Directional monotone comparative statics in function spaces

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Abstract.

We characterize x-directional monotone comparative statics in function spaces, generalizing the results in Barthel and Sabarwal (2018) for finite-dimensional Euclidean spaces, and showing that their proofs generalize to infinite-dimensional spaces. Our generalizations include those for x-directional set order, x-quasisupermodularity, x-single crossing property, and x-basic single crossing property. Differential characterizations are generalized as well. This expands the scope of directional monotone comparative statics to infinite dimensional spaces. We include new examples beyond the scope of earlier results, such as neoclassical growth model, infinite-dimensional consumer theory, and infinite-dimensional directional Le Chatelier principle.

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1 Introduction

Consider the general constrained optimization problem: X is the set of choice variables, T is the set of exogenous parameters, $E \subseteq X$ is the constraint set,

$f : X \times T \to \mathbb{R}$ is the objective function, and the problem is to maximize f over E, that is, $\max_{x \in E} f(x,t)$. The solution set to the problem is

$$M(E,t) := \underset{x \in E}{\operatorname{argmax}} f(x,t) = \{x \in E : f(x,t) \geq f(\hat{x},t)\ \forall \hat{x} \in E\}.$$

The solution set gives rise to the optimal action correspondence, $M : \mathcal{P}(X) \times T \rightrightarrows X$, mapping $(E,t) \mapsto M(E,t)$. A central problem in comparative statics is to understand how M changes with E and t. This is applicable in many models used in economics, game theory, optimization, mathematical control, engineering, business, and other disciplines.

A particularly important case is that of monotone comparative statics. Suppose E, E' are subsets of X ordered by some relation, $E \sqsubseteq E'$, and T is a partially ordered set with relation $\geq$. When is it true that $E \sqsubseteq E'$ and $t' \geq t \Rightarrow \arg\max_E f(\cdot,t) \sqsubseteq \arg\max_{E'} f(\cdot,t')$? Intuitively, when is $\arg\max_E f(\cdot,t)$ weakly increasing in (E,t), or equivalently, when is M weakly increasing in (E,t)?

There is a large literature studying monotone comparative statics using a lattice-based approach. A partial list includes Topkis (1978), Topkis (1979), Bulow et al. (1985), Milgrom and Roberts (1990), Vives (1990), Milgrom and Shannon (1994), Amir (1996a), Amir (1996b), Echenique (2002), Echenique and Sabarwal (2003), Roy and Sabarwal (2008), Roy and Sabarwal (2010), Roy and Sabarwal (2012), Jensen (2010), Acemoglu and Jensen (2015), Monaco and Sabarwal (2016), Jensen (2018), Sabarwal (2021), Balbus et al. (2022), Schlee and Khan (2022), and many more.

The lattice-based approach has several benefits over the classical implicit function approach, including action sets, parameter sets, and constraint sets that may be discrete, objective functions may be discontinuous, and the optimization problem may have multiple solutions. Minimal conditions (some form of semi-continuity over a compact set) are sufficient for existence of optimal solutions. In particular, actions sets and constraint sets are not required to be convex and objective functions are not required to be differentiable, continuous, or quasi-concave.

The lattice-based approach typically uses the strong set order to rank constraint sets in lattices. For nonempty subsets A, B of a lattice X with lattice operations $\wedge$ and $\vee$, A is lower than B in the strong set order, $A \sqsubseteq_s B$, if $\forall a \in A$, $\forall b \in B$, $a \wedge b \in A$ and $a \vee b \in B$. The strong set order is unable to rank some standard constraint sets used in economics, for example a budget set at a lower wealth level and one at a higher wealth level. Quah (2007) defined the $\mathcal{C}_i$ flexible set order as a solution to this problem and characterized monotone comparative statics in the case of parametrized constraint sets. Barthel and Sabarwal (2018) generalized this to the i-directional set order and also allow for parameterized objective functions. Both papers prove results for directional monotone comparative statics in finite-dimensional Euclidean spaces.

