

Determinacy and Indeterminacy in Monetary Policy Rules with Money*

John Keating[†] and Andrew Lee Smith[‡]

Department of Economics
The University of Kansas
Lawrence, KS

This Version: October 30, 2013

Abstract

Does Friedman's k-percent rule guarantee a unique equilibrium outcome? We show analytically the answer to this question is sensitive to the method of aggregation. Focusing on broad measures of money, we show that fixing the growth rate of the true monetary aggregate will generate a unique rational expectations equilibrium. Since the true monetary aggregate is parametric, we show this determinacy result extends to the non-parametric Divisia monetary aggregate growth rule. Interestingly, Friedman's proposal to fix the growth rate of the broad simple-sum monetary aggregate is shown to result in indeterminacy stemming from this aggregate's inaccuracy in tracking the true monetary aggregate. Determinacy regions of interest rate rules reacting to the growth rate of monetary aggregates are also discussed and a novel Taylor principle is shown to hold for such rules when the monetary aggregate is accurately measured. All of these results are presented in the framework of the canonical New-Keynesian model.

Keywords: Friedman's k-percent Rule, Determinacy, Monetary Aggregates, Taylor Rules
JEL codes: C43, E31, E40, E44, E51, E52, E58, E60

*A portion of this paper was presented at the 2012 Missouri Economics Conference at the University of Missouri, sponsored by the Federal Reserve Bank of St. Louis. We would like to thank participants for their helpful comments and Joe Haslag in particular for an insightful conversation.

[†]jkeating@ku.edu

[‡]asmith@ku.edu

1 Introduction

In his classic (1960) work, *A Program for Monetary Stability*, Milton Friedman argued the economy can be stabilized by stabilizing the growth rate of a monetary aggregate at k -percent. Ironically, if such a rule fails to deliver a unique rational expectations equilibrium, fixing the growth rate of a monetary aggregate may induce instability by allowing for sunspot equilibria. Given Friedman’s large influence on monetary policy, this policy proposal has received considerable attention by monetary theorist. Surprisingly though, the question of whether this rule could be implemented in a manner which delivers a unique rational expectations equilibrium remains unanswered. This paper provides an answer to this question in the preferred framework of modern monetary policy, a New-Keynesian model.

We show (analytically) Friedman’s k -percent rule can deliver a unique rational expectations equilibrium when the true monetary aggregate is used. Since this aggregate depends on deep structural parameters, such a rule is not *simple* in the sense defined by (Gali, 2008). A *simple* rule which could actually be used by central banks,¹ would be to fix the growth rate of the Divisia monetary aggregate. We show that such a rule inherits the determinacy properties of the k -percent rule using the true monetary aggregate. Another (more popular) non-parametric aggregate is the simple-sum monetary aggregate which was the method used to compute the M2 aggregate Friedman favored in his work. Interestingly, if this aggregate is used in place of the true aggregate, Friedman’s k -percent rule is likely to result in indeterminacy stemming from simple-sum’s error in tracking the true monetary aggregate. We conclude that Friedman was correct in his judgment that fixing the growth rate of a broad monetary aggregate will stabilize the economy. However, his choice of an aggregation method was flawed.

Most similar to our work, Evans and Honkapohja (2003) analyze determinacy properties of k -percent rules in a New-Keynesian model when money is modeled as a single monetary asset. In this case, money in the model can be interpreted as base money since it earns zero interest. They show numerically that fixing the growth rate of this measure of money is consistent with a unique rational expectations equilibrium under a broad range of values. However Friedman’s k -percent rule calls for the central bank to fix the growth rate of a high level monetary aggregate, not base money. In order to analyze determinacy properties of broad monetary aggregates, we require a model like the one developed by Belongia and Ireland (2012a) which provides a role for both currency and deposits as competing sources of monetary services. We use this as our framework to analyze the determinacy properties of monetary policy rules with broad money. In addition to providing analytic representations of the determinacy regions of various monetary aggregate growth rules, we also analyze interest rate rules reacting to inflation and the growth rate of monetary aggregates. We show that once again, “measurement matters” (Belongia, 1996) from a determinacy standpoint. Interest rate rules reacting to the growth rate of either the true monetary aggregate or the Divisia monetary aggregate satisfy a novel Taylor principle for monetary aggregates. The simple-sum monetary aggregate’s error in tracking the true aggregate prevents interest rate rule reacting to simple-sum from having a similar determinacy region. Instead, such rules have small determinacy regions.

¹Arguably, one reason Friedman advocated this very rule was for its simplicity in implementing.

2 Model

We consider a canonical version of the linearized² New-Keynesian (NK) model. As usual, this model consists of 3 non-policy equations when money is included in the model.

$$\tilde{c}_t = E_t[\tilde{c}_{t+1}] - E_t[\Delta \tilde{a}_{t+1}^{pref}] - E_t[\tilde{r}_t - \tilde{\pi}_{t+1}] \quad (1)$$

$$\tilde{\pi}_t = \kappa[\tilde{c}_t - \tilde{z}_t^{tech}] + \beta E_t[\tilde{\pi}_{t+1}] \quad (2)$$

$$\left(\frac{m^A}{p} \right)_t = \tilde{c}_t - \tilde{u}_t^A + \tilde{v}_t^{md} \quad (3)$$

Equation (1) is the dynamic IS curve, equation (2) is the New-Keynesian Phillips Curve (NKPC), equation (3) is the demand for the monetary aggregate. We depart from the textbook analysis of money. Following Belongia and Ireland (2012a) we model broad money so that (3) is the demand for the aggregate service flow from currency and interest bearing deposits.³ For this reason, (3) includes the user-cost of the monetary aggregate u_t^A which in general depends on a vector of interest rates. This distinction between modeling money as a single monetary assets (such as currency or base money) versus broader measures drives this analysis. In particular, we examine the determinacy properties of monetary policy rules with broad measures of money. We examine monetary aggregate growth rules and interest rate rules which react to the growth rate of the monetary aggregate.

We assume

$$M_t^A = \left[\nu^{\frac{1}{\omega}} (N_t)^{\frac{\omega-1}{\omega}} + (1-\nu)^{\frac{1}{\omega}} (D_t)^{\frac{\omega-1}{\omega}} \right]^{\frac{\omega}{\omega-1}} \quad (4)$$

is the true monetary aggregate where N_t and D_t are non-interest bearing currency and interest bearing deposits respectively. Meanwhile, $\omega > 0$ is the elasticity of substitution between currency and deposits and $0 < \nu < 1$ governs the steady-state share of currency and deposits. The aggregate dual user-cost is correspondingly defined by

$$u_t^A = \left[\nu(R_t - 1)^{\omega-1} + (1-\nu)(R_t - R_t^D)^{\omega-1} \right]^{\frac{1}{\omega-1}}. \quad (5)$$

Following the structural model of Belongia and Ireland (2012a), when modeled with a financial sector, the user cost of deposits can be expressed as

$$u_t^D = R_t - R_t^D = (R_t - 1)\tau_t + x_t \quad (6)$$

where τ_t and x_t are first order auto-regressive processes (in logs) which represent exogenous financial disturbances. Specifically, τ_t represents reserves demand disturbances with mean $\bar{\tau}$ and x_t represents deposit cost disturbances with mean $\bar{X}$. To remain empirically relevant, we will assume throughout the paper that the stochastic processes are bounded so that at all

²Following Woodford's (2003) notation we assume all exogenous disturbances are bounded in amplitude by $\|\xi\|$. All variables with a bar over them denote non-stochastic steady-state values. Also, lowercase variables with a $\sim$ over them denote log-deviations from the non-stochastic steady state.

³The variables $\tilde{a}_t^{pref}$, $\tilde{z}_t^{tech}$ and $\tilde{v}_t^{md}$ represent exogenous preference, technology and money-demand disturbances respectively.

times $R_t > R_t^D > 1$ ensuring that user costs are positive and deposits are “cheaper” than currency within a neighborhood of the non-stochastic steady-state. A log-linear approximation to (5) is given by

$$\tilde{u}_t^A \approx \eta_r \tilde{r}_t + \eta_\tau \tilde{\tau}_t + \eta_x \tilde{x}_t \quad (7)$$

where all the coefficients are positive. Substituting (7) into (3) brings about a four variable dynamic model with three equations (1, 2 and 8).

$$\widetilde{\left(\frac{m^A}{p}\right)}_t = \tilde{c}_t - \eta_r \tilde{r}_t - \eta_\tau \tilde{\tau}_t - \eta_x \tilde{x}_t + \tilde{v}_t^{md} \quad (8)$$

As is standard, we close the model with a specification of monetary policy.

