

EXTENDING THE SCOPE OF MONOTONE COMPARATIVE STATICS RESULTS

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ABSTRACT. Generally we can distinguish between two types of comparative statics problems that have been approached with lattice programming methods. The first type of problem considers the change of the optimal solution to a maximization problem as the objective function changes, the other type the change due to a change in the constraint set. Comparative statics theorems have been developed for both cases under cardinal and ordinal assumptions in the literature; Quah (2007) expanded existing work by making it applicable to optimization problems with a new, weaker order on the constraint sets.

The idea of this paper is to extend the existing comparative statics results to an even broader class of constrained optimization problems. We combine the two previously mentioned types of maximization problems and apply the existing comparative statics theorems to cases with changes in both the objective function and non-lattice constraint sets. Examples and applications from a variety of areas in economics, such as consumer theory, producer theory and environmental economics, are provided as well.

JEL: C61, D11, D21, Q58

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1. INTRODUCTION

Comparative statics of constrained optimization problems is a question at the heart of economic analysis. Oftentimes we are not only interested in the optimizers themselves, but also how they will be affected by changes in exogenous parameters. Generally, we distinguish between two types of comparative statics problems that have been approached with lattice programming methods. The first type of problem considers the change of the optimal solution to a maximization problem as the objective function changes, the other type the change due to a change in the constraint set. Comparative statics theorems have been developed for both cases under stronger and weaker assumptions in various works. Based on Topkis (1978), Vives (1990) uses a lattice-based approach to establish monotone increasing best responses in games with strategic complements under the cardinal assumptions of supermodularity and increasing differences on the objective function when strategy spaces are lattices. Milgrom and Roberts (1990) also focus on monotone comparative statics results under these cardinal assumptions in supermodular games. Milgrom and Shannon (1994) then extend these results to the ordinal case. They show that optimal solutions are nondecreasing in the parameters of the problem, replacing both supermodularity and increasing differences by their ordinal counterparts of quasisupermodularity and the Single Crossing Property. In many cases however, constraint sets in economic optimization problems, such as the budget set in the consumer problem, are not lattices and therefore Milgrom and Shannon's result cannot be applied. Quah (2007) addresses these types of problems and considers comparative statics with respect to changes in non-lattice constraint sets. His result provides necessary and sufficient

conditions for nondecreasing solutions using a weaker set order that, in particular, establishes normality of demand under assumptions on the primitives.

This paper generalizes the existing comparative statics results to even more classes of constrained optimization problems. We extend Milgrom and Shannon's comparative statics result to general, parametrized optimization problems with non-lattice constraint sets, that can be ranked by Quah's weaker set order. This result has nice applications, since a variety of economic optimization problems fall into this class of optimization problems. A natural example from consumer theory is the consumer's utility maximization problem with Stone-Geary preferences. In producer theory, our result can be applied to multiple-plant production problems and price discrimination with capacity constraints. Many additional applications can be found in the area of environmental economics such as production regulation through emissions standards and cost-efficient emissions regulation. Moreover, we can use this comparative statics result to generalize known lattice-based versions of the LeChatelier principle.

The remainder of the paper is organized as follows. Section 2 introduces the theoretical framework, section 3 gives the generalized monotone comparative statics results and section 4 provides a variety of applications.

2. THEORETICAL BACKGROUND

To obtain monotone comparative statics results with respect to changes in the constraint set we need to be able to order these sets. Milgrom and Shannon (1994) uses the strong set order by Veinott (1989).

Definition 1. Strong Set Order

Let X be a lattice and consider two subsets S and S' . Then S' dominates S in the strong set order ($S' \geq_S S$) if and only if for all $x \in S$ and for all $y \in S'$ $x \wedge y \in S$ and $x \vee y \in S'$.

Quah (2007) introduces the following concepts to extend the comparative statics results from Milgrom and Shannon (1994) to problems where the constraint set are not lattices. This case is easily encountered even in simple optimization problems like the consumer's utility maximization problem.

Let X, T be partially ordered sets and let Δ and ∇ be two operations on X . Consider a function $f : X \rightarrow \mathbb{R}$. Then we can define the following properties for f .

Definition 2. (Δ, ∇) -Supermodularity

f is (Δ, ∇) -supermodular if for all $x, y \in X$

$$f(x \nabla y) - f(y) \geq f(x) - f(x \Delta y)$$

Definition 3. (Δ, ∇) -Quasisupermodularity

f is (Δ, ∇) -quasisupermodular if for all $x, y \in X$

$$(1) \ f(x) \geq f(x\Delta y) \Rightarrow f(x\nabla y) \geq f(y)$$

$$(2) \ f(x) > f(x\Delta y) \Rightarrow f(x\nabla y) > f(y)$$

Moreover, consider two non-empty subsets of X , S and S' . The operations Δ and ∇ induce the following set order on S and S' .

Definition 4. (Δ, ∇) -induced strong set order

S' dominates S by the (Δ, ∇) -induced strong set order ($S' \geq_{\Delta, \nabla} S$) if and only if for all $x \in S$ and for all $y \in S'$ $x\Delta y \in S$ and $x\nabla y \in S'$.

After the introduction of the general case, now let $X \subseteq R^l$ be a convex set and define the operations ∇_i^λ and Δ_i^λ on X as follows for λ in $[0, 1]$:

$$x\nabla_i^\lambda y = \begin{cases} y & \text{if } x_i \leq y_i \\ \lambda x + (1 - \lambda)(x \vee y) & \text{if } x_i > y_i \end{cases}$$

$$x\Delta_i^\lambda y = \begin{cases} x & \text{if } x_i \leq y_i \\ \lambda y + (1 - \lambda)(x \wedge y) & \text{if } x_i > y_i \end{cases}$$

Graphically, the points x , y , $x\Delta_i^\lambda y$ and $x\nabla_i^\lambda y$ form a backward-bending parallelogram instead of the rectangle generated by x , y and their join and meet (see Figure 2.1).

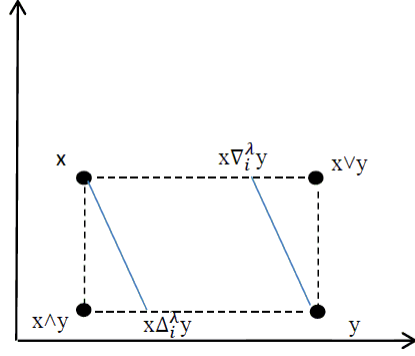


FIGURE 2.1.

Analogously to the previous case, Quah defines properties of a function $f : x \rightarrow \mathbb{R}$ and introduces a new set order with respect to the operations $\nabla_i^\lambda, \Delta_i^\lambda$.

Definition 5. “Parallelogram Inequality”

A function is $(\nabla_i^\lambda, \Delta_i^\lambda)$ - supermodular for some $\lambda \in [0, 1]$ if

$$f(x \nabla_i^\lambda y) - f(y) \geq f(x) - f(x \Delta_i^\lambda y) \text{ for all } x, y \text{ in } X.$$

Definition 6. C_i (C)-Supermodularity

A function $f : X \rightarrow R$ is C_i -supermodular if it is $(\nabla_i^\lambda, \Delta_i^\lambda)$ - supermodular for all $(\nabla_i^\lambda, \Delta_i^\lambda)$ in $C_i = \{(\nabla_i^\lambda, \Delta_i^\lambda) : \lambda \in [0, 1]\}$.

If f is C_i -supermodular for all i , then it is C -supermodular.

Definition 7. C_i (C)-Quasisupermodularity

A function $f : X \rightarrow R$ is C_i -quasisupermodular if it is $(\nabla_i^\lambda, \Delta_i^\lambda)$ - quasisupermodular for all $(\nabla_i^\lambda, \Delta_i^\lambda)$ in $C_i = \{(\nabla_i^\lambda, \Delta_i^\lambda) : \lambda \in [0, 1]\}$, that is

$$(1) \ f(x) \geq f(x\Delta_i^\lambda y) \Rightarrow f(x\nabla_i^\lambda y) \geq f(y)$$

$$(2) \ f(x) > f(x\Delta_i^\lambda y) \Rightarrow f(x\nabla_i^\lambda y) > f(y)$$

If f is C_i -quasisupermodular for all i , then it is C -quasisupermodular.

Quah (2007) shows that C_i -quasisupermodularity follows from properties that can easily be verified. The first one is supermodularity, the other a form of concavity.

Definition 8. i -Concavity

A function f is i -concave if it is concave in direction v for any $v > 0$ with $v_i = 0$.

Proposition 1. (Quah)

The function $f : X \rightarrow R$ is C_i -quasisupermodular if it is supermodular and i -concave.

As in the general case, the operations ∇_i^λ and Δ_i^λ induce a set order, which is weaker than the previously defined strong set order.