A central result in Barthel and Sabarwal (2018) is the following. For each canonical direction $i \in \{1, \ldots, N\}$ in $\mathbb{R}^N$, and using the i-directional set order, $\sqsubseteq_i^{dso}$, on subsets of E of $\mathbb{R}^N$, the solution to the constrained optimization problem, $M(E,t)$, is weakly increasing in (E,t) if, and only, for every $t \in T$, $f(\cdot, t)$ is i-quasisupermodular and satisfies i-single crossing property on its domain $X \subseteq \mathbb{R}^N$, which is a sublattice, and f satisfies basic i-single crossing property on $X \times T$, where T is a poset of exogenous parameters.

We generalize this characterization to subsets of function spaces of the form $\mathbb{R}^X = \{f \mid f : X \to \mathbb{R}\}$ where X is a partially ordered set. Partial order on X formalizes increase in a particular direction $x \in X$. $\mathbb{R}^X$ is equipped with the canonical pointwise partial order and lattice operations: For $f, g \in \mathbb{R}^X$, $f \geq g$, if $\forall x \in X, f(x) \geq g(x)$, $f \vee g$ is defined by $(f \vee g)(x) = \max\{f(x), g(x)\}$ and $f \wedge g$ is defined by $(f \wedge g)(x) = \min\{f(x), g(x)\}$. Our generalization expands the scope of directional monotone comparative statics to a wide class of constrained optimization problems in economics that use infinite dimensional spaces (either spatial or temporal). For example, (a) l_∞ (the space of all bounded real sequences) has been used to model inter-temporal allocation problems over an infinite horizon, (b) L_∞ (the space of all bounded measurable functions on a measure space) has been used to model trading or production under uncertainty, (c) M_X (the space of finite Borel measures on a compact metric space X) has been used to model differentiated commodities, (d) L_2 (the space of all square-integrable functions on a measure space) has been used to model financial instruments with uncertainty, and so on. More generally, our results apply to Archimedean Riesz spaces as every Archimedean Riesz space is lattice isomorphic to an appropriate function space.

Our main result is as follows. Suppose X is a partially ordered set (poset), $\mathbb{X}$ is a sub-lattice of $(\mathbb{R}^X, \geq)$, $(\mathbb{T}, \geq)$ is a poset of exogenous parameters, $\mathbb{X} \times \mathbb{T}$ is endowed with product partial order, and $\Phi : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ is an objective function. For a constraint set of functions $F \subseteq \mathbb{X}$ and a direction $x \in X$, $M(F, t) \equiv \operatorname{argmax}_F \Phi(\cdot, t)$ is weakly increasing in (F, t) in the x-directional set order if, and only if, $\forall t \in \mathbb{T}$, $\Phi(\cdot, t)$ is x-quasisupermodular and satisfies x-single crossing property on $\mathbb{X}$ and Φ satisfies basic x-single crossing property on $\mathbb{X} \times \mathbb{T}$. Our notions of x-directional set order, x-quasisupermodular, x-single crossing property, and basic x-single crossing property are natural generalizations of the corresponding notions in Barthel and Sabarwal (2018). We present differential characterizations as well. We include new examples beyond the scope of earlier results, such as neoclassical growth model, consumer theory, and infinite-dimensional directional Le Chatelier principle.

The paper proceeds as follows: Section 2 defines directional monotone comparative statics in function spaces and presents the main results along with the detailed proof. Section 3 provides a differentiable version using cardinal properties. Section 4 presents several economic applications.