3 Friedman k-percent Rules

In this section we consider the performance of various money growth rules. In particular, we close the model with a constant money growth rule

$$\Delta \tilde{m}_t^A = 0. \quad (9)$$

Such k-percent rules have been examined in New-Keynesian models by Evans and Honkapohja (2003) where they were shown to deliver a unique REE. However, these papers assume the existence of a single monetary asset. This leaves open the question of whether the determinacy properties extend to monetary aggregates. Arguably, this is the relevant case to consider. Friedman for example argued for fixing the growth rate of a broad aggregate such as M2, not the monetary base. In this section we show three main results. First, the k-percent rule is determinate when applied to the true monetary aggregate, regardless of the magnitude of the interest semi-elasticity. Second, this determinacy result extends to rules which replace the true aggregate with the Divisia monetary aggregate. Finally, we show this result does not extend to the Simple-Sum monetary aggregate. Instead, a constant Simple-Sum growth rule is indeterminate unless the interest semi-elasticity is large relative to Simple-Sum’s error in tracking the true monetary aggregate. Consider the dynamic system consisting of (1), (2), (8) and (9).

Proposition 1. *For any $0 < \beta < 1$, for any $\kappa > 0$ and for any $\eta_r \geq 0$ if the central bank follows the policy rule $\Delta \tilde{m}_t^A = 0$ then there exists a unique REE.*

This result shows constant monetary aggregate growth rules always deliver a unique REE, extending the results of Evans and Honkapohja (2003) to monetary aggregates. This result in of itself is of little use to central banks who do not know the underlying parameters of the monetary aggregate. To circumvent this problem, suppose the central bank fixes the growth rate of the Divisia monetary aggregate instead of the true monetary aggregate. Such a rule is more relevant to policy makers as it doesn’t require estimating any parameters.

Definition 1. The growth rate of the Divisia monetary aggregate is defined by:

$$\ln(\mu_t^{Divisia}) = \left(\frac{s_t^N + s_{t-1}^N}{2}\right) \ln\left(\frac{N_t}{N_{t-1}}\right) + \left(\frac{s_t^D + s_{t-1}^D}{2}\right) \ln\left(\frac{D_t}{D_{t-1}}\right), \quad (10)$$

where s_t^N and s_t^D are the expenditure shares of currency and interest bearing deposits respectively defined by:

$$s_t^N = \frac{(r_t - 1)N_t}{(r_t - 1)N_t + (r_t - r_t^D)D_t} \text{ and } s_t^D = 1 - s_t^N. \quad (11)$$

The Divisia monetary aggregate is the expenditure share-weighted growth rate of currency and deposits. Most importantly, the Divisia aggregate is non-parametric and depends only on current and past information. The following lemma shows that for the general CES specification of M_t^A , the Divisia monetary aggregate (locally) tracks the growth rate of the true monetary aggregate to first order accuracy without error.

Lemma 1. *For any $0 < \nu < 1$ and for any $\omega > 0$, the difference between the Divisia monetary aggregate and the true monetary is given by*

$$\ln(\mu_t^{Divisia}) - \Delta \ln(M_t^A) = \mathcal{O}(\|\xi\|^2).$$

The accuracy properties of the Divisia monetary aggregate are well known in Index number theory. Most notably, Divisia (1926) showed in continuous time, the Divisia monetary aggregate tracks any linearly homogenous function without error. Moreover, Diewert (1976) classified the discrete time Divisia monetary aggregate defined above as superlative - meaning that it can track the growth rate of any twice differentiable linearly homogenous function up to second order accuracy without error. This lemma shows that for the CES function in particular, a *linear approximation* to the Divisia monetary aggregate tracks the true monetary aggregate up to first order accuracy without error. This result is very useful in analyzing local determinacy which requires only a linear approximation to the non-linear model.

Formally, suppose the central bank follows the rule

$$\tilde{\mu}_t^{Divisia} = 0 \quad (12)$$

in place of (9). We have the following result. Consider the dynamic system consisting of (1), (2), (8) and (12).

Corollary 1. *For any $0 < \beta < 1$, for any $\kappa > 0$ and for any $\eta_r \geq 0$ if the central bank follows the policy rule $\tilde{\mu}_t^{Divisia} = 0$ then there exists a unique REE.*

The proof follows immediately from combining lemma 1 and proposition 1. Corollary 1 shows the central bank may replace the true aggregate with the Divisia monetary aggregate without any change in determinacy. Intuitively, we acquire this result because determinacy is a local condition and lemma 1 shows that locally, the Divisia monetary aggregate exactly tracks the growth rate of the true monetary aggregate. Therefore, any determinacy properties of the true monetary aggregate growth rules are inherited by rules which instead use the Divisia monetary aggregate. The same can not be said for the more common simple-sum monetary aggregate defined as follows.

Definition 2. The growth rate of the Simple-Sum monetary aggregate is defined by

$$\ln(\mu_t^{\text{Simple-Sum}}) = \ln\left(\frac{N_t + D_t}{N_{t-1} + D_{t-1}}\right). \quad (13)$$

The simple-sum aggregate treats currency and interest bearing deposits as one for one perfect substitutes. Standard microeconomic theory dictates in such a case only the cheapest monetary asset would have a positive demand in equilibrium. Since this is not the case in this model, nor reality, it is perhaps not surprising the simple-sum aggregate will locally track the true aggregate with error. The following lemma defines and quantifies this error.

Lemma 2. For any $0 < \nu < 1$, for any $\omega > 0$ and for any $(\bar{\tau}, \bar{X})$ satisfying $\bar{u}^N = (\bar{R} - 1) > (\bar{R} - 1)\bar{\tau} + \bar{X} = \bar{u}^D > 0$, the difference between the growth rate of the Simple-Sum monetary aggregate and the growth rate of the true monetary aggregate is given by

$$\ln(\mu_t^{\text{Simple-Sum}}) - \Delta \ln(M_t^A) = \psi_r(\omega)\Delta\tilde{r}_t - \psi_x(\omega)\Delta\tilde{x}_t - \psi_\tau(\omega)\Delta\tilde{\tau}_t + \mathcal{O}(\|\xi\|^2) \quad (14)$$

where $(\psi_r(\omega), \psi_x(\omega), \psi_\tau(\omega)) \in \mathbb{R}_{++}^3$

with $\lim_{\omega \rightarrow \infty} (\psi_r(\omega), \psi_x(\omega), \psi_\tau(\omega)) = (0, 0, 0)$.

The proof of the lemma is in the appendix. Intuitively, any change in the relative prices of currency and deposits results in a substitution between these assets from the household. Since the simple-sum monetary aggregate treats these assets as perfect substitutes it is not able to internalize these relative changes in the service flow from the monetary aggregate. Only in the limiting case of perfect substitutes does this error disappear.

From a determinacy standpoint, simple-sum's error in tracking the true monetary aggregate derived from the exogenous financial shocks poses no challenges. However, the error that depends on the endogenous bond rate - $\Delta\tilde{r}_t$ - will alter the determinacy region of rules reacting to the Simple-Sum monetary aggregate. In particular, the following corollary summarizes determinacy properties when the policy rule in (15) is used instead of (9).

$$\tilde{\mu}_t^{\text{Simple-Sum}} = 0. \quad (15)$$

Consider the dynamic system consisting of (1), (2), (8) and (15).

Corollary 2. For any $0 < \beta < 1$, for any $\kappa > 0$ if the central bank follows the policy rule $\tilde{\mu}_t^{\text{Simple-Sum}} = 0$ then there exists a unique REE if and only if $\psi_r(\omega) < \eta_r + \frac{1}{2}$.

The proof of this result is available in the appendix. Important here is the fact that when using the Simple-Sum monetary aggregate in place of the true monetary aggregate (or the Divisia monetary aggregate), a constant monetary aggregate growth rule may not be determinate. In particular, if the error term $\psi_r(\omega)$ is large relative to the interest semi-elasticity of money demand - η_r - then constant Simple-Sum growth rules will result in indeterminacy. Unlike the constant Divisia growth rules, the constant Simple-Sum growth rules place a lower bound on the interest elasticity parameter - a non-policy parameter. Due to the wide ranging estimates of η_r , central banks may be understandably adverse to implementing this rule.

4 Interest Rate Rules with Money

In this section we consider interest rate feedback rules which react to lagged interest rates and the growth rate of a monetary aggregate. Previous studies have included nominal money growth in interest rate rules such as Canova and Menz (2011) and Sims and Zha (2006). Fewer papers have included Divisia monetary aggregates in interest rate feedback rules although this literature is growing. Specifically, Belongia and Ireland (2012b) have estimated structural VAR's with such rules. Also, in a related paper, ? find that such rules may be optimal from a normative perspective if financial shocks are present and the natural rate of interest is unobservable. This result is consistent with the assertion of McCallum and Nelson (2011) that nominal money growth can provide valuable real-time information regarding the natural rate of interest. Despite this interest, there are no results on the determinacy properties of such rules to guide policy makers. With this in mind, assume the central bank sets the policy rate according to

$$\tilde{r}_t = \phi_r \tilde{r}_{t-1} + \phi_m \Delta \tilde{m}_t^A. \quad (16)$$

Using (16) to close the dynamic model defined by (1), (2) and (8) we have the following result.

