Global Games Selection in Games with Strategic Substitutes or Complements

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Abstract

Global games methods are aimed at resolving issues of multiplicity of equilibria and coordination failure that arise in game theoretic models by relaxing common knowledge assumptions about an underlying parameter. These methods have recently received a lot of attention when the underlying complete information game is one of strategic complements (GSC). Little has been done in this direction concerning games of strategic substitutes (GSS), however. This paper complements the existing literature in both cases by extending the global games method developed by Carlsson and Van Damme (1993) to N-player, multi-action GSS and GSC, using a p-dominance condition as the selection criterion. Moreover, this approach is much less restrictive on the conditions that payoffs and the underlying parameter space must satisfy, and therefore serves to cirumvent recent criticisms to global games methods. The second part of this paper generalizes the model by allowing "groups" of players to receive homogenous signals, which, under certain conditions, strengthens the model's power of predictability.

1 Introduction

The global games method serves as an equilibrium selection device for complete information games by embedding them into a class of Bayesian games that exhibit unique equilibrium predictions. This method was pioneered by Carlsson and Van Damme (CvD) (1993) for the case of 2 player, binary action coordination games. In that paper, a complete information game with multiple equilibria was considered, and instead of players observing a specific parameter in the model directly, players were allowed to observe noisy signals about the parameter, transforming it into a Bayesian game. As the signals become more precise, a serially undominated Bayesian prediction emerges, with the interpretation of delivering a unique prediction in a slightly "noisy" version of the original complete information game, resolving the original issue of multiplicity. This method has since been extended to multi-player, multi-action games of strategic complements (GSC) by Frankel, Morris, and Pauzner (FMP)(2003). They observe that if the parameter in question produces dominance regions, where high parameter values correspond to the highest action being strictly dominant for all players and low parameter values correspond to the lowest action being strictly dominant for all players, then as the signals become less noisy, a unique global games prediction emerges. FMP and subsequent work along this line has emphasized two underlying features of global games analysis: uniqueness of selection and noise-independent selection. The former refers to the robustness of a global games selection to the distribution assigned to the noisy parameter, the latter to the uniqueness of the selection as noise shrinks to zero.

Little work in this area has been done in games of strategic substitutes (GSS). Morris (2008) has shown that this case can be much more complex by giving an example of a GSS satisfying the traditional sufficient conditions in FMP yet failing to produce a unique prediction. Harrison (2003) studies scenarios where this difficulty can be overcome by considering binary action aggregative GSS with sufficiently heterogenous players and overlapping dominace regions. Still, the global games solution can only be guaranteed to be a unique Bayesian Nash equilibrium and is not the dominance solvable solution. Although this difficulty is computational in nature, there is also a very important theoretical difficulty that can arise in global games analysis, due to a recent observation by Weinstein and Yildiz (WY) (2007). In games of incomplete information, rationality arguments rely on analyzing a player's hierarchy of beliefs, that is, their belief about the parameter space, what they believe their opponents believe about the parameter space, and so on. In general situations, this information can be identified as a player's type, or a probability measure over the state space and the types of others. Suppose Θ is an underlying parameter space, and consider a complete information situation where players' types assign probability 1 for a specific $\theta^* \in \Theta$, at which player i has multiple rationalizable strategies. WY shows that if the parameter space is "rich" enough, so that any given rationalizable strategy a_i^* for player i is strictly dominant at *some* parameter $\theta^{a_i^*} \in \Theta$, then there is a type for player i that is arbitrarily close to that under the original parameterization θ^* but having a_i^* as the uniquely rationalizable action. This poses a serious criticism to global games analysis: If players' beliefs can be slightly perturbed in a specific way so that any given rationalizable strategy can be justified as the unique rationalizable strategy, how does a modeller know if the global games method is the "right" way to refine the set of equilibria?

One approach, which is alluded to in Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010), is to introduce incomplete information in a "natural" way by introducing uncertainty only to those parameters which are present in the model. This is directly opposed to Basteck, Daniëls, and Heinemann (2012), who have shown that any GSC can be parameterized so that dominance regions are established and the conditions of FMP are met, so that a global games prediction emerges regardless if a model's original parameter space lends itself to such analysis. The latter approach, however, runs the risk of facing the full brunt of the WY critcism, as the following example shows:

Consider a slightly modified version of the technology adoption model considered in Keser, Suleymanova, and Wey (2012). Three agents must mutually decide on whether to adopt an inferior technology A, or a superior technology B. The benefit to each player i of adopting a specfic technology $t = A, B$ is given by

$$U_t(N_t) = v_t + (N_t - 1)$$

where N_t is the total number of players using technology t, and v_t is the stand-alone benefit from using technology t. It is assumed that $v_B > v_A$ in order to distinguish B as the superior technology. Assume for simplicity that $v_A = 1$. Letting $v_B = x$, we have the following payoff matrix:

$P3$: A

		$P2$: A	$P2$: B
$P1$	A	$3,\ 3,\ 3$	$2,\ x,\ 2$
	B	$x,\ 2,\ 2$	$x+1,\ x+1,\ 1$

$P3$: B

		$P2$: A	$P2$: B
$P1$	A	$2,\ 2,\ x$	$1,\ x+1,\ x+1$
	B	$x+1,\ 1,\ x+1$	$x+2,\ x+2,\ x+2$

For $x \in [1,\ 3]$, $(A,\ A,\ A)$ and $(B,\ B,\ B)$ are strict Nash equilibria. Suppose that in order to resolve this issue of multiplicity, a modeller wishes to use the global games approach and introduce uncertainty about the parameter x. For $x > 4$, we have that $(B,\ B,\ B)$ is the strictly dominant action profile. Notice that because we have the parameter restriction $x = v_B > v_A = 1$, no lower dominance region can be established,[1] and therefore in order to apply the FMP framework one must rely on a new parameterization à la Basteck, Daniëls, and Heinemann. But notice, by introducing an arbitrary parameter which produces upper and lower dominance regions in any 2 action game, the richness condition of WY is automatically met, making the global games selection ad hoc.

This paper considers global games analysis in a setting that is much less demanding than the FMP framework in terms of the restrictions that an underlying parameter of uncertainty must satisfy, which, as illustrated above, is often violated. Specifically, the original CvD framework is extended to N-player, multi-action games of either GSS or GSC, where the presence of only one dominance region is required, and need not be one corresponding to the highest or lowest strategies in the action space.[2] We also use iterated deletion of strictly dominated strategies as our solution concept, overcoming the computational difficulties present in Harrison. In their original work, CvD uses a "risk dominance" criteria to determine which equilibria will be selected as the global games prediction, which here is

[1]Likewise, if common knowledge about v_A is relaxed, no upper dominance region would be established.

[2]A state monotonicity assumption is also unnecessary, so that an increase in a parameter need not induce a player to take a higher strategy, as in FMP.

generalized to a p-dominance condition. It is shown, however, that as the number of players grows larger, players must become more certain that a specific equilibrium is played, and the p-dominance condition becomes more restrictive. The second part of this paper considers situations in which the full strength of this condition can be preserved for an arbitrary number of players. It is shown that if in the description of the complete information game of interest, it can be assumed that players can be grouped so that between groups, players receive different signals just as before, but within groups, players are able to share signals, then the power of the global games method to select equilibria can be restrenghtened.

2 Model and Assumptions

The paper will be stated in the case of GSS. When it is needed, the adjustments that are necessary for the results to hold for GSC will be pointed out.

Definition 1. A game $G = (I, (A_i)_{i \in I}, (u_i)_{i \in I})$ of strategic substitutes has the following elements:

- The number of players is finite and given by the set $I = \{1, 2, ..., N\}$.
- Each player i's action set is denoted A_i and is finite and linearly ordered. Let $\overline{a}_i$ and $\underline{a}_i$ denote the largest and the smallest elements in A_i, respectively. Also, for a specific $\tilde{a}_i \in A_i$, denote $\tilde{a}_i^+ = \{a_i \in A_i \,|\, a_i > \tilde{a}_i\}$ and $\tilde{a}_i^- = \{a_i \in A_i \,|\, \tilde{a}_i > a_i\}$.
- Each player's utility function is given by $u_i : A \to \mathbb{R}$.
- (Strategic Substitutes) For each player i, if $a_i' \geq a_i$ and $a_{-i}^{'} \geq a_{-i}$, then

$$u_i(a_i,\, a_{-i}^{'}) - u_i(a_i^{'},\, a_{-i}^{'}) \geq u_i(a_i,\, a_{-i}) - u_i(a_i^{'},\, a_{-i})$$

We will restrict our attention to cases games that exhibit multiple equilibria. Carlsson and Van Damme showed that in 2×2 games with multiple equilibria, if the game can be seen as a specific realization of a parameterized game in which dominance regions exist, then any strict risk-dominant equilibrium can be justified through what is known as a global games selection.

Definition 2. Let $\tilde{a}$ be a Nash equilibrium. Then $\tilde{a}$ is a strict Nash equilibrium if for all i, and for all a_i, $u_i(\tilde{a}_i, \tilde{a}_{-i}) > u_i(a_i, \tilde{a}_{-i})$.

Simply, a Nash equilibrium is strict if each player is best responding uniquely to the other players when they play their part of the equilibrium.

To resolve the issue of multiple equilibria in a normal form game, we hope to "embed" our game into a specific parameterized family of games in which the original game in question is a specific realization of the parameter. We define a family of parameterized games below:

Definition 3. A parameterized game of strategic substitutes $G_X = (I, X, (A_i)_{i\in I}, (u_i)_{i\in I})$ has the following elements:

- $X = [\underline{X}, \overline{X}]$ is a closed interval of $\mathbb{R}$, representing the parameter space.
- $\forall i \in I, \forall a \in A, u_i(a, \cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous function of x. We also make the following convention that $\forall x \geq \overline{X}$, $u_i(a, x) = u_i(a, \overline{X})$, and likewise $\forall x \leq \underline{X}$, $u_i(a, x) = u_i(a, \underline{X})$.[3]
- $\forall x$, we denote $G_X(x) = ((I, X\,(A_i)_{i\in I}, (u_i(\cdot, x))_{i\in I})$ to be the unparameterized game when x is realized. We assume that for all x, $G_X(x)$ is a game of strategic substitutes.
- (Dominance Region) $\exists \tilde{a} \in A$, $\exists D^{\tilde{a}} \subseteq X$ an interval such that

$$D^{\tilde{a}} \subseteq \{x \in X \,|\, \forall i \in I, \forall a_{-i} \in A_{-i}, \forall a_i \in A_i, u_i(\tilde{a}_i, a_{-i}) > u_i(a_i, a_{-i})\}$$

Note that the last requirement states that for some interval $D^{\tilde{a}}$ within X, some $\tilde{a} \in A$ is the dominance solvable strategy at those parameters. By the continuity of payoffs, $D^{\tilde{a}}$ can be assumed to be an open interval, without loss of generality.

With a specific noise structure, a parameterized game of strategic substitutes becomes a Bayesian game. We will call a Bayesian game a global game if it has the specific noise structure defined below, and has the payoff properties as defined in Definition 3.