Definition 9. $C_i(C)$ -flexible Set Order

Let S' and S be subsets of the convex sublattice X . Then S' dominates S in the C_i -flexible set order ($S' \geq_i S$) if for any x in S and y in S' , there exists $(\nabla_i^\lambda, \Delta_i^\lambda)$ in C_i such that $x\nabla_i^\lambda y$ is in S' and $x\Delta_i^\lambda y$ is in S .

S' dominates S in the C -flexible set order ($S' \geq S$) if $S' \geq_i S$ for all i .

Quah (2007) describes the requirements of the C_i -flexible set order as that for any pair of unordered points $x \in S$ and $y \in S'$, we can find $x\Delta_i^\lambda y$ and $x\nabla_i^\lambda y$ such that $x\Delta_i^\lambda y$ is in S and $x\nabla_i^\lambda y$ is in S' , with the four points forming a backward-bending parallelogram.

While Quah (2007) addresses comparative statics with regard to changes in the constraint set using the previously defined framework, Milgrom and Shannon (1994) also includes comparative statics with respect to parameter changes in the objective function. Their result requires an additional assumption on the objective function, which is that for a parametrized objective function, $f : X \times T \rightarrow \mathbb{R}$, where X is a lattice and T is a partially ordered set, f needs to satisfy the Single Crossing Property in (x, t) .

Definition 10. Single Crossing Property

f satisfies the Single Crossing Property (SCP) if for every $x \leq y$ and for every $t \leq t'$

$$f(y, t) \geq f(x, t) \Rightarrow f(y, t') \geq f(x, t')$$

In the following, we introduce new versions of the Single Crossing Property to work with throughout the remainder of the paper.

Definition 11. Strong Single Crossing Property

f satisfies the strong Single Crossing Property (strong SCP) if for every x, y and for every $t \leq t'$

$$f(y, t) \geq f(x, t) \Rightarrow f(y, t') \geq f(x, t')$$

Definition 12. (Δ, ∇) -Single Crossing Property

f satisfies the (Δ, ∇) -Single Crossing Property $((\Delta, \nabla)$ -SCP) if for every x, y with $x \leq_{(\Delta, \nabla)} y$ and for every $t \leq t'$

$$f(y, t) \geq f(x, t) \Rightarrow f(y, t') \geq f(x, t')$$

Definition 13. *i*-Single Crossing Property

f satisfies the *i*-Single Crossing Property (*i*-SCP) if for every x, y with $x_i \leq y_i$ and for every $t \leq t'$

$$f(y, t) \geq f(x, t) \Rightarrow f(y, t') \geq f(x, t')$$

For twice continuously differentiable functions f , this property is implied by $\frac{\partial^2 f}{\partial x_i \partial t} \geq 0$.¹

3. COMPARATIVE STATICS

Consider the following general constrained optimization problem:

$$\max f(x, t) \text{ subject to } x \in S$$

For convenience, denote $M(t, S) := \underset{x \in S}{\operatorname{argmax}} f(x, t)$.

First, consider the general case, where Δ and ∇ are two operations on X . The following results extend Theorem 1 in Quah (2007) to parametrized objective functions. The proofs follow Milgrom and Shannon (1994) with some simple modifications. The first result gives necessary conditions for nondecreasing optimal solutions using the strongest version of the Single Crossing Property.

¹The standard Single Crossing Property is implied by increasing differences, which for twice continuously differentiable functions f is equivalent to $\frac{\partial^2 f}{\partial x_i \partial t} \geq 0$ for all i . As the *i*-Single Crossing Property only focuses on the *i*-th component, it is implied by $\frac{\partial^2 f}{\partial x_i \partial t} \geq 0$.

Theorem 1. *If $f(x, t)$ is (Δ, ∇) -quasisupermodular and has the strong Single Crossing Property in (x, t) and S' dominates S in the (Δ, ∇) -induced strong set order ($S' \geq_{\Delta, \nabla} S$) for $t' \geq t$, then $\operatorname{argmax}_{x \in S} f(x, t)$ is nondecreasing in (t, S) .*

Proof. ($\Rightarrow$) Suppose $S' \geq_{\Delta, \nabla} S$, $t' \geq t$ and let $x \in M(t, S)$, $y \in M(t', S')$. Since $x \in M(t, S)$ and $S' \geq_{\Delta, \nabla} S$, $f(x, t) \geq f(x\Delta y, t)$. By (Δ, ∇) -quasisupermodularity of f , this implies $f(x\nabla y, t) \geq f(y, t)$. As f also has the strong Single Crossing Property in (x, t) , $f(x\nabla y, t) \geq f(y, t)$ implies $f(x\nabla y, t') \geq f(y, t')$ for $t' \geq t$. Since $y \in M(t', S')$ it follows that $x\nabla y \in M(t', S')$.

Similarly, consider $x\Delta y$. Since $y \in M(t', S')$ and $S' \geq_{\Delta, \nabla} S$, it follows that $f(y, t') \geq f(x\nabla y, t')$. By the strong Single Crossing Property, this implies $f(y, t) \geq f(x\nabla y, t)$ and using the (Δ, ∇) -quasisupermodularity of f we have $f(x\Delta y, t) \geq f(x, t)$. Hence $x\Delta y \in M(t, S)$.

Consequently, $\operatorname{argmax}_{x \in S(t)} f(x, t)$ is nondecreasing in (t, S) . □

The next result gives necessary and sufficient conditions for nondecreasing optimal solutions using the (Δ, ∇) -Single Crossing Property for operations Δ and ∇ on X .

Theorem 2. *$f(x, t)$ is (Δ, ∇) -quasisupermodular and has the (Δ, ∇) -Single Crossing Property in (x, t) and S' dominates S in the (Δ, ∇) -induced set order ($S' \geq_{(\Delta, \nabla)} S$) for $t' \geq t$ if and only if $\operatorname{argmax}_{x \in S'} f(x, t') \geq_{(\Delta, \nabla)} \operatorname{argmax}_{x \in S} f(x, t)$ ($\operatorname{argmax}_{x \in S} f(x, t)$ is nondecreasing in (t, S)).*

Proof. ($\Rightarrow$) Suppose $S' \geq_{(\Delta, \nabla)} S$ for $t' \geq t$ and let $x \in M(t, S)$, $y \in M(t', S')$. Since $x \in M(t, S)$ and $S \leq_{(\Delta, \nabla)} S'$, $f(x, t) \geq f(x\Delta y, t)$. By (Δ, ∇) -quasisupermodularity of f , this implies $f(x\nabla y, t) \geq f(y, t)$. As f also has the (Δ, ∇) -Single Crossing

Property in (x, t) , $f(x\nabla y, t) \geq f(y, t) \Rightarrow f(x\nabla y, t') \geq f(y, t')$ for $t' \geq t$. Since $y \in M(t', S')$ it follows that $x\nabla y \in M(t', S')$.

Now suppose $x\Delta y \notin M(S, t)$ and hence $f(x, t) > f(x\Delta y, t)$. (Δ, ∇) -quasisupermodularity of f implies $f(x\nabla y, t) > f(y, t)$ and by the (Δ, ∇) -Single Crossing Property it follows that $f(x\nabla y, t') > f(y, t')$ for any $t' \geq t$. This contradicts the assumption that $y \in M(t', S')$. Therefore, $x\Delta y \in M(t, S)$ and $M(t, S) \leq_{(\Delta, \nabla)} M(t', S')$.

($\Leftarrow$) Fix t . Let x and y be two elements in X and suppose that f is not (Δ, ∇) -quasisupermodular. The only case we need to look at is when x and y are unordered. Also, $x\Delta y \neq x$ and $x\nabla y \neq y$. Let $S = \{x, x\Delta y\}$ and $S' = \{y, x\nabla y\}$. Then $S' \geq_{(\Delta, \nabla)} S$. (Δ, ∇) -quasisupermodularity of f can be violated in the following two ways. First, suppose $f(x, t) \geq f(x\Delta y, t)$, but $f(x\nabla y, t) < f(y, t)$. In this case x is a maximizer of f in S and y maximizes f uniquely in S' , which violates $M(S', t) \geq_{(\Delta, \nabla)} M(S, t)$ (since x and y are unordered). Alternatively, suppose $f(x, t) > f(x\Delta y, t)$, but $f(x\nabla y, t) = f(y, t)$. Now y maximizes f in S' while x is the unique maximizer in S . This again contradicts $M(S', t) \geq_{(\Delta, \nabla)} M(S, t)$. So f is (Δ, ∇) -quasisupermodular.