2 Directional MCS in function spaces

In this section, we formalize the general setup for directional monotone comparative statics in function spaces. Consider the following setup: X is a poset, $(\mathbb{R}^X, \geq)$ is a lattice (in the product order), $\mathbb{X}$ is a sub-lattice of $(\mathbb{R}^X, \geq)$, and $\Phi : \mathbb{X} \to \mathbb{R}$ is a function. In order to compare sets that are (weakly) increasing in a particular direction $x \in X$, consider the following. Given two nonempty sets of functions $F, G \subseteq \mathbb{X}$, G **dominates** F **in the** x**-directional set order**, denoted $F \sqsubseteq_x^{dso} G$, if $\forall f \in F$ and $\forall g \in G$ with $f(x) > g(x)$, $\exists h = s[g - (f \wedge g)] \in \mathbb{X}$ for some $s \in [0,1]$ such that $f + h \in G$ and $g - h \in F$. Moreover, G **dominates** F **in the directional set order**, denoted $F \sqsubseteq^{dso} G$, if $\forall x \in \mathbb{X}$, $F \sqsubseteq_x^{dso} G$.

Intuitively, $F \sqsubseteq_x^{dso} G$ means that for every $f \in F$ and for every $g \in G$, either $f(x) \leq g(x)$ in which case there is a function in $g \in G$ that is weakly higher than f in the x-direction, or $f(x) > g(x)$, in which case there is a function $f + h \in G$ that is (weakly) higher than f in the x-direction. A similar statement holds for existence of a function in F that is weakly lower than g in direction x. The strong set order is stronger than the x-directional set order, as proved in the appendix. Moreover, in the special case when $X = \{1, \ldots, N\}$ with the standard order, our definition specializes to the one in Barthel and Sabarwal (2018)

Complementarity is viewed as a situation in which taking a higher action increases the marginal benefit of taking additional higher actions. This is formalized for objective functions by the well-known properties of supermodularity and quasisupermodularity. Barthel and Sabarwal (2018) propose directional versions of these properties for finite-dimensional Euclidean spaces. We generalize these to function spaces as follows. For direction $x \in X$, a function Φ is x**-supermodular (or** x**-SPM) on** $\mathbb{X}$, if $\forall f, g \in \mathbb{X}$ with $f(x) > g(x)$ we have $\Phi(f \vee g) + \Phi(f \wedge g) \geq \Phi(f) + \Phi(g)$. If Φ is x-supermodular on $\mathbb{X}$ for all $\forall x \in X$ then Φ is supermodular on $\mathbb{X}$. Similarly, the function Φ is x-

quasisupermodular (or, x-QSPM) on $\mathbb{X}$, if $\forall f, g \in \mathbb{X}$ with $f(x) > g(x)$ we have $\Phi(f) \geq \Phi(f \wedge g) \Rightarrow \Phi(f \vee g) \geq \Phi(g)$ and $\Phi(f) > \Phi(f \wedge g) \Rightarrow \Phi(f \vee g) > \Phi(g)$. In particular, x–supermodularity is a cardinal property of Φ and its corresponding ordinal characterization is x–quasisupermodularity. So if the function Φ is x–supermodular on $\mathbb{X}$ then Φ is x–quasisupermodular on $\mathbb{X}$. The converse is not true in general.

Supermodular and quasisupermodular properties are generalized to increasing differences and single crossing property in a natural manner. For $x \in X$, the function Φ satisfies **x-single crossing property (or, x-SCP) on $\mathbb{X}$**, if $\forall f, g \in \mathbb{X}$ with $f(x) > g(x)$ and for every $h = s[g - (f \wedge g)] \in \mathbb{X}$ with $s \geq 0$ and $f + h, g + h \in \mathbb{X}$, (1) $\Phi(f) \geq \Phi(g) \Rightarrow \Phi(f + h) \geq \Phi(g + h)$ and (2) $\Phi(f) > \Phi(g) \Rightarrow \Phi(f + h) > \Phi(g + h)$. Function Φ satisfies **directional single crossing property on $\mathbb{X}$**, if it satisfies x-single crossing property $\forall x \in X$.