[3]Because our analysis will be focused on the interior of X, this is only for simplicity.

Definition 4. A global game $G^v = (G_X,\ f,\ (\varphi_i)_{i\in I})$ is a Bayesian game with the following elements:

- G_X is a parameterized game of strategic substitutes as defined in Definition 3.

- $f : \mathbb{R} \to [0,\ 1]$ is a pdf.

- $\forall i \in I$, φ_i is any continuous pdf whose support lies in the interval $(-\frac{1}{2},\ \frac{1}{2})$. The φ_i are assumed to be independent.

- Each player i receives a signal $x_i = x + v\epsilon_i$, where x is distributed according to f and each ϵ_i is distributed according to φ_i. A subsequent belief about the signals received by each other player j is formed, which is denoted by $f_{i,j}(\cdot|x) : \mathbb{R} \to [0,\ 1]$, where $supp(f_{i,j}(\cdot|x_i)) \subseteq \prod\limits_{j\neq i}(x_i - 2v,\ x_i + 2v)$.[4] Lastly, we require that each φ_i and f are such that $\forall i,\ j,\ x$, and v, $f_{i,j}(\cdot|x)$ is a symmetric distribution.[5]

Note that each global game G^v is characterized by the noise level v of the signal the players recieve. The importance of this will become relevant once the main result is stated. Once the signal is recieved, player i chooses a strategy, hence forming a strategy function $s_i : \mathbb{R} \to A_i$. We denote all of player $i's$ strategy functions by the set S_i. Player i's expected utility from playing strategy a_i against the strategy function s_{-i} after receiving x_i is given by

$$\pi_i(a_i,\ s_{-i},\ x_i) = \int\limits_{x_i-2v}^{x_i+2v} \cdots \int\limits_{x_i-2v}^{x_i+2v} u_i(a_i,\ s_{-i}(x_{-i}),\ x_i)(\prod_{j\neq i} f_{i,j}(x_j|x_i))(\prod_{j\neq i} dx_j)$$

To simplify notation, we let $\triangle u_i(a_i^{'},\ a_i,\ a_{-i},\ x) = u_i(a_i^{'},\ a_{-i},\ x) - u_i(a_i,\ a_{-i},\ x)$ be player $i's$ advantage of playing $a_i^{'}$ over a_i when facing a_{-i} at a given x . Similairly, we write $\triangle\pi_i(a_i^{'},\ a_i,\ s_{-i},\ x)$ for player i's expected advantage from playing $a_i^{'}$ against s_{-i} after receiving signal x.

Much of our analysis will involve charaterizing the set of serially undominated strategies in a global game.

[4]Since $x_i = x + v\epsilon_i$ and $x_j = x + v\epsilon_j$, $x_j = x_i - v\epsilon_i + v\epsilon_j$.

[5]For example, when f is the "improper prior" on $\mathbb{R}$, see Morris and Shin (2003).

Definition 5. Let G^v be a global game. For each player $i \in I$, and each $a_i \in A_i$, define the following:

- $\mathcal{P}^{v,0}_{i,\,a_i} = A_i$, $\mathcal{S}^{v,0}_i = S_i$

- $\forall n > 0$,
  $\mathcal{P}^{v,n}_{i,\,a_i} = \{x \in X \,|\, \forall a_i^{'} \in A_i,\, \forall s_{-i} \in \mathcal{S}^{v,n-1}_{-i},\, \triangle\pi_i(a_i,\, a_i^{'},\, s_{-i},\, x) > 0\}$,
  $\mathcal{S}^{v,n}_i = \{s_i \in \mathcal{S}^{v,n-1}_i \,|\, \forall a_i,\, \forall s_i \,|_{\mathcal{P}^{v,n}_{i,\,a_i}} = a_i\}$

- $\mathcal{P}^{v}_{i,\,a_i} = \bigcup\limits_{n \geq 0} \mathcal{P}^{v,n}_{i,\,a_i}$, $\mathcal{S}^v_i = \bigcap\limits_{n \geq 0} \mathcal{S}^{v,n}_i$

It is an easy fact to check that for each a_i and each n, $\mathcal{P}^{v,n}_{i,\,a_i} \subseteq \mathcal{P}^{v,n+1}_{i,\,a_i}$, $\mathcal{S}^{v,n+1}_i \subseteq \mathcal{S}^{v,n}_i$, and that the set of serially undominated strategies for player i in a global game is a subset of the set $\mathcal{S}^v_i$.

Definition 6. Let G be a game of strategic substitutes. Then a global game $G^v = (G_X,\, f,\, (\varphi_i)_{i \in I})$ is a global games embedding of G iff G_X is such that for some $x \in X$, $G_X(x) = G$.

That is, G^v embeds the perfect information game G if G^v is such that the payoffs in G are realized at some parameter x in G^v's parameter space. If G can be embedded into a global game G^v, then the following process can be followed: At each noise level v, the upper and lower serially undominated strategies can be calculated[6]. As noise becomes small, then at every x in the parameter space, the players are essentially playing a slightly noisy version of the complete information game $G_X(x)$. In particular, if G is our game of interest, then if one of the equilibria in G is always selected by the serially undominated strategies in G^v for arbitrarily small noise at the x where G is realized, we will be justified in choosing this equilibrium in the complete information setting.

[6]In Hoffmann (2012) it is established that any Bayesian game of strategic substitutes has a smallest and a largest strategy profile surviving iterated deletion of dominated strategies.

P-dominance

CvD have shown that under specific conditions, the risk-dominant [7] equilibrium will be the equilibrium selected in the global games procedure. The condition is defined below:

Definition 7. Let $\widetilde{a}$ be a Nash equilibrium at $x \in X$. Then $\widetilde{a}$ is p-dominant for $p = (p_1, p_2, ..., p_N)$ at x if for each player i, and each $\lambda_i \in \triangle(A_{-i})$ such that $\lambda_i(\tilde{a}_{-i}) \geq p_i$, we have that for all a_i,

$$\sum_{a_{-i}} u_i(\widetilde{a}_i,\, a_{-i},\, x)\lambda_i(a_{-i}) \geq \sum_{a_{-i}} u_i(a_i,\, a_{-i},\, x)\lambda_i(a_{-i})$$

or

$$l_i(a_i,\, \lambda_i,\, x) \equiv \sum_{a_{-i}} \triangle u_i(\widetilde{a}_i,\, a_i,\, a_{-i},\, x)\lambda_i(a_{-i}) \geq 0$$

Note that if $\widetilde{a}$ is the dominance solvable solution of the game, it is 0-dominant, and if it is a Nash equilibrium, it is 1-dominant. Therefore, the lower the p_i, the more dominant each player's strategy in the equilibrium profile.

For our purposes, we will call $\tilde{a}$ a $p = (p_1, p_2, ..., p_N)$ dominant equilibrium if p_i is the smallest value for player i that satisfies the definition. This is WLOG because any value larger than p_i will also satisfy the definition. Note that the function l_i defined above is a continuous function of the λ_i, which are just vectors in $[0,\, 1]^{|A_{-i}|}$ whose elements sum to 1. By the continuity of utility in x, $\tilde{p}_i$ is also continuous in x.

Let $\tilde{a} \in A$ be an action profile, and let

$$D^{\tilde{a}_i} = \{x \in X \,|\, \forall a_i,\, \forall a_{-i},\, \triangle u_i(\widetilde{a}_i,\, a_i,\, a_{-i},\, x) > 0\}$$

the set of $x's$ in a parameterized game at which $\tilde{a}_i$ is strictly dominant. We denote $D^{\tilde{a}} = \bigcap_{\forall i} D^{\tilde{a}_i}$, those $x's$ at which $\tilde{a}$ is dominance solvable.

For a global games embedding G^v of a GSS G, let $\underline{s}^v$ and $\bar{s}^v$ denote these smallest and

[7] The term "risk-dominant" as used in CvD is simply p-dominance in the case of a 2×2 game where each $p_i = \frac{1}{2}$.

largest serially undominated profiles. We now state the first of two main theorems:

Theorem 1. *Let G be a GSS, and G^v a global games embedding of G. Suppose $\tilde{a}$ is a Nash equilibrium in G such that the following hold:*

1. *$\tilde{a}$ is a strict Nash equilibrium and $p-$dominant on*

$$P = \left\{ x \in X \mid \forall i,\, j, p_i(x) + p_j(x) < \left(\frac{1}{2}\right)^{|I|-2} \right\}$$

2. $I \subseteq P$ is an open interval such that $I \cap D^{\tilde{a}} \neq \emptyset$.

3. $\exists x \in I$ is such that $G_X(x) = G$.

Then there exists a $\tilde{v} > 0$ such that for all $v \in (0,\, \tilde{v}]$, $\underline{s}^v(x) = \overline{s}^v(x) = \tilde{a}$.

That is, for v small, because action spaces are linearly ordered, any serially undominated strategy in G^v selects $\tilde{a}$ at any x satisfying the conditions of Theorem 2.8.

Note that the condition $p_i(x) + p_j(x) < (\frac{1}{2})^{|I|-2}$ for all i, j becomes more demanding as the number of players gets larger. Section 2.3 of this paper considers a method for resolving this issue.

The following Lemma highlights the role of strategic substitutes in the model. In particular, they allow us to characterize the iterated deletion of strictly dominated strategies.

Lemma 1. *For each player $i \in I$, define*

$$\overline{s}_i^{v,n} = \begin{cases} a_i & , \, if\, x \in \mathcal{P}_{i,\, a_i}^{v,n} \\ \overline{a}_i & , \, otherwise \end{cases} \quad and \quad \underline{s}_i^{v,n} = \begin{cases} a_i & , \, if\, x \in \mathcal{P}_{i,\, a_i}^{v,n} \\ \underline{a}_i & , \, otherwise \end{cases}$$

and similarly $\overline{s}_i^v$ and $\underline{s}_i^v$ by replacing the $\mathcal{P}_{i,\, a_i}^{v,n}$ with $\mathcal{P}_{i,\, a_i}^{v}$. Then,

1. $\forall n$, $\underline{s}_i^{v,n} \leq \underline{s}_i^v \leq \overline{s}_i^v \leq \overline{s}_i^{v,n}$.

2. $\overline{s}_i^{v,n} \to \overline{s}_i^v$ and $\underline{s}_i^{v,n} \to \underline{s}_i^v$ pointwise as $n \to \infty$.

3. *For a given $\tilde{a}_i \in A_i$, then $x \in \mathcal{P}_{i,\, \tilde{a}_i}^{v}$ if and only if*

(a) $\forall a_i \in \tilde{a}_i^+,\ \triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{s}_{-i}^v,\, x) > 0$ and

(b) $\forall a_i \in \tilde{a}_i^-,\ \triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{s}_{-i}^v,\, x) > 0$

Proof. For the first claim, suppose that for some a_i, $x \in \mathcal{P}_{i,\,a_i}^{v,n}$. Then $x \in \underset{n\geq 0}{\cup} \mathcal{P}_{i,\,a_i}^{v,n} = \mathcal{P}_{i,\,a_i}^{v}$, so that $\underline{s}_i^{v,n}(x) = \underline{s}_i^v(x)$. Therefore if $\underline{s}_i^{v,n}(x) > \underline{s}_i^v(x)$, we must have that $x \in (\underset{a_i}{\cup} \mathcal{P}_{i,\,a_i}^{v,n})^C$. But then $\underline{s}_i^{v,n}(x) = \underline{a}_i$, a contradiction. The same argument applies to show $\overline{s}_i^v \leq \overline{s}_i^{v,n}$, and obviously $\underline{s}_i^v \leq \overline{s}_i^v$.