Proposition 2. *For any $0 < \beta < 1$, for any $\kappa > 0$ and for any $\eta_r \geq 0$, if the central bank follows the policy rule $\tilde{r}_t = \phi_r \tilde{r}_{t-1} + \phi_m \Delta \tilde{m}_t^A$ with $\phi_m \geq 0$ and $\phi_r \geq 0$ then there exists a unique REE if and only if $\phi_m + \phi_r > 1$.*

The proof to proposition 2 is available in the appendix. Intuitively, this result extends the Taylor Principle to monetary aggregates. Expectations will remain well-anchored if the central bank reacts more than one for one to nominal aggregate-money growth, in the long-run. Importantly, proposition 1 provides a sufficient condition for determinacy that is independent of the magnitude of the interest semi-elasticity of money demand. This is important because estimates of this parameter vary significantly in the empirical literature. For example Ball (2001) estimates $\eta_r = .05$ while Ireland (2009) estimates $\eta_r = 1.9$. Despite the neutrality of the result with regards to η_r , the ability for a central bank to actually use such a rule is conditional on their ability to measure the unobservable true monetary aggregate. To circumvent this issue suppose instead the central bank replaces the true monetary aggregate with the Divisia monetary aggregate

$$\tilde{r}_t = \phi_r \tilde{r}_{t-1} + \phi_d \tilde{\mu}_t^{Divisia}. \quad (17)$$

Unlike (16), the policy rule in (17) could actually be implemented by a central bank if so desired. The following corollary to proposition 2 shows that central banks can replace $\Delta \tilde{m}^A$ with $\tilde{\mu}_t^{Divisia}$ without any change in the determinacy condition.

Corollary 3. *For any $0 < \beta < 1$, for any $\kappa > 0$ and for any $\eta_r \geq 0$ if the central bank follows the policy rule $\tilde{r}_t = \phi_r \tilde{r}_{t-1} + \phi_d \tilde{\mu}_t^{Divisia}$ with $\phi_d \geq 0$ and $\phi_r \geq 0$ then there exists a unique REE if and only if $\phi_d + \phi_r > 1$.*

The proof of this corollary follows immediately by combining lemma 1 with proposition 2. This result is significant in terms of actually implementing interest rate rules reacting to money growth. Most notably, central banks can use the non-parametric Divisia aggregate in interest rate feedback rules like (17) and guarantee determinacy by setting $\phi_d > 1$. The same can not be said for the more common simple-sum monetary aggregate. In particular, the following corollary summarizes determinacy properties when the policy rule in (18) is used instead of (16) or (17)

$$\tilde{r}_t = \phi_r \tilde{r}_{t-1} + \phi_{ss} \tilde{\mu}^{Simple-Sum}. \quad (18)$$

Corollary 4. *For any $0 < \beta < 1$, for any $\kappa > 0$ and for any $\eta_r \geq 0$ if the central bank follows the policy rule $\tilde{r}_t = \phi_r \tilde{r}_{t-1} + \phi_{ss} \tilde{\mu}^{Simple-Sum}$ with $\phi_{ss} \geq 0$ and $\phi_r \geq 0$ then there exists a unique REE if and only if $|\phi_r + \phi_{ss} - \phi_{ss}[\psi_r(\omega) - \eta_r]| > |1 - \phi_{ss}[\psi_r(\omega) - \eta_r]|$.*

The proof in the appendix follows from Lemma 2 and Proposition 2. This result highlights where simple-sum and Divisia monetary aggregates diverge in terms of determinacy. Namely, the determinacy region for interest rate rules reacting to the Simple-Sum monetary aggregate depend on Simple-Sum's error in tracking the true monetary aggregate. Intuitively, if $\psi_r(\omega)$ is relatively large compared to the interest semi-elasticity of money demand then an upper and lower bound is placed on ϕ_{ss} for determinacy. Meanwhile, if $\psi_r(\omega)$ is relatively small compared to the interest semi-elasticity of money demand then only a lower bound is placed on ϕ_{ss} for determinacy. For example, in the extreme case where the simple-sum and Divisia aggregates coincide (i.e. $\omega \rightarrow \infty$), corollary 3 shows that $\phi_d > 1$ is sufficient for determinacy. However, in the more general case, determinacy under the interest rate rule reacting to simple sum may have a relatively small determinacy region as summarized below for the case in which $\psi_r(\omega) - \eta_r > 1$.

To get an idea of how likely, or unlikely, determinacy is under an interest rate rule reacting to the simple-sum aggregate the figure below plots the determinacy region under the numerical calibration presented below. There are 2 regions of determinacy. One of which occurs in the passive regime and one under the active regime. What stands out the most is how unlikely it is that the condition presented in corollary 4 is satisfied. In fact, determinacy under this rule is more the exception than the norm.

5 Numerical Analysis

More realistic policy rules would include not only a response to the growth rate of a monetary aggregate but also a response to inflation. Unfortunately this slightly more complicated rule places analytic results out of reach so we resort to numerical analysis of determinacy. Many of the parameters in the model are standard so we take their values from the previous literature. We vary the remaining parameters over a reasonable range to understand how determinacy may, or may not, be achieved for the US economy. More precisely, we set $\beta = .99$ and $\kappa = .3$. Two key parameters for determinacy of the Simple-Sum rules are η_r - the interest semi-elasticity of money demand and $\psi_r(\omega)$ - simple-sum's endogenous error in tracking the true monetary aggregate. There is a large literature estimating the interest semi-elasticity of money demand equations. Two more recent studies include Ball (2001)

Table 1: Determinacy Regimes Under $\tilde{r}_t = \phi_r \tilde{r}_{t-1} + \phi_{ss} \tilde{\mu}_t^{Simple-Sum}$

<i>Regime^a</i>	<i>Determinacy Conditions^b</i>
Passive	$\phi_r + \phi_{ss} < 1$
	$\phi_{ss} > \frac{1}{\psi_r(\omega) - \eta_r}$
	$\phi_{ss} > \frac{1}{\psi_r(\omega) - \eta_r - 1} \phi_r$
	$\phi_{ss} > \frac{1}{2(\psi_r(\omega) - \eta_r) - 1} + \frac{1}{2(\psi_r(\omega) - \eta_r) - 1} \phi_r$
Active I	$\phi_r + \phi_{ss} > 1$
	$\phi_{ss} < \frac{1}{\psi_r(\omega) - \eta_r}$
	$\phi_{ss} < \frac{1}{\psi_r(\omega) - \eta_r - 1} \phi_r$
Active II	$\phi_r + \phi_{ss} > 1$
	$\phi_{ss} > \frac{1}{\psi_r(\omega) - \eta_r}$
	$\phi_{ss} < \frac{1}{\psi_r(\omega) - \eta_r - 1} \phi_r$
	$\phi_{ss} < \frac{1}{2(\psi_r(\omega) - \eta_r) - 1} + \frac{1}{2(\psi_r(\omega) - \eta_r) - 1} \phi_r$

^a We use the terms "Passive" and "Active" similar to Leeper (1991) to describe a monetary regime where $\phi_r + \phi_{ss} < 1$ or $\phi_r + \phi_{ss} > 1$ respectively.

^b All of these conditions are sufficient for the existence of a unique REE under the conditions stated in Corollary 4 under the additional assumption that $\psi_r(\omega) - \eta_r > 1$. The numerical analysis in the following section shows this is an empirically relevant case to consider for the U.S. economy.

who estimates $\eta_r = .05$ and Ireland (2009) who estimates $\eta_r = 1.9$ providing a reasonable range of values from $\eta_r \in [0.05, 1.9]$.