Parametric effects are included by positing a poset of exogenous parameters $(\mathbb{T}, \geq)$ and extending the objective function to $\Phi : \mathbb{X} \times \mathbb{T} \rightarrow \mathbb{R}$. For $x \in X$, the function Φ satisfies **basic x-single crossing property on $\mathbb{X} \times \mathbb{T}$**, if $\forall f, g \in \mathbb{X}$ with $f(x) > g(x)$ and for every $t, t' \in T$ with $t' \geq t$, (1) $\Phi(f, t) \geq \Phi(g, t) \Longrightarrow \Phi(f, t') \geq \Phi(g, t')$ and (2) $\Phi(f, t) > \Phi(g, t) \Longrightarrow \Phi(f, t') > \Phi(g, t')$. Function Φ satisfies **basic directional single crossing property on $\mathbb{X} \times \mathbb{T}$**, if it satisfies basic x-single crossing property on $\mathbb{X} \times \mathbb{T}$, $\forall x \in X$.

Directional monotone comparative statics is formalized as follows. For $x \in X$ the function Φ satisfies **x-directional monotone comparative statics (or, x-DMCS) on $\mathbb{X} \times \mathbb{T}$**, if for all non-empty $F, G \subseteq \mathbb{X}$ with $F \sqsubseteq_x^{dso} G$ and $\forall t, t' \in \mathbb{T}$ with $t' \geq t$, $\operatorname{argmax}_F \Phi(\cdot, t) \sqsubseteq_x^{dso} \operatorname{argmax}_G \Phi(\cdot, t')$. Moreover, Φ **satisfies directional monotone comparative statics (or, DMCS) on $\mathbb{X} \times \mathbb{T}$**, if Φ satisfies x-directional monotone comparative statics on $\mathbb{X} \times \mathbb{T}$ for every $x \in X$.

Intuitively, Φ satisfies x-directional monotone comparative statics means that whenever the constraint set F increases in the x-directional set order to the constraint set G and/or the parameter t increases to t', then assuming that both sets of optimizers are nonempty, for every optimizer $f \in \operatorname{argmax}_F \Phi(\cdot, t)$, there is either $g \in \operatorname{argmax}_G \Phi(\cdot, t')$ such that $g(x) \geq f(x)$ or there is non negative h such that $f + h \in \operatorname{argmax}_G \Phi(\cdot, t')$. In both cases there is a function in $\operatorname{argmax}_G \Phi(\cdot, t')$ with x-component weakly higher than $f(x)$. A similar statement holds for every $g \in \operatorname{argmax}_G \Phi(\cdot, t')$.

The following two theorems characterize x-directional monotone comparative statics and directional monotone comparative statics.

Theorem 1. The following statements are equivalent.

1. Φ satisfies x-directional monotone comparative statics on $\mathbb{X} \times \mathbb{T}$.
2. $\forall t \in T$ $\Phi(\cdot, t)$ is x–quasisupermodular on $\mathbb{X}$, satisfies x–single crossing property on $\mathbb{X}$ and exhibits basic x–single crossing property on $\mathbb{X} \times \mathbb{T}$.

Proof. ($\Leftarrow$) Consider any $F, G \subseteq \mathbb{X}$ such that $F \sqsubseteq_x^{dso} G$ and fix $t, t' \in \mathbb{T}$ with $t' \geq t$. Choose arbitrary $f \in \operatorname{argmax}_F \Phi(\cdot, t)$ and $g \in \operatorname{argmax}_G \Phi(\cdot, t')$ where $f(x) > g(x)$. Since $f \in F$, $g \in G$ and $F \sqsubseteq_x^{dso} G$, $\exists h = s[g - (f \wedge g)] = s[(f \vee g) - f] \in \mathbb{X}$ for some $s \in [0, 1]$ such that $g - h \in F$ and $f + h \in G$.