Secondly, let x be given. If some a_i, $x \in \mathcal{P}_{i,\,a_i}^v$, then since $\mathcal{P}_{i,\,a_i}^v = \underset{n\geq 0}{\cup} \mathcal{P}_{i,\,a_i}^{v,n}$, and the $\mathcal{P}_{i,\,a_i}^{v,n}$ are an increasing sequence of sets, $\exists N,\ \forall n \geq N$, $x \in \mathcal{P}_{i,\,a_i}^{v,n}$, so that $\underline{s}_i^{v,n}(x) \to \underline{s}_i^v(x)$. If $x \in (\underset{a_i}{\cup} \mathcal{P}_{i,\,a_i}^v)^C$, since for all n , $\underset{a_i}{\cup} \mathcal{P}_{i,\,a_i}^{v,n} \subseteq \underset{a_i}{\cup} \mathcal{P}_{i,\,a_i}^v$, we must have that $x \in (\underset{a_i}{\cup} \mathcal{P}_{i,\,a_i}^v)^C$, giving convergence. The same arguments can be made to show that $\overline{s}_i^{v,n} \to \overline{s}_i^v$.

For the last claim, suppose $x \in \mathcal{P}_{i,\,\tilde{a}_i}^v$. Since $\mathcal{P}_{i,\,\tilde{a}_i}^v = \underset{n\geq 0}{\cup} \mathcal{P}_{i,\,\tilde{a}_i}^{v,n}$, $\exists N$ such that $x \in \mathcal{P}_{i,\,\tilde{a}_i}^{v,N}$. We now show that for all n, $\overline{s}_i^v$ and $\underline{s}_i^v$ are in $\mathcal{S}_i^{v,n}$. Suppose this is not the case. Then $\exists n,\ \exists a_i,\ \exists x' \in \mathcal{P}_{i,\,a_i}^{v,n}$ such that $\overline{s}_j^v(x') \neq a_i$ or $\underline{s}_j^v(x') \neq a_i$. Since $\forall n$, $\mathcal{P}_{i,\,a_i}^{v,n} \subseteq \mathcal{P}_{i,\,a_i}^v$, then $x' \in \mathcal{P}_{i,\,a_i}^v$ but $\overline{s}_i^v(x') \neq a_i$ or $\underline{s}_i^v(x') \neq a_i$, a contradiction. Thus, since this holds for all n, $\overline{s}_i^v$ and $\underline{s}_i^v$ are in $\mathcal{S}_i^{v,N-1}$, and since $x \in \mathcal{P}_{i,\,\tilde{a}_i}^{v,N}$, $a_i \in \tilde{a}_i^+ \Rightarrow \triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{s}_{-i}^v,\, x) > 0$ and $\forall a_i \in \tilde{a}_i^- \Rightarrow \triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{s}_{-i}^v,\, x) > 0$.

Conversely, suppose that $\forall a_i \in \tilde{a}_i^+,\ \triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{s}_{-i}^v,\, x) > 0$ and $\forall a_i \in \tilde{a}_i^-,\ \triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{s}_{-i}^v,\, x) > 0$. Let $s_{-i}^v \in \mathcal{S}_{-i}^v$. Then $\forall n$, $s_{-i}^v \in \mathcal{S}_{-i}^{v,n}$, and $\overline{s}_{-i}^{v,n} \geq s_{-i}^v \geq \underline{s}_{-i}^{v,n}$. If $a_i \in \tilde{a}_i^+$, then since $s_{-i}^v \geq \underline{s}_{-i}^{v,n}$, by GSS we have $\triangle\pi_i(\tilde{a}_i,\, a_i,\, s_{-i}^v,\, x) \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{s}_{-i}^{v,n},\, x)$. Since $\underline{s}_i^{v,n} \to \underline{s}_i^v$, $\triangle\pi_i(\tilde{a}_i,\, a_i,\, s_{-i}^v,\, x) \geq \underset{n\to\infty}{lim}(\triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{s}_{-i}^{v,n},\, x)) = \triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{s}_{-i}^v,\, x) > 0$. Similarly, if $a_i \in \tilde{a}_i^-$, $\triangle\pi_i(\tilde{a}_i,\, a_i,\, s_{-i}^v,\, x) \geq \underset{n\to\infty}{lim}(\triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{s}_{-i}^{v,n},\, x)) = \triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{s}_{-i}^v,\, x) > 0$. Therefore, $\forall a_i \in A_i,\ \triangle\pi_i(\tilde{a}_i,\, a_i,\, s_{-i}^v,\, x) > 0$, and hence $x \in \mathcal{P}_{i,\,\tilde{a}_i}^v$. $\square$

The next result can be helpful even beyond the scope of our proof. It not only provides us with a method for calculating the individual p_i for which an equilibrium is p- dominant, but also shows that under our setting, the p_i satisfy a useful continuity property when viewed as a function of x. We first calculate such a value of p_i with a fixed a_i. For each player i

and strategy $a_i \neq \tilde{a}_i$, define

$$D^{\tilde{a}_i, a_i} = \{x \in \mathbb{R} \,|\, \forall a_{-i},\, \triangle u_i(\tilde{a}_i,\, a_i\,,\, a_{-i},\, x) > 0\}$$

Then player i's domince region is given by $D^{\tilde{a}_i} = \underset{a_i}{\cap} D^{\tilde{a}_i, a_i}$. The proof of the following Proposition and Corollary are generalized and given in Section 2. When a parameter space X is mentioned, assume an arbitrary global games embedding.

Proposition 1. *Let $\tilde{a}$ be a strict Nash equilibrium on X, and for each player i, and fixed a_i, define*

$$\hat{p}_i(a_i,\, x) = \begin{cases} \underset{\substack{\lambda_i \in \triangle(A_{-i}) \\ l_i(a_i,\, \lambda_i,\, x) = 0}}{max} (\lambda_i(\tilde{a}_{-i})) & ,\, x \notin D^{\tilde{a}_i,\, a_i} \\ \\ 0 & ,\, x \in D^{\tilde{a}_i,\, a_i} \end{cases}$$

Then $p_i(a_i,\, x)$ is an upper semi-continuous function on X.

Corollary 1. *Let $\tilde{a}$ be a strict Nash equilibrium on X, and $p_i : X \to [0,\, 1]$ be the smallest value satisfying this condition for player i at parameter x. Then for all x,*

$$p_i(x) = \underset{a_i}{max}(\hat{p}_i(a_i,\, x))$$

and is upper semi-continuous on all of X.

Corollary 2. *Suppose the conditions of Theorem 2.8 hold. Let $[a,\, b] \subseteq P$, where $a, b \in int(P)$. Then $\exists \bar{v} > 0,\, \forall v \in (0,\, \bar{v}],\, \forall x,\, y \in [a,\, b],\, \forall i,\, j,$*

$$d(x,\, y) < v \Longrightarrow p_i(x) + p_j(y) < \left(\frac{1}{2}\right)^{|I|-2}$$

Proof. Let i and j be given. Let $v*$ be such that $\forall x \in [a,\, b],\; B(x,\, v*) \subseteq P$. For a contradiction, suppose that $\forall v \in (0,\, v*],\, \exists v' \leq v,\, \exists x_{v'},\, y_{v'} \in P$ such that $d(x_{v'},\, y_{v'}) < v'$ and $p_i(x_{v'}) + p_j(y_{v'}) + (\frac{1}{2})^{I-2} \geq (\frac{1}{2})^{I-2}$. Collecting all such $(x_{v'},\, y_{v'})_{v>0}$, and since for all $v',\, x_{v'} \in [a,\, b]$, then passing to a subsequence if necessary, we may assume that $x_{v'} \to x* \in [a,\, b]$. Since $\forall v',\, d(x_{v'},\, y_{v'}) < v'$, then then $y_{v'} \to x*$. By the previous Corollary, since p_i

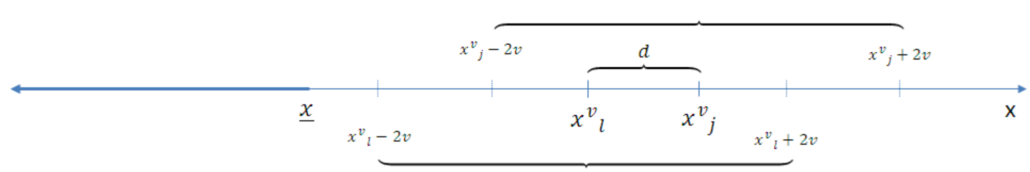
and p_j are upper semi-continuous,

$$p_i(x*) + p_j(x*) \geq \limsup_{v' \to 0} (p_i(x_{v'}) + p_j(y_{v'}) + (\frac{1}{2})^{|I|-2}) \geq (\frac{1}{2})^{|I|-2}$$

contradicting the fact that $x* \in P$. Therefore, $\forall i, j$, there exists a $v_{i,j} > 0$ satisfying the hypothesis. Letting $\bar{v} = \min_{i,j}(v_{i,j})$ gives the result. $\square$

Below is a sketch of the proof of Theorem 1, the full proof in full generality being relegated to Section 2.3.