Now let $S \equiv \{x, \bar{x}\}$ with $x \leq_{(\Delta, \nabla)} \bar{x}$. Then $f(\bar{x}, t) - f(x, t) \geq 0$ implies $\bar{x} \in M(t, S)$. Since $M(t, S) \leq_{(\Delta, \nabla)} M(\bar{t}, S)$ for $\bar{t} \geq t$ it follows that $f(\bar{x}, \bar{t}) - f(x, \bar{t}) \geq 0$ for all $\bar{t} \geq t$. Thus f has the (Δ, ∇) -Single Crossing Property. $\square$

Similarly, we can generalize the main monotone comparative statics result in Quah (2007) for optimization problems with constraint sets that can be ordered by the C_i -flexible set order to problems with parametrized objective functions. We provide necessary and sufficient conditions for non decreasing optimal solutions for parameter changes in the objective function and the above mentioned type of constraint sets;

thus the main theorem of the paper provides properties of the objective function that characterize non-decreasing optimal solutions.

Theorem 3. *$f(x, t)$ is $C_i(C)$ -quasisupermodular and has the i -Single Crossing Property in (x, t) and S' dominates S in the C_i -flexible set order ($S' \geq_i S$) for $t' \geq t$ if and only if $\operatorname{argmax}_{x \in S'} f(x, t') \geq_i \operatorname{argmax}_{x \in S} f(x, t)$ ($\operatorname{argmax}_{x \in S} f(x, t)$ is nondecreasing in (t, S)).*

Proof. The bracketed version follows logically from the unbracketed one; hence the latter will be proven in the following.

($\Rightarrow$) Suppose $S' \geq_i S$ for $t' \geq t$ and let $x \in M(t, S)$, $y \in M(t', S')$. Since $x \in M(t, S)$ and $S \leq_i S'$, $f(x, t) \geq f(x \Delta_i^\lambda y, t)$. By C_i -quasisupermodularity of f , this implies $f(x \nabla_i^\lambda y, t) \geq f(y, t)$ with $x_i > y_i$. As f also has the i -Single Crossing Property in (x, t) , $f(x \nabla_i^\lambda y, t) \geq f(y, t) \Rightarrow f(x \nabla_i^\lambda y, t') \geq f(y, t')$ for $t' \geq t$. Since $y \in M(t', S')$ it follows that $x \nabla_i^\lambda y \in M(t', S')$.

Now suppose $x \Delta_i^\lambda y \notin M(S, t)$ and hence $f(x, t) > f(x \Delta_i^\lambda y, t)$. C_i -quasisupermodularity of f implies $f(x \nabla_i^\lambda y, t) > f(y, t)$ and by the i -Single Crossing Property it follows that $f(x \nabla_i^\lambda y, t') > f(y, t')$ for any $t' \geq t$. This contradicts the assumption that $y \in M(t', S')$. Therefore, $x \Delta_i^\lambda y \in M(t, S)$ and $M(t, S) \leq_i M(t', S')$.

($\Leftarrow$) Fix t . Let x and y be two elements in X and suppose that f is not C_i -quasisupermodular for some $\lambda^* \in [0, 1]$. The only case we need to look at is when $x_i > y_i$ and x and y are unordered. Also, $x \Delta_i^{\lambda^*} y \neq x$ and $x \nabla_i^{\lambda^*} y \neq y$. Let $S = \{x, x \Delta_i^{\lambda^*} y\}$ and $S' = \{y, x \nabla_i^{\lambda^*} y\}$. Then $S' \geq_i S$. C_i -quasisupermodularity of f can be violated in the following two ways. First, suppose $f(x, t) \geq f(x \Delta_i^{\lambda^*} y, t)$, but

$f(x\nabla_i^{\lambda^*}y, t) < f(y, t)$. In this case x is a maximizer of f in S and y maximizes f uniquely in S' , which violates the i -increasing property (since $x_i > y_i$). Alternatively, suppose $f(x, t) > f(x\Delta_i^{\lambda^*}y, t)$, but $f(x\nabla_i^{\lambda^*}y, t) = f(y, t)$. Now y maximizes f in S' while x is the unique maximizer in S . This again contradicts the i -increasing property. So f is C_i -quasisupermodular.

Now let $S \equiv \{x, \bar{x}\}$ with $x_i \leq \bar{x}_i$. Then $f(\bar{x}, t) - f(x, t) \geq 0$ implies $\bar{x} \in M(t, S)$. Since $M(t, S) \leq_i M(\bar{t}, S)$ for $\bar{t} \geq t$ it follows that $f(\bar{x}, \bar{t}) - f(x, \bar{t}) \geq 0$ for all $\bar{t} \geq t$. Thus f has the i -Single Crossing Property. $\square$

Notice that the solution to the optimization may not be unique, as we have made no assumptions to guarantee that. In the case of a solution set, $\underset{x \in S'}{\operatorname{argmax}} f(x, t')$ dominates $\underset{x \in S}{\operatorname{argmax}} f(x, t)$ in the C_i -flexible set order. By Proposition 3 in Quah (2007), this implies that $\underset{x \in S'}{\operatorname{argmax}} f(x, t')$ is i -higher² than $\underset{x \in S}{\operatorname{argmax}} f(x, t)$.

²Quah (2007) defines that a set S' is i -higher than a set S , if whenever both sets are nonempty, for any $x \in S$ there exists $x' \in S'$ such that $x'_i \geq x_i$ and for any $x' \in S'$ there exists $x \in S$ such that $x'_i \geq x_i$.

4. APPLICATIONS

4.1. Parametrized Consumer Utility Maximization Problem. The standard utility maximization problem as discussed in Quah (2007) can be extended by parametrization of the utility function by some parameter θ . It can be written as $\max u(x, \theta)$ subject to $p \cdot x \leq w$. The parameter vector now is two-dimensional, $t = (\theta, w)$.

As shown in Quah (2007), the budget set for $w' \geq w$ does not dominate the initial one by the strong set order, because the join of two arbitrary elements may lie outside of the larger set. However, it does dominate the smaller budget set in the C -flexible set order as illustrated in Figure 4.1.

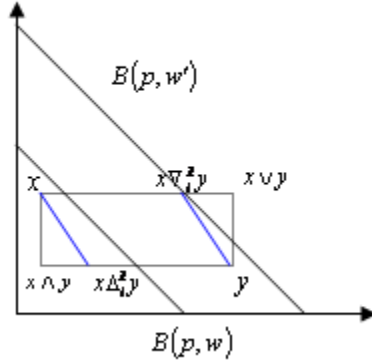


FIGURE 4.1. Budget set

For our comparative statics theorem to apply, the utility function needs to satisfy C_i -quasisupermodularity and the i -Single Crossing Property. By Proposition 2 in Quah (2007), u is C_i -quasisupermodular if it is supermodular and i -concave. The utility function has the i -Single Crossing Property if for all x, x' with $x'_i \geq x_i$, $u(x', \theta) \geq u(x, \theta)$ implies $u(x', \theta') \geq u(x, \theta')$ for $\theta' \geq \theta$. For a twice continuously

differentiable utility function, it is easy to check for increasing differences, that is $\frac{\partial^2 u}{\partial x_i \partial \theta} \geq 0$, which implies the i -Single Crossing Property. If these conditions are satisfied, by Theorem 3, we have nondecreasing solutions to the consumer's utility maximization problem. The class of Stone-Geary utility functions is an example of such parametrized utility functions and in the following possible interpretations of the parameter θ will be discussed.

Example 1. Stone-Geary Utility

Consider utility functions of the form $u(x) = \sum_{i=1}^n \alpha_i \log(x_i - b_i)$ with $\alpha_i > 0$, $x_i - b_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$. In this class of utility functions, the parameters $b_1, \dots, b_n$ can be interpreted as a necessary consumption basket. An interesting question in this context is how the consumer's demand for a good changes if his income increases, but so does his necessary consumption basket. Thus, $t = (b, w) \leq t' = (b', w')$. Theorem 3 readily provides comparative statics results for this case. We can easily check that u is C_i -quasisupermodular³ and satisfies the i -Single Crossing Property.⁴ Thus, since $B(p, w') \geq_i B(p, w)$, our result guarantees nondecreasing demand for good i as income and necessary consumption basket go up.

An application of Stone-Geary preferences can be found in Harbaugh (1998), that analyzes the prestige motive behind charitable donations. The paper considers two possible types of benefits to the donor, intrinsic benefit and prestige benefit. His utility hence depends on the amount of the public good x that he consumes, prestige

³ u is C_i -quasisupermodular if it is supermodular and i -concave. In this case, we see that u is supermodular since $\frac{\partial^2 u}{\partial x_i \partial x_j} = 0$, $\forall i, j, i \neq j$. Moreover, u is i -concave as $\frac{\partial^2 u}{\partial x_j^2} = -\frac{\alpha_j}{(x_j - b_j)^2} \leq 0$, $\forall j \neq i$.

