\Phi_i(\bar{p}, \bar{x}) = \{e_i\}$, hence $e_i \in U$. The proof will be complete if we show that $\Phi_i(p, x) \cap U \neq \emptyset$ for every $(p, x) \in \Omega_i := \{(p, x) : x_i \notin \mathcal{B}_i^{\gamma\rho}(p)\}$, which is an open neighborhood of $(\bar{p}, \bar{x})$ (by Claim

[15] The proof carries on under the following weaker assumption that ker W(q) $\cap -cone(\sum_{i\in\mathcal{I}} Z_i) \subset \cup_{i\in\mathcal{I}} Z_i$,.

6). In fact we only need to show that $e_i \in \Phi_i(p,x)$ since $e_i \in U$. We distinguish two cases. If $\breve{\mathcal{B}}_i^{\gamma\rho}(p) = \emptyset$, then $\Phi_i(p,x) = \{e_i\}$, from the definition of Φ_i. If $\breve{\mathcal{B}}_i^{\gamma\rho}(p) \neq \emptyset$, then $\Phi_i(p,x) = \mathcal{B}_i^{\gamma\rho}(p)$ (from the definition of Φ_i) and it contains e_i since $p\Box(e_i - e_i) \leq W(\bar{q})0 + 0 + \rho(p)$, recalling that $e_i \in X_i$, $0 \in Z_i, 0 \in [0,1]$ (by Assumptions **C** and **F**) and $\rho(p) \geq 0$.

Case 2: $\bar{x}_i \notin \mathcal{B}_i^{\gamma\rho}(\bar{p})$ and $\breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \neq \emptyset$. Recall that $\Phi_i(\bar{p},\bar{x}) \cap U \neq \emptyset$. From the definition of Φ_i, one has $\Phi_i(p,x) = \mathcal{B}_i^{\gamma\rho}(p)$ for all (p,x) in the set

$$\Omega_i' := \{(p,x) \in \prod_{i\in\mathcal{I}} X_i \times B_L : \ x_i \notin \mathcal{B}_i^{\gamma\rho}(p) \text{ and } \breve{\mathcal{B}}_i^{\gamma\rho}(p) \cap U \neq \emptyset\},$$

and we now show that Ω_i' is an open neighborhood of $(\bar{p},\bar{x})$. Indeed, first Ω_i' is open (by Claim 6). Second, to show that Ω_i' contains $(\bar{p},\bar{x})$, we recall that $\bar{x}_i \notin \mathcal{B}_i^{\gamma\rho}(\bar{p})$ (by Assumption of Case (ii)) and it only remains to show that $\breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \cap U \neq \emptyset$; indeed, choose $\breve{x}_i \in \breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \neq \emptyset$ (by Assumption of Case (ii)), and $x_i \in \mathcal{B}_i^{\gamma\rho}(\bar{p}) \cap U = \Phi_i(\bar{p},\bar{x}) \cap U \neq \emptyset$, then one sees that, for $t > 0$ small enough, $t\breve{x}_i + (1-t)x_i \in \breve{\mathcal{B}}_i^{\gamma\rho}(p) \cap U$.

Consequently, Φ_i is lower semicontinuous at $(\bar{p},\bar{x})$ since, for all $(p,x) \in \Omega_i'$ (an open neighborhood of $(\bar{p},\bar{x})$) one has $\emptyset \neq \breve{\mathcal{B}}_i^{\gamma\rho}(p) \cap U \subset \mathcal{B}_i^{\gamma\rho}(p) \cap U = \Phi_i(p,x) \cap U$.

Case 3: $\bar{x}_i \in \mathcal{B}_i^{\gamma\rho}(\bar{p})$. Recall that $\Phi_i(\bar{p},\bar{x}) \cap U \neq \emptyset$, hence we can choose $\tilde{x}_i$ so that

$$\tilde{x}_i \in \Phi_i(\bar{p},\bar{x}) \cap U = \breve{\mathcal{B}}_i^{\gamma\rho}(\bar{p}) \cap P_i(\bar{x}) \cap U.$$

From Claim 6, there exists an open neighborhood M of $\bar{p}$ and an open neighborhood V of $\tilde{x}_i$ such that, for every $p \in M$, one has $\emptyset \neq V \subset \breve{\mathcal{B}}_i^{\gamma\rho}(p) \cap U$. Noticing that $P_i(\bar{x}) \cap V \neq \emptyset$ (since it contains $\tilde{x}_i$), the lower semicontinuity of P_i at $\bar{x}$ (by Assumption **C**) implies that $P_i(x) \cap V \neq \emptyset$ for every x in some open neighborhood N of $\bar{x}$. Consequently

$$\emptyset \neq P_i(x) \cap V \subset P_i(x) \cap \breve{\mathcal{B}}_i^{\gamma\rho}(p) \cap U \subset \mathcal{B}_i^{\gamma\rho}(p) \cap U \text{ for every } (p,x) \in M \times N.$$

Noticing that $\Phi_i(p,x) \subset \mathcal{B}_i^{\gamma\rho}(p)$ (from its definition) we thus deduce that $\Phi_i(p,x) \cap U \neq \emptyset$ for every (p,x) in the neighborhood $M \times N$ of $(\bar{p},\bar{x})$. □

5.3 Proof of Lemma 3.1

We let $\mathcal{W}$ be the closed cone spanned by $\cup_{i\in\mathcal{I}} W(\bar{q})Z_i$, which is a linear space by Assumption **FA**. For every $p \in \mathbb{R}^L$, we let

$\varphi(p) = -p\Box p + \frac{\lambda\cdot(p\Box p)}{\|\lambda\|^2}\lambda$, $w(p) = \text{proj}_{\mathcal{W}}\varphi(p)$ and $\gamma(p) = \text{proj}_{\mathcal{W}^\perp}\varphi(p)$.[16]

The following claim shows that the mapping γ satisfies Assumption $\mathbf{\Gamma_1(\bar{q})}$ of Lemma 3.1, together with other properties that will allow us to show that Assumption $\mathbf{\Gamma_2(\bar{q})}$ also holds.