Proof. (Sketch of Theorem 1) Suppose that for all $x < \underline{x}$, $\tilde{a}$ is strictly dominant for each player, and that $[\underline{x}, \infty) \subseteq P$. Recall that for each player i, the set P_i^v are those $x's$ at which every serially undomanted strategy s_i plays $\tilde{a}_i$. Therefore, define for each player $x_i^v = sup(x \,|\, [\underline{x}, x) \subseteq P_i^v)$, the largest point starting from the dominance region on which player i plays $\tilde{a}_i$ without break. Suppose by way of contradiction that for some some player i, $x_i^v < \infty$. If we consider the lowest of such x_i^v, labelled x_l^v, it must be the case that player l does not observe $\tilde{a}_{-l}$ with the probability $p_l(x_l^v)$ necessary to play $\tilde{a}_i$ unambiguously at x_l^v. Therefore there must be some other x_j^v within $[x_l^v + 2v)$. Likewise, since $x_j^v < \infty$, it must be the case that player j does not observe $\tilde{a}_{-j}$ with the probability $p_j(x_j^v)$ necessary to play $\tilde{a}_j$ unambiguously at x_j^v. If we let $d = x_j^v - x_l^v$, we obtain the following graphical representation:



In the two player case as in CvD, we see that if $v \to 0$, then $d \to 0$, thus x_l^v and x_j^v must converge to some common point, x^*. We then reach an immediate contradiction, since $x^* \in P$, but $p_l(x^*) + p_j(x^*) \geq 1$. However, with more than two players, such a convergence result need not obtain, since as v approaches 0 , it is not clear that the same two players will constitute the two lowest players for each such v. Proposition 1 and Corollary 2 allow us to approximate this convergence result uniformly for all possible players. That is, let $\bar{v} > 0$ be as given in Corollary 2, and let $v \in (0, \bar{v}]$. Then regardless of who the players l and j are, we must have that $p_l(x_l^v) + p_j(x_j^v) < \left(\frac{1}{2}\right)^{|I|-2}$. Since player l is the lowest player, she must see $\tilde{a}_{-l}$ played with probability at least $\left(\frac{1}{2}\right)^{|I|-1}\left(\frac{1}{2} + \frac{d}{2v}\right)$, and since j is the second lowest player, she must see $\tilde{a}_{-j}$ played with probability at least $\left(\frac{1}{2}\right)^{|I|-1}\left(\frac{1}{2} - \frac{d}{2v}\right)$. Then we must have

$$p_l(x_l^v) + p_j(x_j^v) \geq \left(\frac{1}{2}\right)^{|I|-1}\left(\frac{1}{2} + \frac{d}{2v}\right) + \left(\frac{1}{2}\right)^{|I|-1}\left(\frac{1}{2} - \frac{d}{2v}\right) = \left(\frac{1}{2}\right)^{|I|-2},$$

contradicting $v \in (0, \bar{v}]$. □

We now consider an example:

Example 1. Consider again a modified version of the technology adoption model considered in Keser, Suleymanova, and Wey (2012), where the benefit to each player i of adopting a specfic technology $t = A, B$ is given by

$$U_t(N_t) = v_t + \gamma_t(N_t - 1)$$

where N_t is the total number of players using technology t, v_t is the stand-alone benefit from using technology t, and γ_t is the benefit derived from the network effect of adopting the technology of others. Recall that $v_B > v_A$ in order to distinguish B as the superior technology. Assume for simplicity that $v_A = \gamma_A = 1$, $\gamma_B = 3$, and that there is a unit cost for upgrading to the superior technology. Letting $v_B = x$, we have the following payoff matrix:

$P3$: A (left), B (right)

		$P2$: A	$P2$: B
$P1$	E	3, 3, 3	2, x, 2
	S	x, 2, 2	$x+3$, $x+3$, 1

		$P2$: A	$P2$: B
$P1$	A	2, 2, $x-1$	1, $x+3$, $x+2$
	B	$x+3$, 1, $x+2$	$x+6$, $x+6$, $x+6$

For $x \in [1, 3]$, (A, A, A) and (B, B, B) are strict Nash equilibria, and for $x > 4$, (B, B, B) is strictly dominant. Recall that because we have the parameter ristriction $x = v_B > v_A = 1$, no lower dominance region can be established as in the FMP framework, and the same is true about the upper dominance region if CK about v_A is relaxed, and hence the FMP framework does not apply to any "natural" parameters present in the model.

Applying Theorem 2.8,[8] we have that $p_1(x) = p_2(x) = \frac{3-x}{8}$, and $p_3(x) = \frac{4-x}{9}$. In order to satisfy $p_i(x) + p_j(x) < (\frac{1}{2})^{|I|-2}$ for all i, j, we have that (B, B, B) is the global games prediction for any $x > 1.4$.

3 Groups

In this section we allow for the possibility that players can be grouped so that between groups, players receive independent signals, just as before, but within groups, players are able to share signals. To establish notation, we let $\mathcal{G} = \{g_1, g_2, ..., g_N\} \subseteq 2^I$ represent a partitioning of the set of players. For each $a \in A$, we let a_{g_i} denote those actions taken by all players in group g_i, a_{-g_i} to be those actions taken by all players in all groups other than g_i, and $a_{g_{-i}}$ to be those actions taken by all players in group g_i other than player i herself. Similar notation will be used when describing strategy functions.

Definition 8. Let $\widetilde{a} \in A$. Then $\mathcal{G}^{\widetilde{a}} = \{g_1, g_2, ..., g_N\} \subseteq 2^I$ is an $\widetilde{a}-$based partitioning of I if $\forall i \in I$, $i \in g_i$ if the following hold:

1. $u_i(\widetilde{a}_i, a_{g_{-i}}, \widetilde{a}_{-g_i}) > u_i(a_i, a_{g_{-i}}, \widetilde{a}_{-g_i})$, $\forall a_i$, $\forall a_{g_{-i}}$,.

[8]It is easily verified that all assumptions are met for $x > 2$.

2. $\forall j \in g_i$, player j receives signal $x_{g_i} = x + v\epsilon_{g_i}$, where ϵ_{g_i} is distributed according to φ_{g_i}, a continuous distribution with support in $(-\frac{1}{2}, \frac{1}{2})$.

3. $\forall i, j, i \neq j$, φ_{g_i} and φ_{g_j} are independent.

Notice that when all groups are singletons, or we have the trivial $\widetilde{a}-$based partition, the first condition reduces to $\widetilde{a}$ being a strictly dominant equilibrium. As will be shown in subsequent examples, this condition can be quite natural in many applications in an aggregative setting or in network games. The second and third conditions state that all players in a group receive the same noisy signal about the state of nature, which is independent of the signals received by players in other groups. To the best of the knowledge of the author, all previous work on global games has assumed that each player receives signals about an underlying parameter whose error terms are independently distributed. However, this is one extreme end of the spectrum of possible error distributions. That is, when common knowledge about a parameter in a model is relaxed, depending on the parameter under consideration, certain groups of players may share a common attribute which allows them to be privy to the same information about that parameter. As a simple motivation, suppose there are three firms in a Cournot economy, two of them being separated geographically from the third. If common knowledge of a weather forecast-based parameter is relaxed, it is more natural to assume that those in the same geographical region receive the same noisy signal about a weather forecast, but receive only rough knowledge of the signal received in the neighboring region.

Another way to motivate a partitioning is if groups of players receiving independent signals are in a position to share their private information. Suppose that each player j in each group $g_j \in 2^I$ receives an independent signal $x_j = x + v\epsilon_j$, but that groups decide to condition their actions on the average signal within the group

$$x_{g_j} = \frac{1}{|g_j|}\sum_{j \in g_j}^{|g_j|} x + v\epsilon_j = x + \frac{v}{|g_j|}\sum_{j \in g_j}^{|g_j|} \epsilon_j$$

By defining $\epsilon_{g_j} = \frac{1}{|g_j|}\sum_{j \in g_j}^{|g_j|} \epsilon_j$, we see that φ_{g_j} has support in $(-\frac{1}{2}, \frac{1}{2})$ and that the the φ_{g_i} are independent across groups, satisfying the grouping signal requirement.

Below we define the relevant notion of p-dominance, called group p-dominance. For each player i, each $a_i \in A_i$, and each $a_{g_{-i}}, \in A_{g_{-i}}$, let

$$D^{\tilde{a}_i, a_i, a_{g_{-i}}} = \{x \in \mathbb{R} \,|\, \forall a_{-g_i},\, \triangle u_i(\widetilde{a}_i,\, a_i\, a_{g_{-i}},\, a_{-g_i},\, x) > 0\}$$

and $D^{\tilde{a}_i, a_{g_{-i}}} = \bigcap_{a_i} D^{\tilde{a}_i, a_i, a_{g_{-i}}}$. Then $D^{\tilde{a}_i} \equiv \bigcap_{a_{-g_i}} D^{\tilde{a}_i, a_{g_{-i}}}$ is player $i's$ dominance region, or the set of parameters where $\widetilde{a}_i$ is a strictly dominant action for player i.

Definition 9. Let $\widetilde{a}$ be a Nash equilibrium at $x \in X$. Then $\widetilde{a}$ is group p-dominant for $p = (p_1, p_2, ..., p_N)$ at x if for each player i, and each $\lambda_i \in \triangle(A_{-g_i})$ such that $\lambda_i(\tilde{a}_{-g_i}) \geq p_i$, we have that for all a_i, and all $a_{g_{-i}}$,

$$\sum_{a_{-g_i}} u_i(\widetilde{a}_i,\, a_{g_{-i}},\, a_{-g_i},\, x)\lambda_i(a_{-g_i}) \geq \sum_{a_{-g_i}} u_i(\widetilde{a}_i,\, a_{g_{-i}},\, a_{-g_i},\, x)\lambda_i(a_{-g_i})$$

or

$$l_i(a_i,\, a_{g_{-i}},\, \lambda_i,\, x) \equiv \sum_{a_{-g_i}} u_i(\widetilde{a}_i,\, a_{g_{-i}},\, a_{-g_i},\, x)\lambda_i(a_{-g_i}) \geq 0$$

Note that the concept of group p-dominance is fundamentally different than that of p-dominance, in that it implies that a player is only concerned with how often she sees players outside of her group play their part in the equilibrium profile. We now state the second main theorem of this paper.

Theorem 2. *Let G be a GSS, G^v a global games embedding of G, and $\mathcal{G}^{\widetilde{a}} = \{g_1,\, g_2,\, ...,\, g_N\}$ an $\widetilde{a}-$based partitioning of I. Suppose $\tilde{a}$ is a Nash equilibrium in G such that the following hold:*

1. *$\tilde{a}$ is a strict Nash equilibrium and $p-$dominant on*

$$P = \left\{x \in X \,|\, \forall i,\, j,\, p_i^g(x) + p_j^g(x) < (\frac{1}{2})^{|G|-2}\right\}$$

2. $I \subseteq P$ is an open interval such that $I \cap D^{\tilde{a}} \neq \emptyset$.

3. $\exists x \in I$ is such that $G_X(x) = G$.

Then there exists a $\tilde{v} > 0$ *such that for all* $v \in (0, \tilde{v}]$, $\underline{s}^v(x) = \bar{s}^v(x) = \tilde{a}$.

Notice that the p-dominance condition depends on the number of groups that can be established, rather than the number of players in the game. Therefore, it is possible to have a large number of players and have the p-dominance condition be no more restrictive than that of Theorem 1. The following Proposition and Corollary give an explicit formula for the value satisfying the group p-dominance condition for player i at x, denoted $p_i^g(x)$, and that this value is upper semi-continuous in x.

Proposition 2. *Let* $\tilde{a}$ *be a strict Nash equilibrium on* X*, and for each player* i*, and fixed* a_i*,* $a_{g_{-i}}$*, define*

$$\hat{p}_i^g(a_i,\, a_{g_{-i}},\, x) = \begin{cases} \underset{\substack{\lambda_i \in \triangle(A_{-g_i}) \\ l_i(a_i,\, a_{g_{-i}},\, \lambda_i,\, x)=0}}{max} (\lambda_i(\tilde{a}_{-i})) & ,\, x \notin D^{\tilde{a}_i, a_i, a_{g_{-i}}} \\ 0 & ,\, x \in D^{\tilde{a}_i, a_i, a_{g_{-i}}} \end{cases}$$

Then $\hat{p}_i^g(a_i,\, a_{g_{-i}},\, \cdot)$ *is an upper semi-continuous function on* X.