⁴ $\frac{\partial^2 u}{\partial x_i \partial b_i} = \frac{\alpha_i}{(x_i - b_i)^2} \geq 0$

p and amount donated d . The donor's utility maximization problem can then be written as

$$\max u(x, p, d) = \log x + b \log(p + k_1) + c \log(d + k_2)$$

subject to the budget constraint $x + qd \leq w$, where q is the after-tax price of giving and k_1 and k_2 are non-negative constants that capture how much the individual values prestige and his intrinsic benefit. Putting it in the framework of the above discussion of Stone-Geary utility, $-k_1$ and $-k_2$ are the necessary amounts of prestige and intrinsic benefit for the donor. The amount of prestige resulting from a donation depends on how charities report donations and how society converts these reports into prestige. Harbaugh (1998) considers different reporting schemes by the charities, one of which is exact reporting. In this case, prestige is equal to the amount donated ($p = d$). The optimization problem can then be simplified to two variables of choice, the amount of the public good x and the donation d .

As shown above, this class of utility functions satisfies C_i -quasisupermodularity and has the i -SCP in $-k_i$. Since $B(q, w') \geq_i B(q, w)$ for an income increase from w to w' , the optimal donation is nondecreasing by Theorem 3, which is what the empirical analysis in Harbaugh (1998) shows as well. Our result however also adds a conclusion about changes in a person's value for prestige. If a person puts more emphasis on prestige ($-k_1$ increases) as his wealth increases, our comparative statics result yields nondecreasing donations.

4.2. Multiple-Plant Production. Another example that our comparative statics result can be applied to is production allocation in a multiple-plant firm. Basic two firm models can be found in Patinkin (1947) and Sattler and Scott (1982). In the

latter, the firm is allocating production of a given production target $\bar{q}$ between an old and a new plant, where the new plant has lower costs than the old one. The firm faces the following cost minimization problem:

$$\begin{aligned} \min \quad & C(q_{new}, q_{old}) = C_{new}(q_{new}) + C_{old}(q_{old}) \\ \text{subject to} \quad & q_{new} + q_{old} \geq \bar{q}. \end{aligned}$$

Both Patinkin (1947) and Sattler and Scott (1982) mostly focus on the “broken” marginal cost curve as the firm switches from only operating one plant to both plants as total output increases. Our comparative statics theorem on the other hand adds insights under what assumptions on the cost function and exogenous changes, the firm will increase production in both plants in the case that total output is such that both of them are in use.

Example 2. Since we are interested in comparative statics results, we include parameters ω_{new} and ω_{old} in the cost functions that measure the level of technology (with $\frac{\partial MC_i}{\partial \omega_i} \leq 0$) and rewrite the problem as

$$\begin{aligned} \max \quad & \tilde{C}(q_{new}, q_{old}, \omega_{new}, \omega_{old}) = -[C_{new}(q_{new}, \omega_{new}) + C_{old}(q_{old}, \omega_{old})] \\ \text{subject to} \quad & q_{new} + q_{old} \geq \bar{q}. \end{aligned}$$

Now consider a situation where production technology in the old plant gets updated and the firm increases its total production target. Hence, $t' = (\omega'_{new}, \omega'_{old}, \bar{q}') \geq t$, where t' is the new and t the initial parameter vector.

The new constraint set $S(\bar{q}')$ dominates the original one $S(\bar{q})$ in the C_i -flexible set order. Assuming that the cost function at each plant is convex in quantity, the objective function is C_i -quasisupermodular and satisfies the i -SCP for all i ⁵. As

⁵ $\tilde{C}$ is C_i -quasisupermodular if it is supermodular and i -concave.

a result, our comparative statics theorem yields nondecreasing optimal production quantities for either plant in this case.

4.3. Price Discrimination with Capacity Constraints. Another simple application of our comparative statics result can be found in the area of price discrimination with capacity constraints. In the airline and lodging industries, this problem is known as yield management. Belobaba (1987) summarizes yield management research in the airline industry; for examples from the lodging industry see for example Hanks, Cross and Noland (1992). Reece and Sobel (2000) discusses the example of an airline that practices price discrimination while facing a capacity constraint⁶ and addresses the question how airlines should adjust the allocation of seats between the customer groups as demand in one of the market segments increases or if costs of operation increase. They separately consider the cases of fixed non-binding and binding capacity as well as the possibility of capacity adjustment. They find that in the case on non-binding capacity constraints, changes in the marginal cost of operation directly affect prices and optimal quantities. In the case of binding capacity constraints, as long as marginal cost is below the point of intersection of the marginal revenue curves, changes in marginal cost do not affect the optimal capacity allocation; demand changes for one group however influence the optimal quantity and price of the other group. If the firm also optimally chooses capacity in the long

As each cost function is independent of the quantity produced at the other plant, $\frac{\partial^2 \tilde{C}}{\partial q_i \partial q_j} = 0$ for $i \neq j$, thus $\tilde{C}$ is supermodular. Moreover, assuming convex cost functions, $\frac{\partial^2 \tilde{C}}{\partial x_i^2} \geq 0$ for all i . Thus $\tilde{C}$ is C_i -quasisupermodular for all i .

Additionally, since marginal cost is nonincreasing in ω_i at both plants, $\frac{\partial^2 \tilde{C}}{\partial x_i \partial \omega_j} = -\frac{\partial MC_i}{\partial \omega_j} = 0$ for $i \neq j$ and $\frac{\partial^2 \tilde{C}}{\partial x_i \partial \omega_i} = -\frac{\partial MC_i}{\partial \omega_i} \geq 0$ for all i . Therefore the i -SCP holds.

⁶This multiple fare class problem is also briefly discussed in Belobaba (1989).

run, prices will only rise or fall in the long run if marginal cost changes as capacity is adjusted. With our comparative statics result, we can easily address a variety of combinations of the above mentioned scenarios.

Example 3. Consider an airline that price-discriminates between two groups of customers, i.e. business and leisure travelers. Since the number of seats on a plane is limited, the airline faces the following constrained optimization problem:

$$\begin{aligned} \max \quad & p_L(q_L, \phi_L) \cdot q_L + p_B(q_B, \phi_B) \cdot q_B - C(q_L, q_B, \omega) \\ \text{subject to} \quad & q_L + q_B \leq \bar{q} \end{aligned}$$

Since we are interested in comparative statics, the demand functions and the cost function have been parametrized. ϕ_L and ϕ_B capture exogenous demand shocks such as holiday travel or vacation time with $\frac{\partial p_i}{\partial \phi_i} \geq 0$ for $i = L, B$. The cost function parameter ω accounts for changes in transportation cost inputs such as fuel prices and assume marginal cost of transportation is nonincreasing in ω . Moreover, assume linear demand functions with $\frac{\partial p_i}{\partial q_j} \geq 0$, $i = L, B$ and constant marginal costs of transportation.

A straightforward comparative statics question is how the firms optimal allocation between business and leisure travelers changes during peak travel season compared to normal traffic. In anticipation of higher demand, the airline increases its capacity by assigning larger planes to popular routes. Additionally, we can add a decrease in input costs, such as lower kerosine prices to the scenario. Then, $t' = (\phi'_L, \phi'_B, \omega', \bar{q}') \geq t$.

Clearly, the constraint set at higher capacity $S(\bar{q}')$ dominates $S(\bar{q})$ in the C_i -flexible set order. Under the above assumptions on the objective function, π is C_i -quasisupermodular and satisfies the i -SCP. Thus, by Theorem 3, the airline's optimal number of seats allotted to both leisure and business customers is nondecreasing.

When comparing these results to those in Reece and Sobel (2000), we see that when solely focusing on changes in marginal cost our comparative statics theorem yields the same conclusions in the cases of non-binding capacity constraints and when considering capacity adjustments.

Demand changes in the case of fixed and binding capacity are one aspect that cannot be addressed by our result, since at capacity, the marginal cost for an extra seat for either one of the market segments is the marginal revenue of the other. So if as in the above example demand for leisure travel increases during holidays, MR_L and thus MC_B increase. For our result to apply, we need $t' \geq t$, which is not consistent with demand in one market increasing and marginal cost of transportation in the other market increasing.

Besides this special case of binding and fixed capacity, our framework can address comparative statics questions of a more general nature in this model, as it allows for other factors like demand shocks occurring in conjunction with transportation cost and capacity changes. Moreover, unlike the generally used comparative statics approach that requires uniqueness of solution, this result also applies in the case of multiple optimizers, as it could occur for piecewise profit functions that have a linear part.

4.4. Production Regulation by Efficiency Standards. Another area of application with a variety of examples is production regulation, particularly in environmental economics. Production can either be regulated by direct restrictions on production such as quotas or indirectly through efficiency standards, value restrictions, etc.

Production quotas directly impose a limit on the quantity a firm may produce to restrict supply and maintain a certain price level. The value of this limit is set by some regulatory agency, hence it depends on the strictness of the regulator. In the model, the rigidity of regulation is captured by the parameter θ . Additionally, the firm's objective function, its profit, will also be parametrized by ϕ and ω to capture shifts in demand and changes to the firm's costs.