Claim 7 *The mapping $p \to \gamma(p)$ is continuous on $\mathbb{R}^L$ and for all $p \in \mathbb{R}^L$, one has*

(i) $\gamma(p) \in \lambda^\perp \cap \mathcal{W}^\perp \subset \lambda^\perp \cap (\cup_{I\in\mathcal{I}} W(\bar{q})Z_i)^\perp$;

(ii) $-p\Box p \ll \varphi(p) = w(p) + \gamma(p)$ *if* $p \neq 0$;

(iii) If $p \neq 0$, *there exists* i_0 *such that, for* $\tau > 0$ *small enough*

$p\Box(-\tau p) \ll W(\bar{q})z_{i_0} + \tau\gamma(p)$ *for some* $z_{i_0} \in Z_{i_0}$.

Proof. The continuity of the mapping $\gamma : \mathbb{R}^L \to \mathbb{R}^D$ is a consequence of the continuity of the mappings φ (for fixed λ) and $\text{proj}_{\mathcal{W}^\perp}$.

[16]When X is an (arbitrary) subset of $\mathbb{R}^D$, we let $X^\perp := \{w \in \mathbb{R}^D : w \cdot x = 0 \text{ for all } x \in X\}$. When $\mathcal{W}$ is a linear subspace of $\mathbb{R}^D$ and $\varphi \in \mathbb{R}^D$, we denote by $\text{proj}_{\mathcal{W}}\varphi$ (resp. $\text{proj}_{\mathcal{W}^\perp}\varphi$) the orthogonal projection of φ on $\mathcal{W}$ (resp. on $\mathcal{W}^\perp$), that is, the unique $\bar{w} \in \mathcal{W}$ (resp. $\bar{\gamma} \in \mathcal{W}^\perp$) such that $\varphi - \bar{w} \in \mathcal{W}^\perp$ (resp. $\varphi - \bar{\gamma} \in \mathcal{W}$).

Part (i). The proof that $\lambda \cdot \varphi(p) = 0$ is done by simple calculation from the definition of $\varphi(p)$. Recalling that $w(p) \in \mathcal{W} \subset W(\bar{q})\mathcal{Z}_{\mathcal{F}} \subset \lambda^{\perp}$ (from the definition of λ) we deduce that $\gamma(p) = \varphi(p) - w(p)$ also belongs to $\lambda^{\perp}$. Finally, $\gamma(p) \in \mathcal{W}^{\perp} \subset (\cup_{I\in\mathcal{I}} W(\bar{q})Z_i)^{\perp}$ since $\cup_{I\in\mathcal{I}} W(\bar{q})Z_i \subset \mathcal{W}$.

Part (ii). If $p \neq 0$, Notice that $p\Box p > 0$, hence $\lambda \cdot (p\Box p) > 0$ (since $\lambda \gg 0$) and one has $-p\Box p - \varphi(p) = -\frac{\lambda\cdot(p\Box p)}{\|\lambda\|^2}\lambda \ll 0$. Thus, $-p\Box p \ll \varphi(p) = w(p) + \gamma(p)$.

Part (iii). Since $w(p)$ belongs to $\mathcal{W}$, which is the closed cone spanned by $\cup_{I\in\mathcal{I}} W(\bar{q})Z_i$, then $w(p) = \lim_{n\to\infty} t^n w^n$ for some sequence $(t^n) \subset \mathbb{R}_+$ and $(w^n) \subset \cup_{I\in\mathcal{I}} W(\bar{q})Z_i$. By eventually considering a subsequence, we can assume that, for all n, w^n belongs to $W(\bar{q})Z_i$, for some given i independent of n, say $i = 1$; thus $w^n = W(\bar{q})z_1^n$ for some $z_1^n \in Z_1$. From Part (ii), we deduce that there exist an integer n_0, such that, for $n \geq n_0$ $-p\Box p \ll t^n w^n + \gamma(p)$. Fix $n = n_0$, thus, for $\tau \in (0, 1/t^{n_0}]$, one has $p\Box(-\tau p) \ll W(\bar{q})(\tau t^{n_0} z_1^{n_0}) + \tau\gamma(p)$, and $z_1 := \tau t^{n_0} z_1^{n_0} \in Z_1$ (since Z_1 is convex, $0 \in Z_1$, $z_1^{n_0} \in Z_1$, $0 \leq \tau t^{n_0} \leq 1$). □

We now end the proof of Lemma 3.1 by showing that $x_{i_0} := e_{i_0} - \tau p \in \breve{\mathcal{B}}_{i_0}^{\gamma}(p)$ for $\tau > 0$ small enough (where i_0 is defined as in the above claim). Let $r > 0$ such that $B(e_{i_0}, r) \subset X_{i_0}$ (which is possible since $e_{i_0} \in \text{int } X_{i_0}$ from the Strong Survival Assumption **S**). Then, for $\tau > 0$ small enough (as in the above claim) and such that $\tau \leq 1$ and $\tau\|p\| \leq r$ one has $x_{i_0} := e_{i_0} - \tau p \in B_L(e_{i_0}, r) \subset X_{i_0}$ (since $\|\tau p\| \leq r$), and

$p\Box(x_{i_0} - e_{i_0}) = -p\Box\tau p \ll W(\bar{q})z_{i_0} + \tau\gamma(p)$, with $z_{i_0} \in Z_{i_0}$, and $\tau \in [0, 1]$. □

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