Proof. Appendix. □

Corollary 3. *Let* $\tilde{a}$ *satisfy the group* p*-dominance condition, and for each player* i*, let the function* $p_i^g : X \to [0,\, 1]$ *be the smallest value satisfying this condition for player* i *and parameter* x*. Then*

$$p_i^g(x) = \underset{a_i,\, a_{g_{-i}}}{max} \, (p_i^g(a_i,\, a_{g_{-i}},\, x))$$

and is upper semi-continuous on all of X.

Proof. For the first claim, let x be given. Let $\lambda_i \in \triangle(A_{-g_i})$ be such that $\lambda_i(\tilde{a}_{-g_i}) \geq \underset{a_i,\, a_{g_{-i}}}{max} \, (\hat{p}_i^g(a_i,\, a_{g_{-i}},\, x))$. Choose any $a_i^{'}$ and $a_{g_{-i}}^{'}$, giving $\lambda_i(\tilde{a}_{-g_i}) \geq \hat{p}_i^g(a_i^{'},\, a_{g_{-i}}^{'},\, x)$. If $x \in D^{\tilde{a}_i,\, a_i^{'},\, a_{g_{-i}}^{'}}$, we trivially have that $l_i(a_i^{'},\, a_{g_{-i}}^{'},\, \lambda_i,\, x) \geq 0$. If $x \notin D^{\tilde{a}_i,\, a_i^{'},\, a_{g_{-i}}^{'}}$, then by the first part of Lemma 3 in the Appendix, $\lambda_i(\tilde{a}_{-g_i}) \geq \hat{p}_i^g(a_i^{'},\, a_{g_{-i}}^{'},\, x) \Rightarrow l_i(a_i^{'},\, a_{g_{-i}}^{'},\, \lambda_i,\, x) \geq 0$. Thus

$\lambda_i(\tilde{a}_{-g_i}) \geq \underset{a_i,\, a_{g_{-i}}}{max} (\hat{p}_i^g(a_i,\, a_{g_{-i}},\, x))$ implies that for any $a_i^{'}$ and $a_{g_{-i}}^{'}$, $l_i(a_i^{'},\, a_{g_{-i}}^{'},\, \lambda_i,\, x) \geq 0$. Since $p_i^g(x)$ is defined as the smallest value satisfying this property, we must have that $\underset{a_i,\, a_{g_{-i}}}{max} (p_i^g(a_i,\, a_{g_{-i}},\, x)) \geq p_i^g(x)$. For a contradiction, suppose that this inequality is strict. Since for all $x \in D^{\tilde{a}_i}$, $\underset{a_i,\, a_{g_{-i}}}{max} (p_i^g(a_i,\, a_{g_{-i}},\, x)) = 0 \equiv p_i^g(x)$, we must have that $x \notin D^{\tilde{a}_i}$. In particular, let $a_i^{'}$ and $a_{g_{-i}}^{'}$ be such that $x \notin D^{\tilde{a}_i,\, a_i^{'},\, a_{g_{-i}}^{'}}$ and $\underset{a_i,\, a_{g_{-i}}}{max} (\hat{p}_i^g(a_i,\, a_{g_{-i}},\, x)) = \hat{p}_i^g(a_i^{'},\, a_{g_{-i}}^{'},\, x)$ so that $x \notin D^{\tilde{a}_i,\, a_i^{'},\, a_{g_{-i}}^{'}}$ and $\hat{p}_i^g(a_i^{'},\, a_{g_{-i}}^{'},\, x) > p_i^g(x)$. By the second part of Lemma 3, we must have that for any $\lambda_i \in \triangle(A_{-g_i})$ such that $\lambda_i(\tilde{a}_{-g_i}) > p_i^g(a_i^{'},\, a_{g_{-i}}^{'},\, x)$, $l_i(a_i^{'},\, a_{g_{-i}}^{'},\, \lambda_i,\, x) > 0$, contradicting $\hat{p}_i^g(a_i^{'},\, a_{g_{-i}}^{'},\, x) = \underset{\substack{\lambda_i \in \triangle(A_{-g_i}) \\ l_i(a_i,\, a_{g_{-i}},\, \lambda_i,\, x)=0}}{max} (\lambda_i(\tilde{a}_{-g_i}))$. Therefore, $p_i^g(x) = \underset{a_i,\, a_{g_{-i}}}{max} (\hat{p}_i^g(a_i,\, a_{g_{-i}},\, x))$. The fact that $p_i^g(x)$ is upper semi-continuous on X follows from the fact that each $p_i^g(a_i,\, a_{g_{-i}},\, x)$ is. $\square$

In what follows, for player i, we will let $s_{-g_i}^v(x)$ denote the probability with which player i believes that her opponents will play $\tilde{a}_{-g_i}$ according to $s_{-g_i}^v$ if x is observed. Specifically,

$$s_{-g_i}^v(x) = \int_{\mathbb{R}^{|G|-1}} (1_{\{s_{-g_i}^v = \tilde{a}_{-g_i}\}}) f_i(x_{-g_i}|x) dx_{-g_i}$$

Proposition 3. *Suppose the conditions of Theorem 2.17 hold, and let $x \in P$ be such that for some player i, $0 \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{a}_{g_{-i}},\, \underline{s}_{-g_i}^v,\, x)$ for some $a_i \in \tilde{a}_i^+$ or $0 \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{a}_{g_{-i}},\, \overline{s}_{-g_i}^v,\, x)$ for some $a_i \in \tilde{a}_i^-$. Then:*

1. $\exists j \neq g_i$ such that $B(x,\, 2v) \nsubseteq P_j^v$.

2. $p_i^g(x) \geq \underline{s}_{-g_i}^v(x)$ or $p_i^g(x) \geq \overline{s}_{-g_i}^v(x)$, respectively.

Proof. Suppose we are in the former case, the proof of the latter being identical. For the first part, suppose that $\forall j \neq g_i$, $B(x,\, 2v) \subseteq P_j^v$. Then after recieving x, player i knows that $\widetilde{a}_{-g_i}$ is played for sure, and thus we have

$$0 \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{a}_{g_{-i}},\, \underline{s}_{-g_i}^v,\, x) =$$

$$\int_{x_i-2v}^{x_i+2v}\cdots\int_{x_i-2v}^{x_i+2v}\triangle u_i(\tilde{a}_i,\, a_i,\, \underline{a}_{g_{-i}},\, \underline{s}^v_{-g_i},\, x)(\prod_{\substack{g_j\\ j\neq i}} f_{i,j}(x_{g_j}|x))(\prod_{j\neq i} dx_{g_j}) = \triangle u_i(\tilde{a}_i,\, a_i,\, \underline{a}_{g_{-i}},\, \widetilde{a}_{-g_i},\, x)$$

Note that the last term must be strictly positive, a contradiction.

For the second part, applying Fubini's theorem, we have that

$$0 \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{a}_{g_{-i}},\, \underline{s}^v_{-g_i},\, x) = \int_{x_i-2v}^{x_i+2v}\cdots\int_{x_i-2v}^{x_i+2v}\triangle u_i(\tilde{a}_i,\, a_i,\, \underline{a}_{g_{-i}},\, \underline{s}^v_{-g_i},\, x)(\prod_{\substack{g_j\\ j\neq i}} f_{i,j}(x_{g_j}|x))(\prod_{\substack{g_j\\ j\neq i}} dx_{g_j})$$

$$= \int_{[x-2v,\, x+2v]^{|G|-1}} \triangle u_i(\tilde{a}_i,\, a_i,\, \underline{a}_{g_{-i}},\, \underline{s}^v_{-g_i},\, x) f_i(x_{-g_i}|x)dx_{-g_i}$$

$$= \int_{\mathbb{R}^{|G|-1}} \triangle u_i(\tilde{a}_i,\, a_i,\, \underline{a}_{g_{-i}},\, \underline{s}^v_{-g_i},\, x)(\sum_{a_{-g_i}}(1_{\{\underline{s}^v_{-g_l}=a_{-g_l}\}})f_i(x_{-g_i}|x))dx_{-g_i})$$

$$= \sum_{a_{-g_i}} \triangle u_i(\tilde{a}_i,\, a_i,\, \underline{a}_{g_{-i}},\, a_{-g_i},\, x)(\int_{\mathbb{R}^{|G|-1}}(1_{\{\underline{s}^v_{-g_l}=a_{-g_l}\}})f_i(x_{-g_i}|x)dx_{-g_i}).$$

If we define $\lambda_i^{'} \in \triangle(A_{-g_i})$ by $\lambda_i^{'}(a_{-g_i}) = \int_{\mathbb{R}^{|G|-1}}(1_{\{\underline{s}^v_{-g_i}=a_{-g_i}\}})f_i(x_{-g_i}|x)dx_{-g_i}$, we have that $0 \geq l_i(a_i,\, \underline{a}_{g_{-i}},\, \lambda_i^{'},\, x)$. By Lemma 3 in the Appendix, $0 \geq l_i(a_i,\, \underline{a}_{g_{-i}},\, \lambda_i^{'},\, x) \Rightarrow p_i^g(a_i,\, \underline{a}_{g-i},\, x) \geq \lambda_i^{'}(\tilde{a}_{-g_i})$. Also, from the Appendix we have that each $\hat{p}_i(a_i,\, a_{g_{-i}}\, x) = p_i(a_i,\, a_{g_{-i}}\, x)$, so $p_i^g(x) = \max_{a_i,\, a_{g_{-i}}}(\hat{p}_i(a_i,\, a_{g_{-i}}\, x)) = \max_{a_i,\, a_{g_{-i}}}(p_i(a_i,\, a_{g_{-i}}\, x))$. Therefore,

$$p_i^g(x) = \max_{a_i,\, a_{g_{-i}}}(p_i(a_i,\, a_{g_{-i}}\, x)) \geq p_i(a_i,\, \underline{a}_{g-i},\, x) \geq$$

$$\lambda_i^{'}(\tilde{a}_{-i}) = \int_{\mathbb{R}^{|G|-1}}(1_{\{\underline{s}^v_{-g_i}=a_{-g_i}\}})f_i(x_{-g_i}|x)dx_{-g_i} = \underline{s}^v_{-g_i}(x)$$

completing the proof. $\square$

Proof. (Of Theorem 2) First note that Corollary 2 can be stated in the group context by replacing $\left(\frac{1}{2}\right)^{|I|-2}$ with $\left(\frac{1}{2}\right)^{|G|-2}$, the proof being the same. Suppose that all the conditions are satisfied, but for all $\overline{v} > 0$, there is a $v \in (0,\, \overline{v}]$ such that for some serially undominated strategy s and some $\tilde{x}$ lying in an open interval $I \subseteq P$ which intersects $D^{\tilde{a}}$, $s(\tilde{x}) \neq \tilde{a}$.