A very general version of the firm's constrained optimization problem can then be written as follows:

$$\max \quad \pi = V(x_R, x_{UR}, \phi) - C(x_R, x_{UR}, \omega) \text{ subject to } x_R \leq \bar{q}(\theta),$$

where x_R denotes the regulated commodities the firm produces and x_{UR} is the vector of all unregulated goods the firm produces.

Naturally, a question of interest is how policy changes, economic shocks or technological progress affect the firm's optimal output, that is what impact do changes of the parameter vector $t = (\phi, \omega, \theta)$ have on the optimal solutions.

Denote the constraint set depending on the parameter vector t by $S(t) = \{x \in \mathbb{R}^n \mid x_R \leq \bar{q}(\theta)\}$. It can easily be seen that for $t' \geq t$ with $\theta' \geq \theta$, $S(t')$ dominates $S(t)$ in the strong set order. This type of problem can be addressed using the result by Milgrom and Shannon (1994).

Instead of imposing a direct limit on the quantity that a firm may produce of a certain good, an alternative policy to monitor output is to establish efficiency standards

or value restrictions. For an efficiency standard, a weighted sum of all commodities, with efficiency based weights, needs to lie below an upper bound determined by the regulator. In the case of a value restriction, the value of the firm's output may not exceed a given limit.

The constraint set depending on the parameter vector in this case can be written as $S(t) = \{x \in \mathbb{R}^n \mid \alpha \cdot x \leq \bar{\theta}\}$, where α represents either the vector of weights or prices. Notice that these constraint sets are not lattices like in the previous case of a production quota, hence the set at a higher parameter t' does not dominate the initial one at t in the strong set order. However, $S(t')$ dominates $S(t)$ in the C_i -flexible set order. Therefore, this type of problem cannot be addressed by existing results.

Our comparative statics theorem however does apply in this case, given that the firm's objective function satisfies the previously named assumptions of C_i -quasisupermodularity and the i -Single Crossing Property. For example, this is the case for simple linear demand and cost functions where demand for goods i and j is unrelated and marginal cost of good i is nonincreasing in the parameter ω .⁷

Example 4. Consider a car producer that produces two types of cars, one with high fuel efficiency and one with low fuel efficiency. The produced quantities of each type are denoted x_H and x_L . Let ϕ_H , ϕ_L and θ be parameters that capture changes in demand for high and low efficiency cars and the strictness of regulation for the production of the fuel-inefficient model. Demand for either car models increases in

⁷See Appendix for a more detailed discussion of C_i -quasisupermodularity and the i -SCP for profit functions.

ϕ_i , so $\frac{\partial p_i}{\partial \phi_i} > 0$. Moreover, let low values of θ imply a “green” mindset, which results in stricter regulation and lower production limits.

In the case of an efficiency standard, the optimization problem of the firm can be written as

$$\begin{aligned} \max \quad & \pi = p_H(\phi_H)x_H + p_L(\phi_L)x_L - C(x_H, x_L) \\ \text{subject to} \quad & \alpha_H x_H + \alpha_L x_L \leq \eta(\theta), \end{aligned}$$

where α_H and α_L are weights based on energy consumption and $\eta(\theta)$ is an upper bound.

A question of interest now is how an increase in all parameters from $t = (\phi_H, \phi_L, \theta)$ to $t' > t$, that represents a change to a generally less environmentally conscious attitude in society, affects the firm’s optimal output for both types of cars. It seems straightforward that less emphasis on the environment leads to laxer regulation standards for inefficient cars. Also, a less “green” attitude by society increases the demand for cars. Demand for low efficiency cars is higher because people are not willing to pay a more expensive, but also more efficient car. On the other hand, people that would not buy cars at all in the greener mindset and rely solely on public transportation, bicycles and walking may now buy high efficiency cars.

First of all, notice that the new constraint set for $\theta' > \theta$ dominates the previous one in the C_i -flexible set order as illustrated in Figure 4.2, while these constraint sets cannot be ranked in the strong set order.

As previously pointed out, the profit function π is C_i -quasisupermodular if the demand and cost function are linear and if either demand of good i and good j are unrelated or if $\frac{\partial p_i}{\partial x_j} \geq 0$ and $\frac{\partial p_j}{\partial x_i} \geq 0$. Clearly, this is the case if both markets are

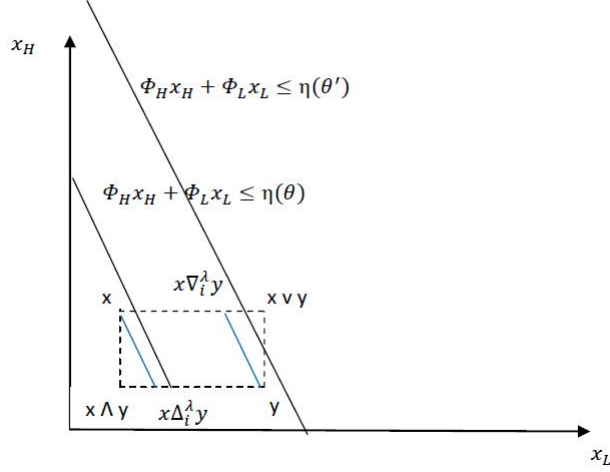


FIGURE 4.2. Efficiency Standard Constraint Sets

perfectly competitive and therefore prices are determined by the market and costs are linear.

Moreover, the profit function also needs to satisfy the *i*-Single Crossing Property. If π is twice continuously differentiable, it is easy to check if it exhibits the cardinal concept of increasing differences, which implies the *i*-Single Crossing Property. The conditions under which the profit function has increasing differences are $\frac{\partial^2 \pi}{\partial x_L \partial \phi_L} = \frac{\partial p_L}{\partial \phi_L} \geq 0$ and $\frac{\partial^2 \pi}{\partial x_H \partial \phi_H} = \frac{\partial p_H}{\partial \phi_H} \geq 0$. These conditions are consistent with the idea of the model that demand is increasing in the parameters ϕ_L and ϕ_H .

Hence the conditions for Theorem 3 are satisfied and we have nondecreasing optimal solutions. Thus production of both high and low efficiency cars is nondecreasing if society cares less about reducing emissions and the environment.

Other possible interpretations of changes in the parameters ϕ_L and ϕ_H in this example are economic conditions such as changes in household wealth or more or

less favorable interest rates that lead to increases or decreases in the demand for cars. Moreover, we can parametrize the cost function by ω_H and ω_L as well to account for changes in production technology or government taxes and subsidies for either type.

4.5. Emissions Standards and Production Decisions. Helfand (1991) examines the effect of various different forms of emissions standards on a firm's optimal output decision. While not all of them fit in our framework, some of the findings can be replicated and extended to more general comparative statics questions using our result. Helfand (1991) finds that the direction of input adjustments after the introduction of a standard depends on the sign of the cross-partial of the production function, which is the change in the marginal product of one input as another one changes.

Example 5. Consider a firm with a production function $f(x_1, x_2)$ and output is non-decreasing with nonincreasing marginal returns for each input ($\frac{\partial f}{\partial x_i} \geq 0$, $\frac{\partial^2 f}{\partial x_i^2} \leq 0$ for all i). During the production process the firm also produces pollution $A(x_1, x_2)$. First consider the case where both inputs contribute to pollution, so $\frac{\partial A}{\partial x_i} > 0$. Furthermore, suppose the firm has a linear cost function and that the market for this good is perfectly competitive. Without any regulatory constraints the firm maximizes its profit

$$\pi = p \cdot f(x_1, x_2) - \omega_1 x_1 - \omega_2 x_2.$$

One type of standard discussed in Helfand (1991) is the case of a set amount of a specific input. Here, the regulator puts a limit on how much of a polluting

input can be used in the production process or requires a minimum amount of a pollution-abating input. Suppose input 1 increases pollution, while input 2 reduces emissions.

First, suppose the regulator restricts the use of the polluting input. So the standard takes the form $x_1 \leq \bar{A}_1$. For $\bar{A}_1' \geq \bar{A}_1$, the constraint set $S(\bar{A}_1')$ dominates $S(\bar{A}_1)$ in the strong set order. Alternatively, a minimum amount of the pollution-reducing input could be required, so $x_2 \geq \bar{A}_2$. Again, for $\bar{A}_2' \geq \bar{A}_2$, $S(\bar{A}_2')$ dominates $S(\bar{A}_2)$ in the strong set order.

While these types of standards can be addressed using the result of Milgrom and Shannon (1994), there are other ones that do not fall into this framework. For example, Helfand (1991) also considers the case where the regulator introduces an emissions standard $\bar{A}$ that limits the total amount of allowed pollution by this firm in a given period of time. Moreover, assume that pollution is proportional to the amount of input used in the production process. We can then write this constraint as $a_1x_1 + a_2x_2 \leq \bar{A}$. As both inputs cause pollution, $a_1, a_2 > 0$.