To gain some insight regarding the size of simple-sum's error term in tracking the true monetary aggregate, we interpret

$$\ln(N_t + D_t) - \ln(M_t^A) = \beta_0 + \beta_1 \tilde{r}_t + \beta_2 \tilde{x}_t + \beta_3 \tilde{\tau}_t + \varepsilon_t \quad (19)$$

as a linear econometric relationship. In doing so we implicitly assume $\mathcal{O}(\|\xi\|^2)$ from the linearization is a constant plus an unforecastable error. The estimates of β_1 will provide a calibrated value for $\psi_r(\omega)$. We estimate (19) using monthly data from 1967:01 to 2012:09 from the St. Louis Fed's FRED database. We use data on Simple-Sum M2 and the M2 MSI (Divisia) series to form the dependent variable. As for the independent variables we first construct a reserves ratio series - $\tilde{\tau}_t$ - using the St. Louis Adjusted Reserves series and the non-currency components of M2. We then use this and the commercial and industrial loan rate, the M2 own rate and the structural equation in (6) to back out a time series for $\tilde{x}_t$. In this simple model, $\tilde{r}_t$ is simultaneously equal to the loan rate, the benchmark rate and the policy rate. Hence, interpreting $\tilde{r}_t$ empirically is not straightforward. For this reason, we estimate (19) twice where first we use the 3-Month Treasury Bill secondary market series for $\tilde{r}_t$ and then we examine the robustness of our estimates by using the commercial and industrial loan rate for $\tilde{r}_t$. Table 2 reveals the point-estimates are surprisingly robust to both specifications.

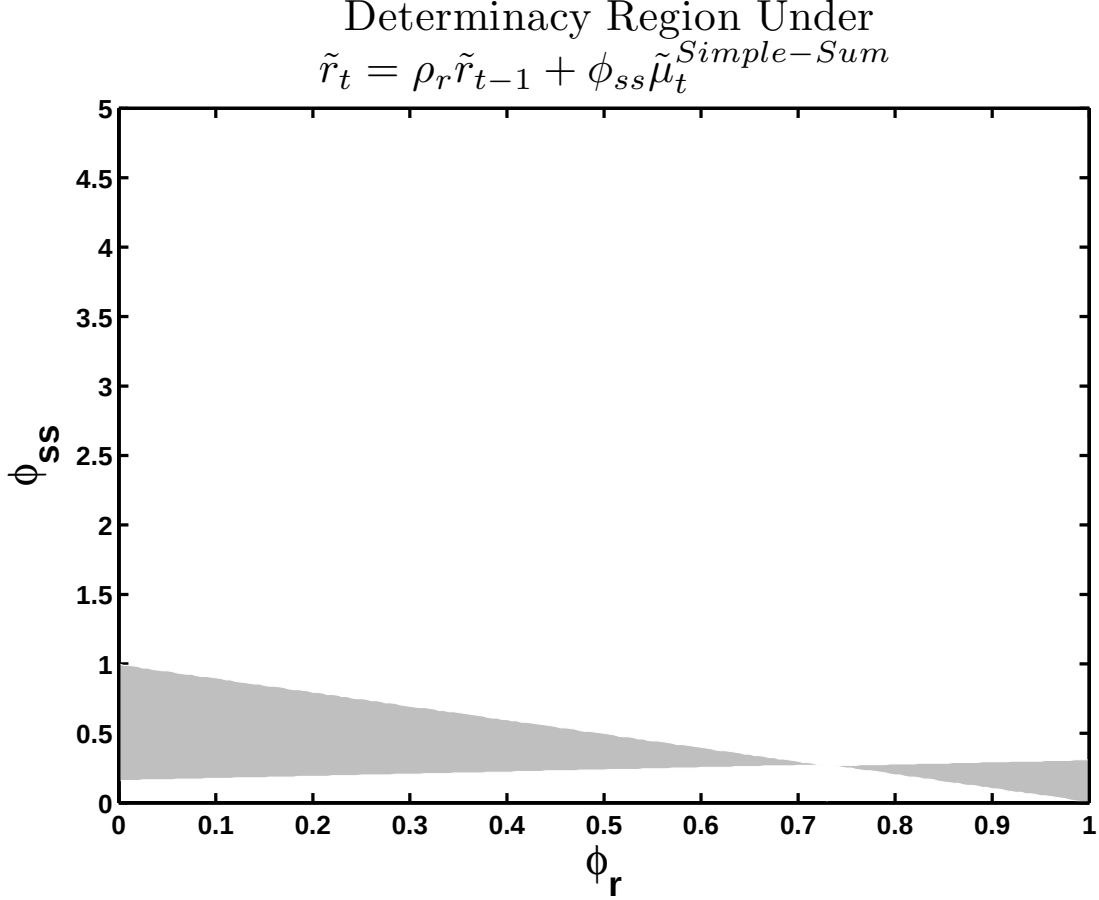


Figure 1: The shaded areas are determinate while the white area is indeterminate. The determinacy region above is graphed for $\eta_r = 1.9$ with the other parameters fixed at the values presented in Table 3.

The results in Table 2 support the theoretical findings in lemma 2 as the estimated coefficients all have the predicted sign. Moreover, the key parameter for the determinacy of the constant Simple-Sum growth rule is estimated to be significantly larger than the interest semi-elasticity estimates of Ball (2001) and Ireland (2009) over both the full-sample and the sub-sample⁴. To give further credence to the above estimation, the value of $\hat{\beta}_1$ over the 1984:01 - 2007:09 period is nearly identical to the value of $\psi_r(\omega)$ obtained from the calibration put forth by Belongia and Ireland (2012a).

Hence, we calibrate values as follows for the determinacy analysis that follows. It is immediately clear that over the range of values in Table 3 the necessary and sufficient condition for the determinacy of the constant simple-sum growth rule in corollary 2 is not met. Moreover, Figure 1 above plots the determinacy region under the interest rate rule reacting to the lagged interest rate and the growth rate of simple-sum and shows the determinacy region comprises a small percentage of the parameter space. However, more realistic policy rules

⁴The full sample estimates include heterogeneous monetary regimes and the financial crisis. For this reason, we estimate equation (19) over a sub-sample which roughly captures the *Great Moderation* over which the Fed's policy can be characterized by a predictable interest rate rule with a low inflation target.

Table 2: Econometric Estimation of Simple-Sum Error Term ^a

$\ln(N_t + D_t) - \ln(M_t^A) = \beta_0 + \beta_1 \tilde{r}_t + \beta_2 \tilde{x}_t + \beta_3 \tilde{\pi}_t + \varepsilon_t$				
1967:01-2012:09	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
<i>Measuring $\tilde{r}_t$ with the 3-Month T-Bill Rate</i>				
OLS Estimates	-0.2025	11.0377	-0.3444	-0.0509
Standard-Error	0.0164	2.7619	0.0492	0.0164
<i>Measuring $\tilde{r}_t$ with the C & I Loan Rate</i>				
OLS Estimates	-0.1518	9.8770	-0.3669	-0.0659
Standard-Error	0.0128	2.5236	0.0560	0.0153
1984:01-2007:09	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
<i>Measuring $\tilde{r}_t$ with the 3-Month T-Bill Rate</i>				
OLS Estimates	-0.0462	5.8694	-0.0589	-0.0289
Standard-Error	0.0019	0.6701	0.0071	0.0079
<i>Measuring $\tilde{r}_t$ with the C & I Loan Rate</i>				
OLS Estimates	-0.0462	5.6362	-0.0669	-0.0170
Standard-Error	0.0019	0.6427	0.0074	0.0071

^a The computed standard errors are Newey-West HAC using $T^{\frac{1}{4}}$ lags.

focus on inflation as well. A question of more practical interest may be: How does including a response to the growth rate of money affect the well known determinacy properties of interest rate rules reacting to lagged the interest rate and inflation? Concretely, we consider rules such as

$$\tilde{r} = \phi_r \tilde{r}_{t-1} + \phi_m \Delta \tilde{m}_t^A + \phi_\pi \tilde{\pi}_t \quad (20)$$

where we will once again focus on implementable rules which replace the growth rate of the true monetary aggregate with either the growth rate of the Divisia monetary aggregate or the growth rate of simple-sum. The main results of this section further emphasize the importance of using the Divisia monetary aggregate in policy making as the determinacy regions are large and invariant to non-policy parameters such as the interest semi-elasticity of money demand. Importantly, including a response to the growth rate of the Divisia monetary aggregate doesn't destabilize the economy by inducing indeterminacy in an already determinate rule. The same can not be said when the simple-sum aggregate is used instead.

Interest Rate Rules Reacting to the Lagged Interest Rate, Divisia and Inflation

First consider rules which replace the the true aggregate in (20) with the Divisia monetary aggregate⁵. We find the results in corollary 3 above extend to rule reacting to inflation as

⁵Of course, Lemma 1 implies the determinacy properties are equivalent under both the true aggregate and the Divisia monetary aggregate.

Table 3: Calibrated Parameters

<i>Parameter</i>	<i>Calibrated Value</i>
β	.99
κ	.30
$\psi_r(\omega)$	5.6
η_r	[0.05, 1.9]
ϕ_r	[0, 1]
ϕ_d	[0, 5]
ϕ_{ss}	[0, 5]
ϕ_π	[0, 5]

well. By adjusting the nominal rate more than 1 for 1 to changes in the growth rate of the Divisia monetary aggregate and inflation in the long run, the central bank can bring about a unique REE as figure 2 concisely shows. We summarize this in the result below.