Let $\underline{x} \in I \cap D^{\tilde{a}}$, and since P is an open interval, let v' be such that $B(\underline{x},\, 2v') \cup B(\tilde{x},\, 2v') \subseteq I$. Let v'' satisfy the conditions of Corollary 2, and let $\bar{v} = min(\frac{v'}{2},\, \frac{v''}{2})$. For a contradiction, let $v \in (0,\, \bar{v}]$ violate the Theorem, as described above. Since $\tilde{x} \in I/D^{\tilde{a}}$ and $D^{\tilde{a}}$ is an interval, we can assume WLOG that $\tilde{x}$ lies to the right of $D^{\tilde{a}}$[9] For each player j, define $x^v_j = sup(x \,|\, [\underline{x},\, x) \subseteq P^v_j)$. Note that by continuity in x and since $\underline{x} \in D^{\tilde{a}}$, these are well-defined. Also note that at each x^v_j, $p^g_j(x^v_j) \geq \bar{s}^v_{-g_i}(x^v_j)$ or $p^g_j(x^v_j) \geq \bar{s}^v_{-g_i}(x^v_j)$. To see this, we show that there is some $a_j \in \tilde{a}^+_j$ such that $0 \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{a}_{g_{-j}},\, \bar{s}^v_{-g_j},\, x^v_j)$ or $a_j \in \tilde{a}^-_j$ such that $0 \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{a}_{g_{-j}},\, \bar{s}^v_{-g_j},\, x^v_j)$. If this is not true, by GSS, note that since $(\overline{a}_{g_{-j}},\, \bar{s}^v_{-g_j}) \geq \bar{s}^v_{-j}$ and $\underline{s}^v_{-j} \geq (\underline{a}_{g_{-j}},\, \underline{s}^v_{-g_j})$, then for all $a_j \in \tilde{a}^+_j$ we have $\triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{s}^v_{-j},\, x^v_j) \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \underline{a}_{g_{-j}},\, \underline{s}^v_{-g_j},\, x^v_j) > 0$ and for all $a_j \in \tilde{a}^-_j$ we have $\triangle\pi_i(\tilde{a}_i,\, a_i,\, \bar{s}^v_{-j},\, x^v_j) \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{a}_{g_{-j}},\, \bar{s}^v_{-g_j},\, x^v_j) > 0$. Since each term on the right hand side is continuous in x, $\exists\bar{\varepsilon} > 0,\, \forall\varepsilon \in (0,\, \bar{\varepsilon}]$, $\triangle\pi_i(\tilde{a}_i,\, a_i,\, \bar{s}^v_{-j},\, x^v_j + \varepsilon) > 0$ and $\triangle\pi_i(\tilde{a}_i,\, a_i,\, \bar{s}^v_{-j},\, x^v_j + \varepsilon) > 0$ for each $a_j \in \tilde{a}^+_j$ and $a_j \in \tilde{a}^-_j$, respectively. By Lemma 1, P^v_j can be extended to the right of x^v_j, contradicting the definition of x^v_j. Thus WLOG suppose there is some $a_j \in \tilde{a}^-_j$ such that $0 \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{a}_{g_{-j}},\, \bar{s}^v_{-g_j},\, x^v_j)$. By Proposition 3, we have that $p^g_i(x^v_j) \geq \bar{s}^v_{-g_i}(x^v_j)$.

Let x^v_l be the smallest of the x^v_j. Since by the above discussion we may assume $0 \geq \triangle\pi_i(\tilde{a}_i,\, a_i,\, \overline{a}_{g_{-j}},\, \bar{s}^v_{-g_j},\, x^v_j)$ for some $a_l \in \tilde{a}^-_l$, and by Proposition 3 the set $L^v_l \equiv \{j \notin g_l \,|\, \exists x \in B(x^v_l,\, 2v)/P^v_j\}$ is non-empty. Define $\hat{x} \equiv \underset{j \in L^v_l}{inf}\left(x \in B(x^v_l,\, 2v)/P^v_j\right)$, and let $j \in L^v_l$ be such that $\hat{x} = inf\left(x \in B(x^v_l,\, 2v)/P^v_j\right)$. Then $\hat{x}$ satisfies the following properties:

1. $\hat{x} = x^v_j$: Suppose $x^v_j > \hat{x}$. By the definition of $\hat{x}$, for every $\varepsilon > 0$ there is some $x \notin P^v_j$ such that $x < \hat{x} + \varepsilon$. Setting $\varepsilon = x^v_j - \hat{x} > 0$, we have the existence of some $x \notin P^v_j$ such that $x < \hat{x} + (x^v_j - \hat{x})$, or $x < x^v_j$, contradicting the definition of x^v_j. Thus $\hat{x} \geq x^v_j$. If $\hat{x} > x^v_j$, then since $\hat{x} \in [x^v_l,\, x^v_l + 2v)$, we have that $x^v_l + 2v > \hat{x} > x^v_j \geq x^v_l$. By definition of x^v_j there must be some $x' \in [x^v_j,\, \hat{x})$ such that $x' \notin P^v_j$, contradicting the definition of $\hat{x}$, and giving the result.

2. $p^g_j(\hat{x}) \geq \bar{s}^v_{-g_j}(\hat{x})$ or $p^g_j(\hat{x}) \geq \underline{s}^v_{-g_j}(\hat{x})$: Since $\hat{x} = x^v_j$, this follows from the discussion above.

[9] Or, $\forall x \in D^{\tilde{a}}$, $\tilde{x} > x$.

3. $\forall m \neq l$, $\hat{x} \leq x_m^v$: Suppose for some j we have $\hat{x} > x_m^v$. The contradiction is the same as in the second half of part 1.

Finally, denoting $\hat{x} = x_j^v$ and assuming that $p_j^g(x_j^v) \geq \overline{s}_{-g_j}^v(x_j^v)$ with no loss in generality for the remainder of the proof, note that

$$F_l(x_{-l}^v|x_l^v) = \int\limits_{x_{-g_l} \leq x_{-g_l}^v} f_l(x_{-g_l}|x_l^v)dx_{-g_l} \leq \int\limits_{[x_l^v-2v,\, x_l^v+2v]} 1_{\{\overline{s}_{-g_l}^v = \tilde{a}_{-g_l}\}} f_l(x_{-g_l}|x_l^v)dx_{-g_l} = \overline{s}_{-g_l}^v(x_l^v)$$

and likewise $F_j(x_{-g_j}^v|x_j^v) \leq \overline{s}_{-g_j}^v(x_j^v)$. We then have

$$p_l^g(x_l^v) + p_j^g(x_j^v) \overset{(1)}{\geq} \overline{s}_{-g_l}^v(x_l^v) + \overline{s}_{g_j}^v(x_j^v) \overset{(2)}{\geq} F_l(x_{-g_l}^v|x_l^v) + F_j(x_{-g_j}^v|x_j^v)$$

$$\overset{(3)}{=} \prod_{\substack{g_i \\ i\neq l}} F_l(x_{g_i}^v|x_l^v) + \prod_{\substack{g_i \\ i\neq j}} F_l(x_{g_i}^v|x_l^v) \overset{(4)}{\geq} F_l(x_j^v|x_l^v) \prod_{\substack{g_i \\ i\neq l,j}} F_j(x_{g_i}^v|x_j^v) + F_j(x_l^v|x_j^v) \prod_{\substack{g_i \\ i\neq l,j}} F(x_{g_i}^v|x_j^v)$$

$$= (\prod_{\substack{g_i \\ i\neq l,j}} F_j(x_{g_i}^v|x_j^v)) \left(F_l(x_j^v|x_l^v) + F_j(x_l^v|x_j^v)\right) = \prod_{\substack{g_i \\ i\neq l,j}} F_j(x_{g_i}^v|x_j^v) \overset{(5)}{\geq} \left(\frac{1}{2}\right)^{|G|-2}$$

Inequality(1) follows from Proposition 3, inequality (2) from the discussion above, inequality (3) from the independence of the signals, inequality (4) from the fact that $x_l^v \leq x_j^v$, and inequality (5) from the fact that it cannot be guaranteed that all other x_i^v lie strictly above x_j^v. Therefore, $p_l^g(x_l^v) + p_j^g(x_j^v) \geq \left(\frac{1}{2}\right)^{|G|-2}$. But since v satisfies the conditions of Corallary 2, we must have $p_l^g(x_l^v) + p_j^g(x_j^v) < \left(\frac{1}{2}\right)^{|G|-2}$, a contradiction.

□

Note that Theorem 1 follows immediately, which has the same set-up as Theorem 2 but with the "trivial" partitioning of each player receiving their own signal. We now consider examples

Example 2. Consider the following version of the Brander-Spencer model, where a foreign firm (F_f) decides whether to remain in (R) or leave (L) a market consisting of two domestic firm (F_d), who must decide whether to enter (E) or stay out (S) of the market. The domestic firms receives a government subsidy $s \geq 0$ whereas the foreign firms do not. Suppose we have a simplified payoff matrix given by the following:

F_f

R

		F_d	
		E	S
F_d	E	$-3+s,\ -3+s,\ -3$	$2+s,\ 0,\ 2$
	S	$0,\ 2+s,\ 2$	$0,\ 0,\ 3$

L

		F_d	
		E	S
F_d	E	$3+s,\ 3+s,\ 0$	$3+s,\ 0,\ 0$
	S	$0,\ 3+s,\ 0$	$0,\ 0,\ 0$

This is a game of strategic substitutes parameterized by $s \geq 0$. We see that for $s \in [0, 3]$, the Nash equilibria are given by (E, S, R), (S, E, R), and (E, E, L), and that for $s > 3$, (E, E, L) is the dominance solvable strategy. Note that the parameter restriction $s \geq 0$ prevents us from establishing a lower dominance region, thus traditional global games methods cannot be applied. It is easily checked that the conditions of Theorem 1 are satisfied, but calculating the p_i function for player 3 gives $p_3(s) = \frac{1}{2}$, $\forall s$. Note that automatically the condition

$$p_i(s) + p_j(s) < \left(\frac{1}{2}\right)^{|I|-2} = \frac{1}{2},\ \forall i,\ j$$

of Theorem 1 cannot be applied, and thus we are unable to settle the issue of multiple equilibra at all.

However, we see that we can group players 1 and 2 together to form an (E, E, L)- based partitioning according to Definition 8. The interpretation is that the two domestic firms receive the same signal of the subsidy, whereas the foreign firm receives a signal independent of the two domestic firms. Calculating the p_i^g functions, we have that $p_1^g(s) = p_2^g(s) = \frac{3-s}{5}$, and $p_3^g(s) = \frac{1}{2}$. Thus the condition

$$p_i^g(s) + p_j^g(s) < \left(\frac{1}{2}\right)^{|G|-2} = 1,\ \forall i,\ j$$

of Theorem 2 holds for all $s > \frac{1}{2}$, allowing us to establish (E, E, L) as the equilibrium selection at those parameters.