With regard to comparative statics, the natural question to ask is how changes in the emissions standard affect the firm's input decision. With the comparative statics result in Theorem 3 we can go a little bit further than that and examine how changes in the parameter vector $t = (p, -\omega_1, -\omega_2, \bar{A})$ affects the optimal input decision of the firm. Suppose t increases to $t' = (p', -\omega_1', -\omega_2', \bar{A}') \geq t$, that is the price for the finished product increases, input prices decrease and regulation is loosened. Clearly, the new constraint set $S(\bar{A}')$ dominates $S(\bar{A})$ in the C_i -flexible set order. For Theorem 3 to apply, the objective function needs to satisfy C_i -quasisupermodularity and

the i -SCP. For the above profit function, this will be the case given that $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$, $i \neq j$ ⁸. Then the firm's optimal input quantities are nondecreasing in t .

Hence, when we limit our attention to changes in the emissions standard, we find the same results as Helfand (1991). If $\frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$ and the Hessian is negative semi-definite, which in our approach is needed for C_i -quasisupermodularity, more polluting inputs and less of emissions-abating inputs are used when standards are looser (or non-existent) than under stricter regulation. Moreover, our comparative statics theorem can address a more general version of these constrained optimization problems. Price and technology parameters can be included in the objective function and our result addresses changes in optimal inputs for changes in the parameter vector consisting of price, technology and emissions standard.

In addition to that, Helfand (1991) restricts attention to unique optimal solutions of the firm's constrained optimization problem by assuming that the profit function is strictly concave. Our result goes beyond that, as it does not require strict concavity and provides comparative statics results also in the case of multiple solutions.

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π is C_i -quasisupermodular if it is supermodular and i -concave. The former is the case if $\frac{\partial^2 \pi}{\partial x_i \partial x_j} = p \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0$. By assumption, $\frac{\partial^2 f}{\partial x_i^2} \leq 0$; thus $\frac{\partial^2 \pi}{\partial x_i^2} = p \cdot \frac{\partial^2 f}{\partial x_i^2} \leq 0$ for all i and π is i -concave for all i in the 2 goods case. In the general case, the Hessian

$$H_{-i} = \begin{bmatrix} f_{1,1} & \cdots & f_{1,i-1} & f_{1,i+1} & \cdots & f_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{i-1,1} & \cdots & f_{i-1,i-1} & f_{i-1,i+1} & \cdots & f_{i-1,n} \\ f_{i+1,1} & \cdots & f_{i+1,i-1} & f_{i+1,i+1} & \cdots & f_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{n,1} & \cdots & f_{i-1,n} & f_{i+1,n} & \cdots & f_{n,n} \end{bmatrix}$$

needs to be negative semi-definite.

Moreover, $\frac{\partial^2 \pi}{\partial x_i \partial p} = \frac{\partial f}{\partial x_i} \geq 0$, $\frac{\partial^2 \pi}{\partial x_i \partial (-\omega_j)} = 0$ for $i \neq j$ and $\frac{\partial^2 \pi}{\partial x_i \partial (-\omega_i)} = 1 \geq 0$ for all i . Hence the i -SCP holds.

4.6. Emissions Standards and Technological Innovation. Another example in the context of production regulation is technological innovation under emissions standards. Bruneau (2004) compares incentives for innovation in emissions-reducing technology for emissions and performance standards. In the course of the discussion, the paper also addresses changes in output and emissions abatement due to technical innovation.

Example 6. Consider the case of an emissions standard, that restricts emissions by a firm to $\bar{e}$. Suppose units are scaled such that emissions e rise/abatement a falls one-to-one with output q , so $e = q - a$. Then a firm, when choosing optimal output and abatement levels faces the constraint $q - a \leq \bar{e}$.

Abatement however is costly, but these costs can be lowered by technological innovation. This is represented by the abatement cost $kC(a)$, where $k \leq 1$ is a technology parameter. Moreover, assume that the market price P depends on industry output Q .

The firm then solves the following constraint profit maximization problem:

$$\begin{aligned} \max_{q,a} \quad & \pi = P \cdot q - C(q) - kC(a) \\ \text{s.t.} \quad & q - a \leq \bar{e} \end{aligned}$$

Bruneau (2004) uses quadratic specifications for both cost functions ($C(q) = \frac{1}{2}cq^2$ and $kC(a) = \frac{1}{2}ka^2$) and first considers the simplified case where production costs

are equal to zero. He finds that in both cases, technological progress from $k = 1$ to $k < 1$ leads to increases in output and abatement.

Our comparative statics result can be applied in this context to replicate the comparative statics results in Bruneau (2004) and provides conditions on the profit function that yields nondecreasing optimal solutions. Technical innovation means a decrease in k (or increase in $-k$). For Theorem 3 to apply, the profit function needs to satisfy C_i -quasisupermodularity and the i -Single Crossing Property.⁹ We can easily check that π is C_i -quasisupermodular, as it is supermodular ($\frac{\partial^2 \pi}{\partial q \partial a} = \frac{\partial^2 \pi}{\partial a \partial q} = 0$) and i -concave ($\frac{\partial^2 \pi}{\partial q^2} \leq 0$ for linear demand functions, $\frac{\partial^2 \pi}{\partial a^2} \leq 0$). It also has the i -SCP, since $\frac{\partial^2 \pi}{\partial q \partial (-k)} = 0$ and $\frac{\partial^2 \pi}{\partial a \partial (-k)} = a \geq 0$. Thus the profit function satisfies the assumptions for Theorem 3 and therefore yields nondecreasing optimal solutions for output and abatement.

With our result, we can easily extend the results in Bruneau (2004) by considering comparative statics with respect to additional parameters such as demand shocks or production costs, as well as changes in the constraint set due to regulatory adjustments to the emissions standard $\bar{e}$. Rewrite the profit function as $\pi = P(\alpha) \cdot q - \frac{1}{2}cq^2 - \frac{1}{2}ka^2$ and let $t = (\alpha, -c, -k, \bar{e})$. An increase in the parameter vector to t' can be interpreted as a positive demand shock and technological progress that lowers production and abatement costs, while the regulator loosens the emissions standard. Theorem 3 yields nondecreasing optimal solutions for output and abatement, since we can check that the i -SCP for the additional parameters is satisfied and the constraint set at $\bar{e}'$ dominates the one at $\bar{e}$ in the C_i -flexible set order.

⁹As the constraint set remains unchanged, we do not have to check if the sets can be ranked by the C_i -flexible set order.

4.7. Ethanol Quota. In the Energy Independence and Security Act of 2007, the Renewable Fuel Standard (RFS) requires the blenders to use increasing amounts of renewable fuel, which in practice mainly means ethanol made from corn. The policy aims to reduce dependence on oil for gasoline production and replace petroleum gasoline by renewable fuels such as ethanol. For conventional cars, fuel can be blended with up to 10% to 15% of ethanol, flex-fuel vehicles can also run on 85% ethanol.

Example 7. In this example adapted from Zhang, Qiu and Wetzstein (2010), consider the gasoline blending sector. Blenders provide two types of fuel, E85 (85% ethanol and 15% petroleum gasoline) and E_γ , which contains γ % ethanol and $1 - \gamma$ % petroleum gasoline. Currently the maximum blend is regulated at 15%. To consider possible adjustments of this “blend wall”, let $0.1 \leq \gamma \leq 0.2$. Let $p_{85}(E_{85}, \phi)$ and $p_\gamma(E_\gamma, \phi)$ denote inverse market demand for E85 and E_γ , with ϕ being a parameter that captures changes in demand. The blender uses e_{85} and g_{85} in ethanol and petroleum gasoline in the blending process, which is proportional. Hence, $\frac{e_{85}}{e_{85} + g_{85}} = 0.85$ or after rearranging terms, $g_{85} = \frac{3}{17}e_{85}$. Similarly for E_γ , $g_\gamma = \frac{1-\gamma}{\gamma}e_\gamma$. The production technologies $y_{85}(e_{85})$ and $y_\gamma(e_\gamma)$ of blended fuel can be expressed only in terms of amount of ethanol used because of the proportional production process and output is assumed to be increasing with diminishing marginal returns.

Moreover, under the RFS, each blender has to meet his mandated ethanol quota of e_0 , that is his total quantity of ethanol $e = e_{85} + e_\gamma$ must exceed e_0 . The blenders profit maximization problem can then be written as follows.

$$\begin{aligned} \max \quad & \pi = p_{85}(E_{85}, \phi) \cdot y_{85}(e_{85}) + p_\gamma(E_\gamma, \phi) \cdot y_\gamma(e_\gamma) - c_G\left(\frac{3}{17}e_{85} + \frac{1-\gamma}{\gamma}e_\gamma\right) - c_e \cdot (e_{85} + e_\gamma) \\ & \text{subject to } e_{85} + e_\gamma \geq e_0. \end{aligned} \quad ^{10}$$

Zhang, Qiu and Wetzstein (2010) finds that under certain conditions on demand elasticities, an increase in the “blend wall” γ likely delays the shift towards flex-fuel vehicles and leads to an increase in E_γ supply and a decrease in E_{85} provided.