Conjecture 1. *For any $\phi_r > 0$, $\phi_d > 0$ and $\phi_\pi > 0$, if the central bank follows the policy rule $\tilde{r}_t = \phi_r \tilde{r}_{t-1} + \phi_d \tilde{\mu}_t^{Divisia} + \phi_\pi \tilde{\pi}_t$ then there exists a unique REE if and only if $\phi_r + \phi_d + \phi_\pi > 1$.*

Figure 2: The shaded areas are determinate while the white area is indeterminate. The determinacy region above is found to hold for all values of $0 < \beta < 1$, $\kappa > 0$ and $\eta_r \geq 0$.

We present this as a conjecture since we are only able to verify this statement numerically. One interpretation of this result is that including a response to the growth rate of the Divisia monetary aggregate in a policy rule reacting to inflation and lagged interest rates will not disrupt the determinacy properties of the original rule. For example, in Keating and Smith (2013) we show that adding a response to the growth rate of the Divisia monetary aggregate to an interest rate rule reacting only to inflation is welfare enhancing as it provides information regarding the unobservable natural rate of interest. Here we stress that such a change in policy will not destabilize the economy since we can ensure the existence of a unique REE under this rule.

Interest Rate Rules Reacting to the Lagged Interest Rate, Simple-Sum and Inflation

Now we analyze determinacy properties of nominal interest rate rules reacting to inflation as well as the growth rate of simple-sum and lagged interest rates. Conventional wisdom suggests that reacting aggressively to inflation will stabilize the economy by ensuring a unique REE - the so called Taylor principle. Although this is true in this model, we find that augmenting an interest rate rule which satisfies the Taylor principle with a positive response to the growth rate of simple-sum may destabilize the economy by inducing indeterminacy. Simply put, reacting to the growth rate of simple-sum is likely to be destabilizing regardless of the inflation or lagged interest rate response. Figure 3 displays this by plotting the

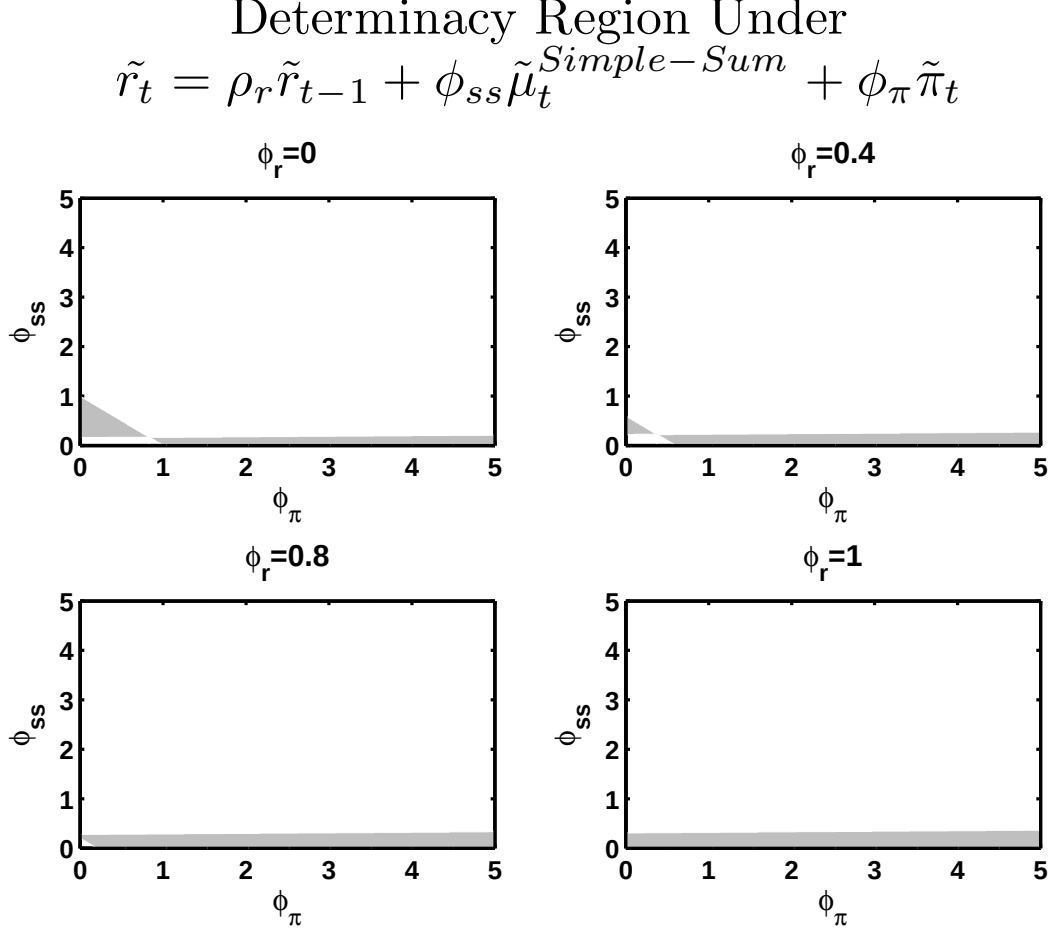


Figure 3: The shaded areas are determinate while the white area is indeterminate. The determinacy region above are evaluated for $\psi_r(\omega) = 5.6$.

determinacy region for the policy rule

$$\tilde{r} = \phi_r \tilde{r}_{t-1} + \phi_{ss} \tilde{\mu}_t^{Simple-Sum} + \phi_\pi \tilde{\pi}_t. \quad (21)$$

Comparing figures 2 and 3 highlights the difference between placing the growth rate of the Divisia aggregate and the simple-sum aggregate in an interest rate reaction function. Most noticeably, when the Divisia aggregate is embedded in the interest rate rule the determinacy regions are large and independent of non-policy parameters such as the interest semi-elasticity of money demand. On the other hand, when the simple-sum aggregate is placed in the interest rate reaction function the economy is more likely to be indeterminate than determinate. Moreover, the determinacy regions depend critically on non-policy parameters including the interest semi-elasticity of money demand and the portion of simple-sum's error related to the policy rate - $\psi_r(\omega)$. To emphasize this point, Figure 4 plots the determinacy regions under the policy defined by (21) for $\psi_r(\omega) = 3$ - nearly three standard deviations below the estimated value.

Figure 4 highlights how sensitive the determinacy regions are to this parameter. Very similar results are found if we analyze determinacy under a much larger value of $\eta_r = 3.8$,

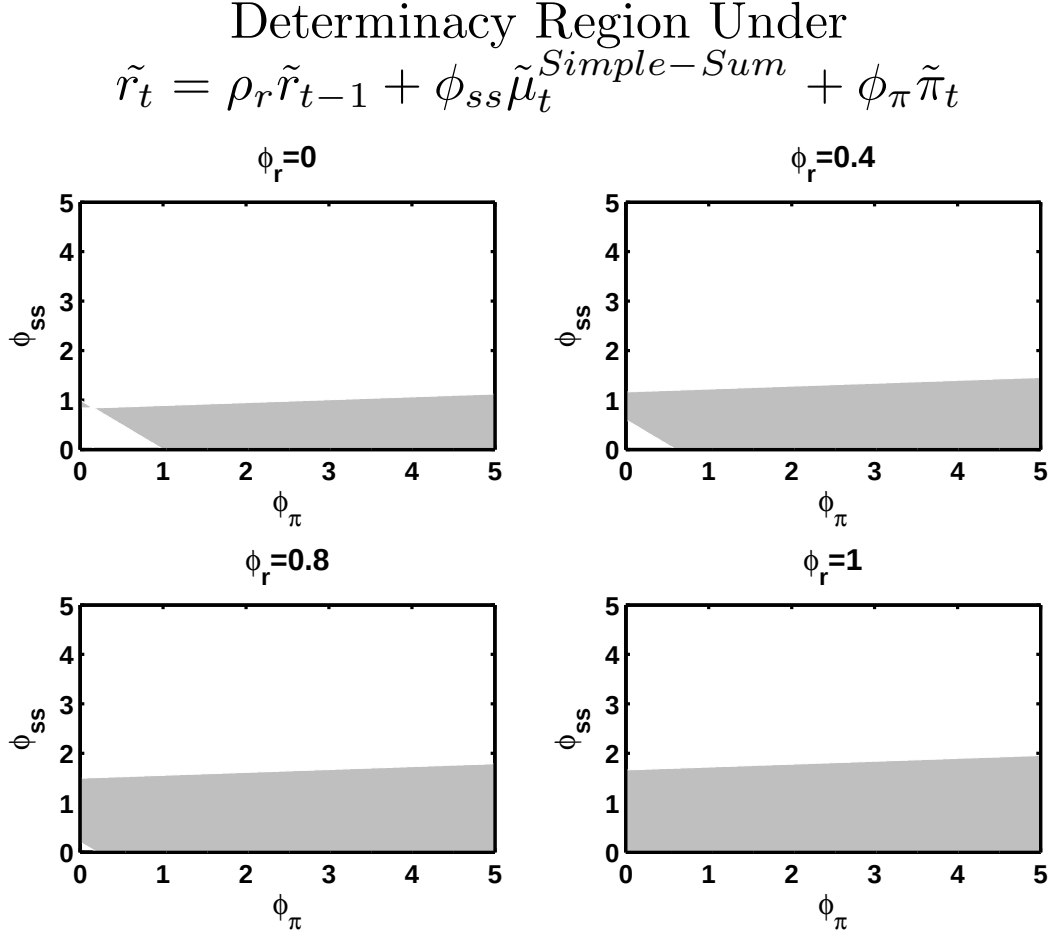


Figure 4: The shaded areas are determinate while the white area is indeterminate. The determinacy region above are evaluated for $\psi_r(\omega) = 3$ - nearly three standard deviations below the estimated value.