Example 3. Recall the payoff matrix given in Example 1:

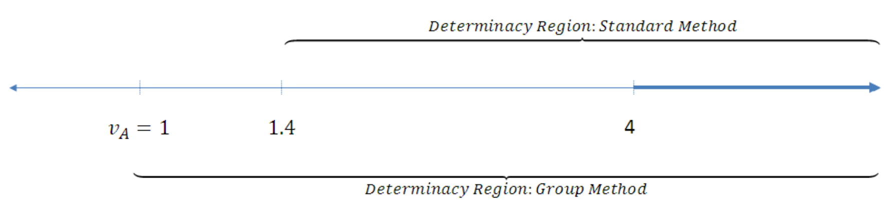
P3 = A, P2 columns:

		A	B
$P1$	E	3, 3, 3	2, x, 2
	S	x, 2, 2	$x+3$, $x+3$, 1

P3 = B, P2 columns:

		A	B
$P1$	A	2, 2, $x-1$	1, $x+3$, $x+2$
	B	$x+3$, 1, $x+2$	$x+6$, $x+6$, $x+6$

In that example, it was shown that$(B,\ B, B)$ is the global games prediction for any $x > 1.4$. However, if the modeller notices that in the description of the game, players 1 and 2 are already users of technology B , it is more natural to assume that they receive the same signal regarding v_B, but independent of the one that player 3, the user of technology A, receives. It is also easily verified that grouping players 1 and 2 in this way forms a $(B,\ B,\ B)-$based partitioning of I . Applying Proposition 2, we find that $p_1^g(x) = p_2^g(x) = \frac{3-x}{4}$, and $p_3^g(x) = \frac{4-x}{9}$. Satisfying the condition that $p_i^g(x) + p_j^g(x) < (\frac{1}{2})^{|G|-2}$ for all $i,\ j$ gives us that $(B,\ B,\ B)$ is the global games prediction for all $x > 1$. That is, allowing for the oberservation that players may naturally share information about the underlying parameter of uncertainty in question, we see that the predictability power of the global games method increases. In fact, since the assumption in the model is that $v_B > v_A = 1$, grouping eliminated multiplicity from the model completely.



4 Appendix

Below it's shown that for each $a_{g_{-i}}$, $p_i^g(a_i,\ a_{g_{-i}},\ x)$ is an upper semi-continuous function on all of X.

Lemma 2. *Suppose $\tilde{a}$ is a strict, group $p-$dominant Nash equilibrium and $x \notin D^{\tilde{a}_i, a_i, a_{g_{-i}}}$*

for some player $i \in I$, $a_i \in A_i$, and $a_{g-i} \in A_{g-i}$. Then $\exists \lambda_i \in \triangle(A_{-g_i})$ such that $l_i(a_i, a_{g_{-i}}, \lambda_i, x) = 0$.

Proof. Since $\tilde{a}$ is a strict Nash equilibrium, we have that $l_i(a_i, a_{g_{-i}}, 1_{\tilde{a}_{-g_i}}, x) > 0$. Suppose for all $\lambda_i \in \triangle(A_{-g_i})$, $l_i(a_i, a_{g_{-i}}, \lambda_i, x) > 0$. Then for all a_{-g_i}, $\triangle u_i(\tilde{a}_i, a_i, a_{g_{-i}}, a_{-g_i} x) > 0$, contradicting the fact that $x \notin D^{\tilde{a}_i, a_i, a_{g_{-i}}}$. Thus for some $\bar{\lambda}_i$, $l_i(a_i, , a_{g_{-i}}, \bar{\lambda}_i, x) \leq 0$. Consider the set of probability measures

$$\mathcal{Z} = \left\{ \lambda_i^\alpha = \begin{cases} \alpha + (1-\alpha)\bar{\lambda}_i(\tilde{a}_{-g_i}) & if\, a_{-g_i} = \tilde{a}_{-g_i} \\ (1-\alpha)\bar{\lambda}_i(a_{-g_i}) & if\, a_{-g_i} \neq \tilde{a}_{-g_i} \end{cases} \mid \alpha \in [0,\, 1] \right\}$$

Note that $\lambda_i^1 = 1_{\tilde{a}_{-g_i}}$ and $\lambda_i^0 = \bar{\lambda}_i$, giving us $l_i(a_i, a_{g_{-i}}, \lambda_i^1, x) > 0$, $l_i(a_i, a_{g_{-i}}, \lambda_i^0, x) \leq 0$, and that $l_i(a_i, a_{g_{-i}}, \lambda_i^\alpha, x)$ is a continuous function of α. By the intermediate value theorem, $\exists \bar{\alpha} \in [0,\, 1)$ such that $l_i(a_i, a_{g_{-i}}, \lambda_i^{\bar{\alpha}}, x) = 0$, giving the result. $\square$

Lemma 4 will show that the set $\{\lambda_i \in \triangle(A_{-g_i}) \,|\, l_i(a_i, a_{g_{-i}}, \lambda_i, x) = 0\}$ is compact, and since by the last lemma it is non-empty, the value $\hat{p}_i^g(a_i, a_{g_{-i}},\ x) \equiv \underset{\substack{\lambda_i \in \triangle(A_{-g_i}) \\ l_i(a_i, a_{g_{-i}}, \lambda_i, x) = 0}}{max} (\lambda_i(\tilde{a}_{-g_i}))$ is well defined. We now show that for all such x, $p_i^g(a_i, a_{g_{-i}},\ x) = \hat{p}_i^g(a_i, a_{g_{-i}},\ x)$.

Lemma 3. *Suppose that $x \notin D^{\tilde{a}_i, a_i, a_{g_{-i}}}$. Then $p_i^g(a_i, a_{g_{-i}},\ x) = \hat{p}_i^g(a_i, a_{g_{-i}},\ x)$.*

Proof. It is first shown that $p_i^g(a_i, a_{g_{-i}},\ x) \leq \hat{p}_i^g(a_i, a_{g_{-i}},\ x)$. In order to do this, we show that for all $\lambda_i^{'} \in \triangle(A_{-g_i})$ such that $\lambda_i^{'}(\tilde{a}_{-g_i}) \geq \hat{p}_i^g(a_i, a_{g_{-i}},\ x)$, $l_i(a_i, a_{g_{-i}}, \lambda_i^{'}, x) \geq 0$. Because $p_i^g(a_i, a_{g_{-i}},\ x)$ is defined as the lowest value that satisfies this property, the conclusion follows. Suppose $\lambda_i^{'} \in \triangle(A_{-g_i})$ is such that $\lambda_i^{'}(\tilde{a}_{-g_i}) \geq \hat{p}_i^g(a_i, a_{g_{-i}},\ x)$, but $l_i(a_i, a_{g_{-i}}, \lambda_i^{'}, x) < 0$. Since $\tilde{a}$ is a strict Nash equilibrium, we have that $l_i(a_i, a_{g_{-i}}, 1_{\tilde{a}_{-g_i}}, x) > 0$. Consider the set of probability measures

$$\mathcal{Z} = \left\{ \lambda_i^\alpha = \begin{cases} \alpha + (1-\alpha)\lambda_i^{'}(\tilde{a}_{-g_i}) & if\, a_{-g_i} = \tilde{a}_{-g_i} \\ (1-\alpha)\lambda_i^{'}(a_{-g_i}) & if\, a_{-g_i} \neq \tilde{a}_{-g_i} \end{cases} \mid \alpha \in [0,\, 1] \right\}$$

and note that $\lambda_i^1 = 1_{\tilde{a}_{-g_i}}$ and $\lambda_i^0 = \lambda_i^{'}$, giving us $l_i(a_i, a_{g_{-i}}, \lambda_i^1, x) > 0$, $l_i(a_i, a_{g_{-i}}, \lambda_i^0, x) <$

0 . Since $l_i(a_i,\, a_{g_{-i}},\, \lambda_i^{\alpha},\, x)$ is a continuous function of α, by the intermediate value theorem $\exists\bar{\alpha} \in (0,\,1)$ such that $l_i(a_i,\, a_{g_{-i}}\, \lambda_i^{\bar{\alpha}},\, x) = 0$. Since $\lambda_i^{\bar{\alpha}}(\tilde{a}_{-g_i}) = \bar{\alpha} + (1-\bar{\alpha})\lambda_i^{'}(\tilde{a}_{-g_i}) > \lambda_i^{'}(\tilde{a}_{-g_i})$, we have found a $\lambda_i^{\bar{\alpha}}$ such that $l_i(a_i,\, a_{g_{-i}},\, \lambda_i^{\bar{\alpha}},\, x) = 0$ but $\lambda_i^{\bar{\alpha}}(\tilde{a}_{-g_i}) > \lambda_i^{'}(\tilde{a}_{-g_i}) \geq \hat{p}_i^g(a_i,\, a_{g_{-i}},\, x)$, contradicting the definition of $\hat{p}_i^g(a_i,\, a_{g_{-i}},\, x)$. Hence $p_i^g(a_i,\, a_{g_{-i}},\, x) \leq \hat{p}_i^g(a_i,\, a_{g_{-i}},\, x)$.

To show equality, suppose that $p_i^g(a_i,\, a_{g_{-i}},\, x) < \hat{p}_i^g(a_i,\, a_{g_{-i}},\, x)$. We will show that for any $\lambda_i^{''}$ such that $\lambda_i^{''}(\tilde{a}_{-g_i}) > p_i^g(a_i,\, a_{g_{-i}},\, x)$, we must have that $l_i(a_i,\, a_{g_{-i}},\, \lambda_i^{''},\, x) > 0$. Hence if $p_i^g(a_i,\, a_{g_{-i}},\, x) < \hat{p}_i^g(a_i,\, a_{g_{-i}},\, x)$ is true, any $\lambda_i^{'}$ such that $\lambda_i^{'}(\tilde{a}_{-g_i}) = \hat{p}_i^g(a_i,\, a_{g_{-i}},\, x)$ must be such that $l_i(a_i,\, a_{g_{-i}},\, \lambda_i^{''},\, x) > 0$, a direct contradition to the existence of $\hat{p}_i^g(a_i,\, a_{g_{-i}},\, x)$. Let $\mathcal{A} = \underset{a_{-g_i}}{argmin}(\triangle u_i(\tilde{a}_i,\, a_i,\, a_{g_{-i}},\, a_{-g_i}\, x))$. Note that $\triangle u_i(\tilde{a}_i,\, a_i,\, a_{g_{-i}},\, \tilde{a}_{-g_i},\, x)$ is not part of this set, for if it were, then for all a_{-g_i} we'd have $\triangle u_i(\tilde{a}_i,\, a_i,\, a_{g_{-i}},\, a_{-g_i}\, x) \geq \triangle u_i(\tilde{a}_i,\, a_i,\, a_{g_{-i}},\, \tilde{a}_{-g_i},\, x) > 0$, contradicting the fact that $x \notin D^{\tilde{a}_i, a_i,\, a_{-g_i}}$. Let λ_i be such that $\lambda_i(\tilde{a}_{-g_i}) = p_i^g(a_i,\, a_{g_{-i}},\, x)$, and assign to all a_{-g_i} in $\mathcal{A}$ the probability $\frac{1-\lambda_i(\tilde{a}_{-g_i})}{|\mathcal{A}|}$. By the definition of $\tilde{a}_i$ being $p_i^g(a_i,\, a_{g_{-i}},\, x)$ dominant, we must have that $l_i(a_i,\, a_{g_{-i}},\, \lambda_i,\, x) \geq 0$. Now let $\lambda_i^{'}$ be such that $\lambda_i^{'}(\tilde{a}_{-g_i}) > p_i^g(a_i,\, a_{g_{-i}},\, x)$, and assign to all a_{-g_i} in $\mathcal{A}$ the probability $\frac{1-\lambda_i^{'}(\tilde{a}_{-g_i})}{|\mathcal{A}|}$. Note that we have

$$l_i(a_i,\, a_{g_{-i}},\, \lambda_i^{'},\, x) = \triangle u_i(\tilde{a}_i,\, a_i,\, a_{g_{-i}},\, \tilde{a}_{-g_i},\, x)\lambda_i^{'}(\tilde{a}_{-g_i}) + \left(\frac{1-\lambda_i^{'}(\tilde{a}_{-g_i})}{|\mathcal{A}|}\right)\sum_{\mathcal{A}} \triangle u_i(\tilde{a}_i,\, a_i,\, a_{g_{-i}},\, a_{-g_i}\, x) >$$

$$\triangle u_i(\tilde{a}_i,\, a_i,\, a_{g_{-i}},\, \tilde{a}_{-g_i},\, x)\lambda_i(\tilde{a}_{-g_i}) + \left(\frac{1-\lambda_i(\tilde{a}_{-g_i})}{|\mathcal{A}|}\right)\sum_{\mathcal{A}} u_i(\tilde{a}_i,\, a_i,\, a_{g_{-i}},\, a_{-g_i}\, x) = l_i(a_i,\, \lambda_i,\, x) \geq 0.$$