Our result allows us to consider the impact of raising the “blend wall” γ and increasing the ethanol quota. The constraint set $S(e'_0)$ under the new policy dominates the original one, $S(e_0)$ in the C_i -flexible set order. As discussed in the appendix, the profit function is C_i -quasisupermodular for linear demand and cost functions and unrelated demand. The i -Single Crossing Property is satisfied for p_{85} and p_γ increasing in ϕ and we can check that $\frac{\partial^2 \pi}{\partial e_\gamma \partial \gamma} \geq 0$.

Thus, Theorem 3 applies and yields nondecreasing optimal solutions for both e_{85} and e_γ ; hence the policy change favoring the use of renewable fuel leads to an increase in both E85 and regular fuel. Our result is the same as in Zhang, Qiu and Wetzstein (2010) for regular fuel and also yields the conclusion that these policies promoting renewable fuels don’t necessarily accelerate the transition to flex-fuel vehicles. The advantages of our approach lie in the ability to easily consider a more general problem

¹⁰Unlike Zhang, Qiu and Wetzstein (2010), this example considers the case of a constrained optimization problem with an ethanol quota without the option to buy and sell permits, which eliminates the constraint. The unconstrained problem can be addressed by existing comparative statics result, this type of constrained optimization problem however cannot.

with additional parameters and changes in the constraint set as well as assumptions on the objective function instead of demand elasticities.

4.8. Cost-Efficient Emissions Regulation. Another application in environmental economics is the problem of cost-efficient emissions regulation. The goal of the regulator is to limit the amount of total emissions of a given pollutant. As the costs to reduce emissions vary among producers, the regulator's goal is to achieve the desired emissions limit at the lowest total abatement cost. Theoretical models for efficient emissions allocation can first be found in Baumol and Oates (1971) or Montgomery (1972) and continue to serve as a baseline model as for example in Muller and Mendelsohn (2009). Cost-effectiveness has been the main criterion for the design of permit markets and therefore the cost-efficient allocation continues to be of interest as the baseline comparison even as regulation is moving more and more towards market-based approaches and away from the traditional command-and-control regulation.

The design of regulatory standards depends on the class of pollutants. First, consider the class of uniformly mixed assimilative pollutants, such as greenhouse gases. These types of pollutants do not accumulate in the atmosphere over time and their concentration only depends on the total amount of emissions regardless the source and its location. In the following example, consider a version of this model adapted from Montgomery (1972) and Tietenberg (2006).

Example 8. Suppose there are J producers that emit a uniformly mixed assimilative pollutant, for example carbon dioxide. The regulator sets the target emissions limit

at $\bar{A}$. The relationship between firms' emissions and total pollution can be described as

$$A = a + b \sum_{j=1}^J (\bar{e}_j - r_j),$$

where A is pollution per year, $\bar{e}_j$ denotes the uncontrolled emission rate of firm j and r_j is firm j 's emissions reduction. The parameters a and b represent “background pollution” from other sources and the degree of proportionality between emissions and total pollution. Moreover, assume that each firm's cost function $C_j(r_j, \omega)$ is continuous and that the marginal cost of emissions reduction is increasing. The parameter ω captures technological change. The regulators problem can then be written as

$$\begin{aligned} \max \quad & \tilde{C} = - \sum_{j=1}^J C_j(r_j, \omega) \\ \text{subject to} \quad & a + b \sum_{j=1}^J (\bar{e}_j - r_j) \leq \bar{A}, \quad r_j \geq 0 \end{aligned}$$

Now consider an increase in the parameter vector $t = (\omega, -\bar{A})$ caused by new emissions reducing technology and a reduction of the pollution target by the regulator.

The constraint set $S(-\bar{A}')$ dominates $S(-\bar{A})$ by the C_i -flexible set order and the objective function $\tilde{C}$ is supermodular ($\frac{\partial^2 \tilde{C}}{\partial r_i \partial r_j} = 0$, for all i, j) and i -concave (in the two firm case, $\frac{\partial^2 \tilde{C}}{\partial r_j^2} = -\frac{\partial MC_j}{\partial r_j} < 0$ since marginal abatement cost is increasing; in the general case, we need to verify that the Hessian H_{-i} is negative semi-definite). Moreover, $\frac{\partial^2 \tilde{C}}{\partial r_j \partial \omega} = -\frac{\partial MC_j}{\partial \omega} \geq 0$ as technological progress decreases the firm's marginal cost of emissions reduction and hence $\tilde{C}$ exhibits the i -Single Crossing Property. Therefore, all conditions of Theorem 3 are satisfied and the cost-effective, optimal amounts of emissions reduction are nondecreasing for all firms.

Our result can be used in this setting to illustrate the effect of regulation regime changes and technological progress in emissions abatement on the cost-efficient emissions allocation. Under natural assumptions on the primitive, our theorem provides comparative statics results for a common baseline model in emissions regulation. As this is the allocation that other regulatory systems are designed to attain as well, it also suggests how these changes affect the outcome in today's market-based approaches.

Unlike the standard comparative statics approach that requires uniqueness of solution, this result can also be applied in cases with multiple optimizers. This could occur in the case of a piecewise cost function that is partly linear (see Figure 4.3).

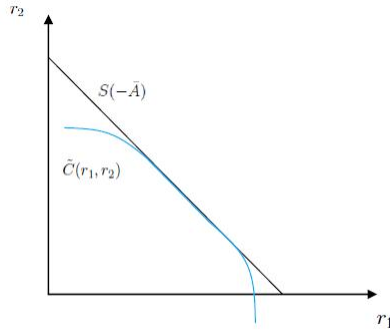


FIGURE 4.3. Optimal emissions allocation with multiple solutions

4.9. LeChatelier Principle.

Example 9. The LeChatelier principle in economics expresses the idea that long run demand is more responsive to price changes than demand in the short run. For example, consider the impact of an input price reduction on a firm's demand for that input. The LeChatelier principle now says, that the demand increase in the short run, when some factors of production are fixed, is smaller than the increase in the long run, when all factors can be adjusted freely. Milgrom and Roberts (1996) introduces a lattice-theory based global LeChatelier principle, which is extended to additional classes of constraints faced by the firm in the short run in Quah (2007).

Quah (2007) points out the two different types of comparative statics problems associated with the short run and long run factor demand adjustment following a price change. In the short run, the firm's objective function changes while the constraint set, based on the optimal point before the price change, stays the same. Between the short run and the long run, the objective function remains the same, but the short run constraints no longer exist; hence the constraint set changes.

Thus we are considering the following two increases of the parameter vector $t = (\theta, S)$, consisting of the cost parameter θ and the constraint set S . In the short run, t increases to $t' = (\theta', S) \geq t$ as the cost parameter changes. As the constraint set changes in the long run, the parameter vector increases again from t' to $t'' = (\theta', S')$.

The firm's objective function can be written as $\pi(x, \theta) = V(F(x)) - C(x, \theta)$, where x is the firm's input vector, $F(x)$ its production function and V the revenue from output. Quah (2007) specifies the firm's cost function as $C(x, \theta) = p \cdot x - \theta x_1$, where p denotes the input price vector and $\theta > 0$ is the considered price reduction for input 1. In the short run, the firm faces constraints, since not all inputs can be varied without

costs. Thus, this constraint set S needs to include the pre-price change optimal input vector x^* . So, $S = \{x \in \mathbb{R}_+^l : x^* \in S\}$. Thus the short run adjustment from x^* to x^{**} is a comparative statics problem where the parameter θ in the objective function changes, but the constraint set remains unchanged at S . In the long run, the change of the constraint set from S to S' leads to the change in input demand from x^{**} to x^{***} .

The proof of the following proposition is in parts similar to the proof of Proposition 8 in Quah (2007).

Proposition 2. *Let x^* maximize $\pi(x, \theta)$ subject to $x \in \mathbb{R}_+^l$. Suppose also, that solutions x^{**} and x^{***} to the problems*

- (1) *maximize $\pi(x, \theta')$ subject to $x \in S$ and*
- (2) *maximize $\pi(x, \theta')$ subject to $x \in S' = \mathbb{R}_+^l$*

where $\theta' \geq \theta$ exist.