twice the estimated value by Ireland (2009). This suggests what matters most for the determinacy properties of simple-sum oriented policy rules are not the policy parameters, but size of the error term $\psi_r(\omega)$ relative to the interest semi-elasticity η_r .

6 Conclusion

The ability to ensure a unique rational expectations equilibrium is a fundamental question of any policy rule. We show that Friedman's k-percent rule is determinate so long as the monetary aggregate is measured accurately. Ironically if the preferred aggregate of Friedman, a broad simple-sum aggregate, is used in his k-percent rule the economy is likely to be unstable due to self-fulfilling expectations. This problem can be remedied in practice by using the non-parametric Divisia monetary aggregate in the k-percent rule. We conclude that Friedman was correct in general about the stabilizing properties of constant monetary aggregate growth rules, but care must be taken when measuring the monetary aggregate. To be fair to Friedman, the development of Divisia monetary aggregates was nascent during much of his work. Interestingly, Friedman and Schwartz were aware of the flaws in the

simple-sum aggregation method (1971, pp. 151-152):

This [simple summation] procedure is a very special case of the more general approach. In brief, the general approach consists of regarding each asset as a joint product having different degrees of 'moneyness,' and defining the quantity of money as the weighted sum of the aggregated value of all assets, the weights for individual assets varying from zero to unity with a weight of unity assigned to that asset or assets regarded as having the largest quantity of 'moneyness' per dollar of aggregate value. The procedure we have followed implies that all weights are either zero or unity. The more general approach has been suggested frequently but experimented with only occasionally. We conjecture that this approach deserves and will get much more attention than it has so far received.

Although much of modern monetary policy emphasis interest rates, the systematic nature of all policy rules have their roots in Friedman's k-percent rule. This paper presents a formal analysis of the ability of this simple rule to bring about a unique rational expectations equilibrium. The results presented stress that, "measurement matters" (Belongia, 1996) when it comes to the determinacy of monetary aggregate growth rules. Placed in the larger literature stressing the differences between Divisia and simple-sum monetary aggregates,⁶ the case against the use of monetary aggregates in policy making may be brought into question. For example, Barnett and Chauvet (2011b) highlight how the growth rate of simple-sum and Divisia monetary aggregates differed during the monetarist's experiment. When combined with the theoretical results in this paper, a natural question arises: Had accurate measures of broad money been used (such as the Divisia aggregates) during the monetarist's experiment, would the use of monetary aggregates fallen from favor among central bankers?

⁶For more general research examining the Divisia monetary aggregate's properties relative to alternative simple-sum measures see the following works. At paper length Barnett and Chauvet (2011b); Belongia (1996) and at book length Barnett and Singleton (1987); Belongia and Binner (2000); Barnett and Serletis (2000); Barnett and Chauvet (2011a); Barnett (2012).

References

- BALL, L. (2001): “Another look at long-run money demand,” *Journal of Monetary Economics*, 47(1), 31–44.
- BARNETT, W. A. (2012): *Getting it Wrong: How Faulty Monetary Statistics Undermine the Fed, the Financial System, and the Economy*, vol. 1 of *MIT Press Books*. The MIT Press.
- BARNETT, W. A., AND M. CHAUVET (2011a): *Financial Aggregation And Index Number Theory*, vol. 2 of *World Scientific Books*. World Scientific Publishing Co. Pte. Ltd.
- (2011b): “How better monetary statistics could have signaled the financial crisis,” *Journal of Econometrics*, 161(1), 6–23.
- BARNETT, W. A., AND A. SERLETIS (2000): *The Theory of Monetary Aggregation*, Contributions to Economic Analysis. Elsevier.
- BARNETT, W. A., AND K. J. SINGLETON (1987): *The microeconomic theory of monetary aggregation*. Cambridge University Press.
- BELONGIA, M. T. (1996): “Measurement Matters: Recent Results from Monetary Economics Reexamined,” *Journal of Political Economy*, 104(5), 1065–83.
- BELONGIA, M. T., AND J. M. BINNER (2000): *Divisia Monetary Aggregates: Theory and Practice*. Palgrave.
- BELONGIA, M. T., AND P. N. IRELAND (2012a): “The Barnett Critique After Three Decades: A New Keynesian Analysis,” NBER Working Papers 17885, National Bureau of Economic Research, Inc.
- (2012b): “Quantitative Easing: Interest Rates and Money in the Measurement of Monetary Policy,” Boston College Working Papers in Economics 801, Boston College Department of Economics.
- CANOVA, F., AND T. MENZ (2011): “Does Money Matter in Shaping Domestic Business Cycles? An International Investigation,” *Journal of Money, Credit and Banking*, 43(4), 577–607.
- DIEWERT, W. E. (1976): “Exact and superlative index numbers,” *Journal of Econometrics*, 4(2), 115–145.
- DIVISIA, F. (1926): “L’indice monétaire et la théorie de la monnaie,” *Revue d’écon polit*, LX(01), 49–81.
- EVANS, G. W., AND S. HONKAPOHJA (2003): “Friedman’s Money Supply Rule vs. Optimal Interest Rate Policy,” *Scottish Journal of Political Economy*, 50(5), 550–566.
- FRIEDMAN, M. (1960): *A Program for Monetary Stability*. Fordham University Press.

- FRIEDMAN, M., AND A. J. SCHWARTZ (1971): *A Monetary History of the United States, 1867-1960*. Princeton University Press.
- GALI, J. (2008): *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework*. Princeton University Press.
- IRELAND, P. N. (2009): “On the Welfare Cost of Inflation and the Recent Behavior of Money Demand,” *American Economic Review*, 99(3), 1040–52.
- KEATING, J., AND A. L. SMITH (2013): “Price Versus Financial Stability: A role for money in Taylor rules?,” WORKING PAPERS SERIES IN THEORETICAL AND APPLIED ECONOMICS 201307, University of Kansas, Department of Economics.
- LEEPER, E. M. (1991): “Equilibria under ‘active’ and ‘passive’ monetary and fiscal policies,” *Journal of Monetary Economics*, 27(1), 129–147.
- MCCALLUM, B. T. (1998): “Solutions to linear rational expectations models: a compact exposition,” *Economics Letters*, 61(2), 143–147.
- MCCALLUM, B. T., AND E. NELSON (2011): “Money and Inflation: Some Critical Issues,” in *Handbook of Monetary Economics*, ed. by B. M. Friedman, and M. Woodford, vol. 3 of *Handbook of Monetary Economics*, chap. 3, pp. 97–153. Elsevier.
- SIMS, C. A., AND T. ZHA (2006): “Were There Regime Switches in U.S. Monetary Policy?,” *American Economic Review*, 96(1), 54–81.
- WOODFORD, M. (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*. Princeton University Press.

A Proofs

In this section we present the proofs to our results stated in the paper. We rely on the micro-founded model described in Belongia and Ireland (2012a), however we assume additively separable preferences between consumption and monetary services allowing us to obtain analytic results for the determinacy regions. Also, to arrive at the standard dynamic IS and NKPC, we slightly alter the timing of the transactions for the representative household.