Finally, let $\lambda_i^{''}$ be arbitrary but satisfying $\lambda_i^{''}(\tilde{a}_{-g_i}) = \lambda_i^{'}(\tilde{a}_{-g_i})$. Because $l_i(a_i,\, a_{g_{-i}},\, \lambda_i^{'},\, x) \geq 0$ gives the smallest value for such probability measures, we have that $l_i(a_i,\, a_{g_{-i}},\, \lambda_i^{''},\, x) \geq l_i(a_i,\, a_{g_{-i}},\, \lambda_i^{'},\, x) > l_i(a_i,\, a_{g_{-i}},\, \lambda_i,\, x) \geq 0$, or $l_i(a_i,\, a_{g_{-i}},\, \lambda_i^{''},\, x) > 0$, giving the result. $\square$

From Lemmas 2 and 3, we have established that on X,

$$p_i^g(a_i,\, a_{g_{-i}},\, x) = \begin{cases} \underset{\substack{\lambda_i \in \triangle(A_{-g_i}) \\ l_i(a_i,\, a_{g_{-i}},\, \lambda_i,\, x)=0}}{max} (\lambda_i(\tilde{a}_{-i})) & ,\, x \notin D^{\tilde{a}_i, a_i,\, a_{g_{-i}}} \\ 0 & ,\, x \in D^{\tilde{a}_i, a_i,\, a_{g_{-i}}} \end{cases}$$

We now establish the upper semi-continuity of $p_i^g(a_i,\, a_{g_{-i}},\, x)$, which is done in two steps.

Lemma 4. *$p_i^g(a_i,\, a_{g_{-i}},\, x)$ is an upper semi-continuous function on $X \cap (D^{\tilde{a}_i, a_i, a_{g_{-i}}})^C$.*

Proof. Recall the Maximum Theorem[10]: If $p_i^g(a_i,\, a_{g_{-i}},\, x) = \underset{\substack{\lambda_i \in \triangle(A_{-g_i}) \\ l_i(a_i,\, a_{g_{-i}},\, \lambda_i,\, x)=0}}{max} (\lambda_i(\tilde{a}_{-i}))$ is such that $\varphi : X \rightrightarrows \triangle(A_{-g_i})$, $\varphi(x) = \{\lambda_i \in \triangle(A_{-g_i}) \,|\, l_i(a_i,\, a_{g_{-i}},\, \lambda_i,\, x) = 0\}$ is upper hemi-continuous with non-empty, compact values, and $f : gf(\varphi) \to \mathbb{R}$ defined as $f(x,\, \lambda_i) = \lambda_i(\tilde{a}_{-g_i})$ is upper semi-continuous, then $p_i^g(a_i,\, a_{g_{-i}},\, \cdot)$ is upper semi-continuous. We show one-by-one that these conditions are met:

(i) Let $(x^n,\, \lambda_i^n)_{n=1}^\infty \subseteq gf(\varphi)$ be such that $(x^n,\, \lambda_i^n) \to (x,\, \lambda)$. Then $\underset{n\to\infty}{lim}(f(x^n,\, \lambda_i^n)) = \underset{n\to\infty}{lim}(\lambda_i^n(\tilde{a}_{-g_i})) = \lambda_i(\tilde{a}_{-g_i}) = f(x,\, \lambda_i)$, showing that f is continuous, and therefore upper semi-continuous.

(ii) By Lemma 2, $\varphi : X \rightrightarrows \triangle(A_{-g_i})$ is non-empty valued. To see that it is compact valued, let x be given and suppose $(\lambda_i^n)_{n=1}^\infty \subseteq \varphi(x)$ is such that $\lambda_i^n \to \lambda_i$. Because $\triangle(A_{-g_i})$ is closed, $\lambda_i \in \triangle(A_{-g_i})$. Because $l_i(\cdot,\, x) : \triangle(A_{-g_i}) \to \mathbb{R}$ is continuous, $l_i(a_i,\, a_{g_{-i}},\, \lambda_i,\, x) = 0$, and hence $\varphi(x)$ is closed-valued. Since $\varphi(x) \subseteq \triangle(A_{-g_i})$, it is therefore compact valued.

Finally, we see that φ is upper hemi-continuous. Recall that a correspondence with compact and Hausdorff range space has a closed graph if and only if it is upper hemi-continuous and closed valued. It therefore suffices to show that φ has a closed graph. To that end, suppose $(x^n,\, \lambda_i^n)_{n=1}^\infty \subseteq gf(\varphi)$ is such that $(x^n,\, \lambda_i^n) \to (x,\, \lambda_i)$. Because X is closed, $x \in X$. Because $\triangle(A_{-i})$ is closed, $\lambda_i \in \triangle(A_{-i})$. Lastly, because $l_i : X \times \triangle(A_{-i}) \to \mathbb{R}$ is continuous, $l_i(a_i,\, a_{g_{-i}},\, \lambda_i,\, x) = \underset{n\to\infty}{lim}(l_i(a_i,\, a_{g_{-i}},\, \lambda_i^n,\, x^n)) = 0$, and thus $(x,\, \lambda_i) \in gf(\varphi)$, so that $gf(\varphi)$ is closed, completing the Lemma. □

Lemma 5. *$p_i^g(a_i,\, a_{g_{-i}},\, x)$ is an upper semi-continuous function on all of X.*

Proof. Let $(x_n)_{n=1}^\infty \subseteq X$ be such that $x_n \to x$. It's shown that $\underset{n}{limsup}(p_i^g(a_i,\, a_{g_{-i}},\, x_n)) \leq p_i^g(a_i,\, a_{g_{-i}},\, x)$ by considering two cases:

[10] Aliprantis, Lemma 17.30

Case 1. Suppose $\exists K > 0, \forall n \geq K, \ x_n \in D^{\tilde{a}_i, a_i, a_{g-i}}$. If $x \in D^{\tilde{a}_i, a_i, a_{g-i}}$, then

$$\underset{n}{limsup}(p_i^g(a_i, \ a_{g-i}, \ x_n)) = \underset{k \geq 1 n \geq k}{infsup}(p_i^g(a_i, \ a_{g-i}, \ x_n)) \leq \underset{n \geq K}{sup}(p_i^g(a_i, \ a_{g-i}, \ x_n)) = 0 \leq p_i^g(a_i, \ a_{g-i}, \ x)$$

by definition. If $x \notin D^{\tilde{a}_i, a_i, a_{g-i}}$, then $\underset{n \geq K}{sup}(p_i^g(a_i, \ a_{g-i}, \ x_n)) = p_i^g(a_i, \ a_{g-i}, \ x)$. Thus

$$\underset{n}{limsup}(p_i^g(a_i, \ a_{g-i}, \ x_n))) = \underset{k \geq 1 n \geq k}{infsup}(p_i^g(a_i, \ a_{g-i}, \ x_n)) \leq \underset{n \geq K}{sup}(p_i^g(a_i, \ a_{g-i}, \ x_n)) = p_i^g(a_i, \ a_{g-i}, \ x)$$

giving the result.

Case 2. Suppose $\forall K > 0, \exists k_K \geq K, \ x_{k_K} \notin D^{\tilde{a}_i, a_i, a_{g-i}}$. Let $\tilde{N} \subseteq \mathbb{N}$ be those indices such that $n \in \tilde{N} \Rightarrow x_n \notin D^{\tilde{a}_i, a_i, a_{g-i}}$, and define $m : \mathbb{N} \to \mathbb{N}$ by $m(n) = \underset{\substack{j \in \tilde{N} \\ j \geq n}}{min}(j)$. Define the sequence $(x_n^{'})_{n=1}^{\infty}$ by the formula $x_n^{'} = x_{m(n)}$. Then we have the following:

1. $\forall n, \ p_i^g(a_i, \ a_{g-i}, \ x_n^{'}) \geq p_i^g(a_i, \ a_{g-i}, \ x_n)$: Let n be given. If $x_n \in D^{\tilde{a}_i, a_i, a_{g-i}}$, then $p_i^g(a_i, \ a_{g-i}, \ x_n^{'}) \geq 0 = p_i^g(a_i, \ a_{g-i}, \ x_n)$. If $x_n \notin D^{\tilde{a}_i, a_i, a_{g-i}}$, then $x_n^{'} = x_{m(n)} = x_n$, so the inequality follows.

2. $x_n^{'} \to x$: Let $\epsilon > 0$ be given. Since $x_n \to x, \ \exists K, \forall n \geq K, \ |x_n - x| < \epsilon$. Since for all $n, \ m(n) = \underset{\substack{j \in \tilde{N} \\ j \geq n}}{min}(j) \geq n$, then for all $n \geq K, \ |x_n^{'} - x| = |x_{m(n)} - x| < \epsilon$, giving convergence.

Finally, $\underset{n}{limsup}(p_i^g(a_i, \ a_{g-i}, \ x_n)) \leq \underset{n}{limsup}(p_i^g(a_i, \ a_{g-i}, \ x_n^{'}))) \leq p_i^g(a_i, \ a_{g-i}, \ x)$, where the first inequality follows from 1., and the second inequality follows from 2., $(x_n^{'})_{n=1}^{\infty} \subseteq (D^{\tilde{a}_i, a_i, a_{g-i}})^C$, and by Lemma 2.25, since $p_i^g(a_i, \ a_{g-i}, \ \cdot)$ is upper semi-continuous on $(D^{\tilde{a}_i, a_i, a_{g-i}})^C$. This completes the lemma. $\square$

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