*Then $x_i^{***} \geq x_i^{**} \geq x_i^*$ if π is C_i -quasisupermodular, satisfies the i -Single Crossing Property and $X_s = \{x \in \mathbb{R}_+^l : x \geq s \text{ for some } s \in S\} \geq_i S$.*

Proof. First consider the increase from $t = (\theta, S)$ to $t' = (\theta', S)$. Since π is C_i -quasisupermodular, satisfies the i -Single Crossing Property and the constraint set remains unchanged, $x_i^{**} \geq x_i^*$ by Theorem 3.

Now t' increases to $t'' = (\theta', S')$. Given that π is C_i -quasisupermodular, has the i -SCP and $\operatorname{argmax}_{x \in S'} \pi(x, \theta')$ exists, there is $\bar{x} \in \operatorname{argmax}_{x \in S'} \pi(x, \theta')$ such that $\bar{x} \geq x^*$. Since $x^* \in S$, $\bar{x} \in \operatorname{argmax}_{x \in X_s} \pi(x, \theta')$.

Moreover, we know that $x^{**} \in \operatorname{argmax}_{x \in S} \pi(x, \theta')$. Since $X_S \geq_i S$, by Theorem 3 there is $x^{***} \in \operatorname{argmax}_{x \in X_S} \pi(x, \theta')$ such that $x_i^{***} \geq x_i^{**}$. Lastly, $x^{***} \in \operatorname{argmax}_{x \in S'} \pi(x, \theta')$, because $\bar{x} \in X_S$ and $\bar{x} \in \operatorname{argmax}_{x \in S'} \pi(x, \theta')$. $\square$

We can easily check that the above objective function is C_i -quasisupermodular if $V \circ F$ is C_i -quasisupermodular. In the case of a perfectly competitive output market, this is the case when all inputs are complementary to each other and the production function displays decreasing marginal product for each input.¹¹ Moreover, the objective function needs to satisfy the i -SCP. This is the case whenever marginal costs of production are nonincreasing in θ .¹²

The advantages in this version of the LeChatelier principle is that like Milgrom and Roberts (1996), it allows for more general parameter changes than Quah (2007) (who only considers the very specific case described above) while also permitting larger classes of constraint sets than Milgrom and Roberts (1996) like Quah (2007). The result in Milgrom and Roberts (1996) is limited to constraint sets S and X_S that can be ranked in the strong set order, which is the case, for example, when certain inputs are held fixed in the short run. Quah (2007) however allows for more general, economically plausible constraint sets. For example, suppose there is one good that

¹¹For supermodularity, we need $\frac{\partial^2 \pi}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_j} \left(\frac{\partial V}{\partial F} \frac{\partial F}{\partial x_i} \right) \geq 0$. Assuming a perfectly competitive output market, $\frac{\partial V}{\partial F} = P$, where P is the market price of output. Then $\frac{\partial^2 \pi}{\partial x_i \partial x_j} = P \cdot \frac{\partial^2 F}{\partial x_i \partial x_j}$, which is nonnegative if $\frac{\partial^2 F}{\partial x_i \partial x_j} \geq 0$.

For i -concavity in the simple two goods case with a linear cost function, we need $\frac{\partial^2 \pi}{\partial x_j^2} \leq 0$, for all $j \neq i$. In a perfectly competitive environment, $\frac{\partial^2 \pi}{\partial x_j^2} = P \cdot \frac{\partial^2 F}{\partial x_j^2}$, which is nonpositive if $\frac{\partial^2 F}{\partial x_j^2} \leq 0$. In the general case, we need to verify that the Hessian of the production function, H_{-i}^F , is negative semi-definite and that the Hessian of the cost function, H_{-i}^C , is positive semi-definite.

¹²The i -SCP is satisfied if $\frac{\partial^2 \pi}{\partial x_i \partial \theta} = -\frac{\partial^2 C(x, \theta)}{\partial x_i \partial \theta} \geq 0$. This is the case whenever $\frac{\partial MC(x, \theta)}{\partial \theta} \leq 0$.

serves different roles in the production process and is therefore considered as several inputs. If in the short run, the total quantity used of these inputs cannot be changed, the constraint set can be written as $S = \{x \in \mathbb{R}_+^l : \sum_{i=m}^l x_i = \sum_{i=m}^l x_i^*\}$. In this case, X_S and S cannot be ranked by the strong set order; however, X_S dominates S in the C -flexible set order.

Appendix

C_i -Quasisupermodularity and the i -SCP of Profit Functions

The objective function is C_i -quasisupermodular if it is supermodular and i -concave (Proposition 2, Quah (2007)). π is supermodular if $\frac{\partial^2 \pi}{\partial x_i \partial x_j} \geq 0$ for all i, j and $i \neq j$. This is equivalent to $\frac{\partial^2 V}{\partial x_i \partial x_j} - \frac{\partial^2 C}{\partial x_i \partial x_j} \geq 0$. Thus the profit function is supermodular if marginal revenue of good i is nondecreasing in x_j and marginal cost of good i is nonincreasing in x_j . We can decompose the total revenue in two parts, the revenue from regulated goods and revenue from unregulated goods. So, $V(x_R, x_{UR}, \phi) = p_R(x_R, x_{UR}, \phi_R) \cdot x_R + p_{UR}(x_R, x_{UR}, \phi_{UR}) \cdot x_{UR}$. Hence,

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 p_i}{\partial x_i \partial x_j} \cdot x_i + \frac{\partial p_i}{\partial x_j} + \frac{\partial^2 p_j}{\partial x_i \partial x_j} \cdot x_j + \frac{\partial p_j}{\partial x_i}.$$

For linear demand, $\frac{\partial^2 p_i}{\partial x_i \partial x_j} = \frac{\partial^2 p_j}{\partial x_i \partial x_j} = 0$. Therefore, marginal revenue of good i is nondecreasing in x_j for linear demand if either demand of good i and good j are unrelated or if $\frac{\partial p_i}{\partial x_j} \geq 0$ and $\frac{\partial p_j}{\partial x_i} \geq 0$. By Definition 7, the objective function is i -concave if it is concave in all variables other than i .

So for the firm's profit function, the Hessian,

$$H_{-i} = \begin{bmatrix} \pi_{11} & \cdots & \pi_{1,i-1} & \pi_{1,i+1} & \cdots & \pi_{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi_{i-1,1} & \cdots & \pi_{i-1,i-1} & \pi_{i-1,i+1} & \cdots & \pi_{i-1,n} \\ \pi_{i+1,1} & \cdots & \pi_{i+1,i-1} & \pi_{i+1,i+1} & \cdots & \pi_{i+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \pi_{n,1} & \cdots & \pi_{i-1,n} & \pi_{i+1,n} & \cdots & \pi_{n,n} \end{bmatrix}$$

needs to be negative semi-definite.

In the simple case of only 2 commodities and linear cost and demand functions, we only need to check that $\frac{\partial^2 \pi}{\partial x_j^2} \leq 0$ for all $j \neq i$. For linear cost and demand functions, $\frac{\partial^2 \pi}{\partial x_j^2} = \frac{\partial^2 V}{\partial x_j^2} - \frac{\partial^2 C}{\partial x_j^2} = \frac{\partial^2 V}{\partial x_j^2} = \frac{\partial^2 p_j}{\partial x_j^2} \cdot x_j + 2 \frac{\partial p_j}{\partial x_j} + \frac{\partial^2 p_i}{\partial x_j^2} \cdot x_i = 2 \frac{\partial p_j}{\partial x_j} \leq 0$.

Hence in this simple case, π is C_i -quasisupermodular if the demand and cost function are linear and if either demand of good i and good j are unrelated or if $\frac{\partial p_i}{\partial x_j} \geq 0$ and $\frac{\partial p_j}{\partial x_i} \geq 0$.

The i -Single Crossing Property is satisfied if $\frac{\partial^2 \pi}{\partial x_i \partial \phi_i} \geq 0$, $\frac{\partial^2 \pi}{\partial x_i \partial \omega} \geq 0$ and $\frac{\partial^2 \pi}{\partial x_i \partial \theta} \geq 0$. Again assuming a linear demand function,

$$\begin{aligned} \frac{\partial^2 \pi}{\partial x_i \partial \phi_i} &= \frac{\partial^2 V}{\partial x_i \partial \phi_i} = \frac{\partial^2 p_i}{\partial x_i \partial \phi_i} \cdot x_i + \frac{\partial p_i}{\partial \phi_i} = \frac{\partial p_i}{\partial \phi_i}, \\ \frac{\partial^2 \pi}{\partial x_i \partial \omega} &= -\frac{\partial^2 C}{\partial x_i \partial \omega} = -\frac{\partial MC_i}{\partial \omega} \text{ and} \\ \frac{\partial^2 \pi}{\partial x_i \partial \theta} &= 0. \end{aligned}$$

Thus, we need $\frac{\partial p_i}{\partial \phi_i} \geq 0$ and marginal cost of good i to be nonincreasing in ω for the i -Single Crossing Property to hold.

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