Proposition 1

Consider the dynamic system defined by equations (1), (2), (8) and (9),

$$AE_t[\tilde{w}_{t+1}] = B\tilde{w}_t + Cs_t.$$

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \eta_r & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \kappa & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where $\tilde{w}_t = \left[\tilde{c}_t, \tilde{\pi}_t, \tilde{r}_{t-1}, \left(\frac{\tilde{m}^A}{p} \right)_{t-1} \right]^T$ so that the system has two non-predetermined variables and two predetermined variables. Therefore, the system will have a unique rational expectations equilibrium (REE) if, and only if, two roots λ which satisfy $|B - \lambda A| = 0$ lie outside the unit circle and remaining two roots lie inside the unit circle McCallum (1998). The generalized eigenvalues of the matrix pencil $B - \lambda A$ are given by

$$\Lambda = \begin{bmatrix} 0 \\ \frac{1}{2} \frac{\beta+1+\kappa - \sqrt{(\beta-1)^2 + 2\beta\kappa + 2\kappa + \kappa^2}}{\beta} \\ \frac{1+\eta_r}{\eta_r} \\ \frac{1}{2} \frac{\beta+1+\kappa + \sqrt{(\beta-1)^2 + 2\beta\kappa + 2\kappa + \kappa^2}}{\beta} \end{bmatrix}. \quad (22)$$

The proof to proposition 1 follows from (A.1) since $|\lambda_i| < 1$ for $i \in \{1, 2\}$, $|\lambda_4| > 1$ and finally $|\lambda_3| > 1$ since $\eta_r \geq 0$ by assumption.

Lemma 1

For this result, first notice the demand for currency and deposits respectively are given by

$$N_t = \nu M_t^A \left(\frac{u_t^A}{u_t^N} \right)^\omega \quad (23)$$

and

$$D_t = (1 - \nu) M_t^A \left(\frac{u_t^A}{u_t^D} \right)^\omega. \quad (24)$$

From Definition 1, we have the growth rate of the Divisia monetary aggregate is given by

$$\ln(\mu_t^{Divisia}) = \left(\frac{s_t^N + s_{t-1}^N}{2} \right) \ln \left(\frac{N_t}{N_{t-1}} \right) + \left(\frac{s_t^D + s_{t-1}^D}{2} \right) \ln \left(\frac{D_t}{D_{t-1}} \right).$$

Using Fisher's factor reversal (which the true price and quantity duals satisfy by definition) we have that

$$M_t^A u_t^A = N_t u_t^N + D_t u_t^D$$

Therefore we have the difference between the growth rate of the Divisia aggregate and the true monetary aggregate is given by

$$\begin{aligned} & \ln(\mu_t^{Divisia}) - \Delta \ln(M_t^A) \\ &= \omega \left[\Delta \ln(u_t^A(u_t^N, u_t^D)) - \frac{1}{2} (s_t^N + s_{t-1}^N) \Delta \ln(u_t^N) - \frac{1}{2} (s_t^D + s_{t-1}^D) \Delta \ln(u_t^D) \right] \\ &\equiv E^D(\ln(u_t^N), \ln(u_t^D), \ln(u_{t-1}^N), \ln(u_{t-1}^D)). \end{aligned}$$

To verify the claim, take a first order approximation of the right-hand side around the steady state where variables with an over-bar denote steady state values.

$$\begin{aligned} & \ln(\mu_t^{Divisia}) - \Delta \ln(M_t^A) \\ &= \left[\frac{\partial E}{\partial \ln(u_t^N)} \right]_{ss} \tilde{u}_t^N + \left[\frac{\partial E}{\partial \ln(u_t^D)} \right]_{ss} \tilde{u}_t^D \\ &+ \left[\frac{\partial E}{\partial \ln(u_{t-1}^N)} \right]_{ss} \tilde{u}_{t-1}^N + \left[\frac{\partial E}{\partial \ln(u_{t-1}^D)} \right]_{ss} \tilde{u}_{t-1}^D + \mathcal{O}(\|\xi\|^2) \\ &= \left[\frac{\partial E}{\partial \ln(u_t^N)} \right]_{ss} \tilde{u}_t^N + \left[\frac{\partial E}{\partial \ln(u_t^D)} \right]_{ss} \tilde{u}_t^D \\ &- \left[\frac{\partial E}{\partial \ln(u_t^N)} \right]_{ss} \tilde{u}_{t-1}^N - \left[\frac{\partial E}{\partial \ln(u_t^D)} \right]_{ss} \tilde{u}_{t-1}^D + \mathcal{O}(\|\xi\|^2) \end{aligned}$$

Finally, notice that

$$\begin{aligned} \left[\frac{\partial E}{\partial \ln(u_t^N)} \right]_{ss} &= \left[\frac{\partial \ln(u_t^A)}{\partial \ln(u_t^N)} - \frac{1}{2} (s_t^N + s_{t-1}^N) \right]_{ss} \\ &= \frac{\nu (\bar{u}^N)^{1-\omega}}{\nu (\bar{u}^N)^{1-\omega} + (1-\nu) (\bar{u}^D)^{1-\omega}} - \frac{\nu (\bar{u}^N)^{1-\omega}}{(\bar{u}^A)^{1-\omega}} \\ &= 0 \end{aligned}$$

and

$$\begin{aligned}
\left[\frac{\partial E}{\partial \ln(u_t^D)} \right]_{ss} &= \left[\frac{\partial \ln(u_t^A)}{\partial \ln(u_t^D)} - \frac{1}{2}(s_t^D + s_{t-1}^D) \right]_{ss} \\
&= \frac{(1-\nu)(\bar{u}^D)^{1-\omega}}{\nu(\bar{u}^N)^{1-\omega} + (1-\nu)(\bar{u}^D)^{1-\omega}} - \frac{(1-\nu)(\bar{u}^D)^{1-\omega}}{(\bar{u}^A)^{1-\omega}} \\
&= 0
\end{aligned}$$

which verifies our claim since we have $\ln(\mu_t^{Divisia}) - \Delta \ln(M_t^A) = \mathcal{O}(\|\xi\|^2)$.

Lemma 2

Much like the proof of lemma 1, substitute the factor demands for currency and deposits defined in equations (23) and (24) into the definition of the Simple-Sum aggregate in definition 2 to arrive at

$$\begin{aligned}
&\ln(\mu_t^{Simple-Sum}) - \Delta \ln(M_t^A) \\
&= \omega \Delta \ln(u_t^A(u_t^N, u_t^D)) + \Delta \ln(\nu(u_t^N)^{-\omega} + (1-\nu)(u_t^D)^{-\omega}) \\
&\equiv E^{ss}(\ln(u_t^N), \ln(u_t^D), \ln(u_{t-1}^N), \ln(u_{t-1}^D)).
\end{aligned}$$

Now take a first-order Taylor expansion of the right hand side around the non-stochastic steady state.

$$\begin{aligned}
&E^{ss}(\ln(u_t^N), \ln(u_t^D), \ln(u_{t-1}^N), \ln(u_{t-1}^D)) \\
&= \omega \left[s^N - \frac{\nu(\bar{u}^N)^{-\omega}}{\nu(\bar{u}^N)^{-\omega} + (1-\nu)(\bar{u}^D)^{-\omega}} \right] \Delta \tilde{u}_t^N \\
&+ \omega \left[s^D - \frac{(1-\nu)(\bar{u}^D)^{-\omega}}{\nu(\bar{u}^N)^{-\omega} + (1-\nu)(\bar{u}^D)^{-\omega}} \right] \Delta \tilde{u}_t^D + \mathcal{O}(\|\xi\|^2) \\
&= \omega \left[\frac{\nu(\bar{u}^N)^{1-\omega}}{\nu(\bar{u}^N)^{1-\omega} + (1-\nu)(\bar{u}^D)^{1-\omega}} - \frac{\nu(\bar{u}^N)^{-\omega}}{\nu(\bar{u}^N)^{-\omega} + (1-\nu)(\bar{u}^D)^{-\omega}} \right] \Delta \tilde{u}_t^N \\
&+ \omega \left[\frac{(1-\nu)(\bar{u}^D)^{1-\omega}}{\nu(\bar{u}^N)^{1-\omega} + (1-\nu)(\bar{u}^D)^{1-\omega}} - \frac{(1-\nu)(\bar{u}^D)^{-\omega}}{\nu(\bar{u}^N)^{-\omega} + (1-\nu)(\bar{u}^D)^{-\omega}} \right] \Delta \tilde{u}_t^D + \mathcal{O}(\|\xi\|^2)
\end{aligned}$$

Let $\alpha = \bar{u}^D / \bar{u}^N \in (0, 1)$, then we have

$$\begin{aligned}
& E^{ss}(\ln(u_t^N), \ln(u_t^D), \ln(u_{t-1}^N), \ln(u_{t-1}^D)) \\
&= \omega \left[\frac{\nu(\alpha)^{\omega-1}}{\nu(\alpha)^{\omega-1} + (1-\nu)} - \frac{\nu(\alpha)^\omega}{\nu(\alpha)^\omega + (1-\nu)} \right] \Delta \tilde{u}_t^N \\
&+ \omega \left[\frac{(1-\nu)}{\nu(\alpha)^{\omega-1} + (1-\nu)} - \frac{(1-\nu)}{\nu(\alpha)^\omega + (1-\nu)} \right] \Delta \tilde{u}_t^D + \mathcal{O}(\|\xi\|^2) \\
&= \omega \left[\frac{\nu(\alpha)^{\omega-1}(\nu(\alpha)^\omega + (1-\nu)) - \nu(\alpha)^\omega(\nu(\alpha)^{\omega-1} + (1-\nu))}{(\nu(\alpha)^\omega + (1-\nu))(\nu(\alpha)^{\omega-1} + (1-\nu))} \right] \Delta \tilde{u}_t^N \\
&+ \omega \left[\frac{(1-\nu)(\nu(\alpha)^\omega + (1-\nu)) - (1-\nu)(\nu(\alpha)^{\omega-1} + (1-\nu))}{(\nu(\alpha)^\omega + (1-\nu))(\nu(\alpha)^{\omega-1} + (1-\nu))} \right] \Delta \tilde{u}_t^D + \mathcal{O}(\|\xi\|^2) \\
&= \omega \left[\frac{\alpha^\omega [(\alpha)^{-1}\nu(1-\nu) - \nu(1-\nu)]}{(\nu(\alpha)^\omega + (1-\nu))(\nu(\alpha)^{\omega-1} + (1-\nu))} \right] \left[\frac{r}{u^N} - \left(\frac{r\tau}{u^D} \right) \right] \Delta \tilde{r}_t \\
&- \omega \left[\frac{\alpha^\omega [(\alpha)^{-1}\nu(1-\nu) - \nu(1-\nu)]}{(\nu(\alpha)^\omega + (1-\nu))(\nu(\alpha)^{\omega-1} + (1-\nu))} \right] \left[\left(\frac{1}{\alpha} \right) \Delta \tilde{r}_t + \left(\frac{x}{u^D} \right) \Delta \tilde{x}_t \right] + \mathcal{O}(\|\xi\|^2) \\
&\equiv \psi(\omega) \left[\frac{ru^D - r\tau u^N}{u^N u^D} \right] \Delta \tilde{r}_t - \psi(\omega) \left(\frac{1}{\alpha} \right) \Delta \tilde{r}_t - \psi(\omega) \left(\frac{x}{u^D} \right) \Delta \tilde{x}_t + \mathcal{O}(\|\xi\|^2) \\
&\equiv \psi_r(\omega) \Delta \tilde{r}_t - \psi_\tau(\omega) \Delta \tilde{r}_t - \psi_x(\omega) \Delta \tilde{x}_t + \mathcal{O}(\|\xi\|^2).
\end{aligned}$$

To complete the proof, first notice that since $(\tau, x) > 0$ we have that $\psi_r(\omega) > 0$ and moreover since $\psi(\omega) > 0$ we have that $(\psi_r(\omega), \psi_x(\omega)) \gg (0, 0)$. Finally, to obtain the limiting result apply L'Hospital's rule to the function

$$h(\omega) = \omega e^{\omega \ln(\alpha)} = \frac{\omega}{e^{-\omega \ln(\alpha)}}$$

to find that $\lim_{\omega \rightarrow \infty} h(\omega) = 0$ which implies $\lim_{\omega \rightarrow \infty} \psi(\omega) = 0$ verifying the claim.

Corollary 2

By Lemma 2, the constant Simple-Sum growth rule implies a policy rule of the form

$$0 = \Delta \tilde{\mu}_t^{\text{Simple-Sum}} = \Delta \tilde{m}_t^A + \psi_r(\omega) \Delta \tilde{r}_t - \psi_\tau(\omega) \Delta \tilde{r}_t - \psi_x(\omega) \Delta \tilde{x}_t.$$

We can rearrange this expression as an interest rate feedback rule.

$$\tilde{r}_t = \tilde{r}_{t-1} - \frac{1}{\psi_r(\omega)} \Delta \tilde{m}_t^A - \frac{\psi_\tau(\omega)}{\psi_r(\omega)} \Delta \tilde{r}_t - \frac{\psi_x(\omega)}{\psi_r(\omega)} \Delta \tilde{x}_t \quad (25)$$

This rule is part of a more general class of rules studied in section 4. More generally, consider rules of the form

$$\tilde{r}_t = \phi_r \tilde{r}_{t-1} + \phi_m \Delta \tilde{m}_t^A$$

which is equation (16) where we drop the stochastic terms from (25) since they have no impact on the determinacy properties. The dynamic system consisting of (1), (2), (8) and (25) can be expressed as

$$AE_t[\tilde{w}_{t+1}] = B\tilde{w}_t + Cs_t.$$

$$A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -\beta & 0 & 0 \\ 0 & 0 & 1 & -\phi_m \\ 0 & 0 & \eta_r & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \kappa & -1 & 0 & 0 \\ 0 & \phi_m & \phi_r & -\phi_m \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

where $\tilde{w}_t = \left[\tilde{c}_t, \tilde{\pi}_t, \tilde{r}_{t-1}, \left(\frac{m}{p} A \right)_{t-1} \right]^T$ so that the system has two non-predetermined variables and two predetermined variables. Therefore, the system will have a unique rational expectations equilibrium (REE) if, and only if, two roots λ which satisfy $|B - \lambda A| = 0$ lie outside the unit circle and remaining two roots lie inside the unit circle McCallum (1998). The generalized eigenvalues of the matrix pencil $B - \lambda A$ are given by

$$\Lambda = \begin{bmatrix} 0 \\ \frac{\frac{1}{2}(\beta+1+\kappa-\sqrt{(\beta-1)^2+2\beta\kappa+2\kappa+\kappa^2})}{\beta} \\ \frac{\phi_r+\phi_m+\eta_r\phi_m}{1+\eta_r\phi_m} \\ \frac{\frac{1}{2}(\beta+1+\kappa+\sqrt{(\beta-1)^2+2\beta\kappa+2\kappa+\kappa^2})}{\beta} \end{bmatrix}. \quad (26)$$

The proof for corollary 2 follows by letting $\phi_r = 1$ and $\phi_m = -\frac{1}{\psi_r(\omega)}$. In which case we have that $|\lambda_i| < 1$ for $i \in \{1, 2\}$, $|\lambda_4| > 1$ and finally $|\lambda_3| > 1$ if and only if

$$\left| \frac{\psi_r(\omega) - \eta_r - 1}{\psi_r(\omega) - \eta_r} \right| > 1.$$

This condition will be satisfied if and only if $\psi_r(\omega) < \eta_r + \frac{1}{2}$ allowing for a possibly infinite generalized eigenvalue for $\psi_r(\omega) = \eta_r$.

Proposition 2

This result follows from examining the eigenvalues presented in (26) above which considers the dynamic model consisting of (1), (2), (8) and (16). We have that $|\lambda_i| < 1$ for $i \in \{1, 2\}$, $|\lambda_4| > 1$ and finally $|\lambda_3| > 1$ if and only if

$$\left| \frac{\phi_r + \phi_m + \eta_r\phi_m}{1 + \eta_r\phi_m} \right| > 1.$$

Since we are assuming all terms are non-negative this is equivalent to having that $|\lambda_3| > 1$ if and only if $\phi_r + \phi_m > 1$.

Corollary 4

Applying lemma 2 to the interest rate rule in (18) we have the policy rule

$$\tilde{r}_t = \phi_r \tilde{r}_{t-1} + \phi_{ss} [\Delta \tilde{m}_t^A + \psi_r(\omega) \Delta \tilde{r}_t - \psi_\tau(\omega) \Delta \tilde{\tau}_t - \psi_x(\omega) \Delta \tilde{x}_t].$$

Rearranging this expression this expression we have a policy rule of the form

$$\tilde{r}_t = \left[\frac{\phi_r - \phi_{ss} \psi_r(\omega)}{1 - \phi_{ss} \psi_r(\omega)} \right] \tilde{r}_{t-1} + \left[\frac{\phi_{ss}}{1 - \phi_{ss} \psi_r(\omega)} \right] \Delta \tilde{m}_t^A \quad (27)$$

where we drop the exogenous shock terms since they have no affect on the determinacy properties of the policy rule. Since the rule in (27) is of the same general form as (16), we can analyze the eigenvalues in (26) above. We then have that $|\lambda_i| < 1$ for $i \in \{1, 2\}$, $|\lambda_4| > 1$ and finally $|\lambda_3| > 1$ if and only if

$$\left| \frac{\phi_r + \phi_{ss} - \phi_{ss} [\psi_r(\omega) - \eta_r]}{1 - \phi_{ss} [\psi_r(\omega) - \eta_r]} \right| > 1$$

or equivalently

$$|\phi_r + \phi_{ss} - \phi_{ss} [\psi_r(\omega) - \eta_r]| > |1 - \phi_{ss} [\psi_r(\omega) - \eta_r]|.$$