As case 1, suppose $s = 1$. Then $h = g - (f \wedge g) = (f \vee g) - f$. Hence $f \wedge g = g - h \in F$ and $f \vee g = f + h \in G$. Since $f \in \operatorname{argmax}_F \Phi(\cdot, t)$, it follows that $\Phi(f, t) \geq \Phi(f \wedge g, t)$. As Φ is x–quasisupermodular on $\mathbb{X}$ and satisfies basic x–single crossing property on $\mathbb{X} \times \mathbb{T}$, we can write

$$\Phi(f, t) \geq \Phi(f \wedge g, t) \Longrightarrow \Phi(f \vee g, t) \geq \Phi(g, t) \Longrightarrow \Phi(f \vee g, t') \geq \Phi(g, t')$$

However $\Phi(g, t') \geq \Phi(f \vee g, t')$ as $g \in \operatorname{argmax}_G \Phi(., t')$. Hence $\Phi(g, t') = \Phi(f \vee g, t')$ and $f + h = f \vee g \in \operatorname{argmax}_G \Phi(\cdot, t')$. In particular, $\Phi(f \vee g, t') \ngtr \Phi(g, t')$. Again as Φ is x–quasisupermodular on $\mathbb{X}$ and satisfies basic x–single crossing property on $\mathbb{X} \times \mathbb{T}$, we can write

$$\Phi(f \vee g, t') \ngtr \Phi(g, t') \Longrightarrow \Phi(f, t') \ngtr \Phi(f \wedge g, t') \Longrightarrow \Phi(f, t) \ngtr \Phi(f \wedge g, t)$$

Thus $\Phi(f \wedge g, t) \geq \Phi(f, t)$. Hence $\Phi(f \wedge g, t) = \Phi(f, t)$ and $g - h = f \wedge g \in \operatorname{argmax}_F \Phi(\cdot, t)$. Since f, g are arbitrary $\operatorname{argmax}_F \Phi(\cdot, t) \sqsubseteq_x^{dso} \operatorname{argmax}_G \Phi(\cdot, t')$ as required.

As case 2, suppose $s < 1$. Without the loss of generality assume $g(y) \geq f(y)\ \forall y \in X, y \neq x$. Then for $h = s[g - (f \wedge g)]$ we have $h(x) = s[g(x) - (f(x) \wedge g(x))] = s[g(x) - g(x)] = 0$ and $h(y) = s[g(y) - f(y)]$ and hence $h \geq 0$ .Now we claim that $f \wedge g = f \wedge (g - h)$. Note that $(f \wedge g)(x) = g(x)$ and $(f \wedge g)(y) = f(y)$. Then $(f \wedge (g - h))(x) = f(x) \wedge (g - h)(x) = f(x) \wedge [g(x) - h(x)] = f(x) \wedge [g(x) - 0] = f(x) \wedge g(x) = g(x)$. Now $(g - h)(y) = g(y) - h(y) = g(y) - sg(y) + sf(y) = (1 - s)g(y) + sf(y) \geq (1 - s)f(y) + sf(y) = f(y)$. Hence $(f \wedge (g - h))(y) = f(y) \wedge (g - h)(y) = f(y)$. Hence $f \wedge g = f \wedge (g - h)$. Define the direction vector $\hat{h} = \frac{s}{1-s}[(g - h) - f \wedge (g - h)]$. Then $h = s[g - (f \wedge g)] = s[(g - h) - (f \wedge g)] + sh = s[(g - h) - f \wedge (g - h)] + sh$ and hence $h = \frac{s}{1-s}[(g - h) - f \wedge (g - h)] = \hat{h}$. Now as $f \in \operatorname{argmax}_F f(\cdot, t)$, $g - h \in F$, Φ satisfies x–single crossing property on $\mathbb{X}$ and Φ satisfies basic x–single crossing property on $\mathbb{X} \times \mathbb{T}$ we can write

$$\Phi(f, t) \geq \Phi(g - h, t) \Longrightarrow \Phi(f + h, t) \geq f(g, t) \Longrightarrow \Phi(f + h, t') \geq \Phi(g, t')$$

Then $g, f + h \in \operatorname{argmax}_G \Phi(\cdot, t')$ and consequently $\Phi(f + h, t') = \Phi(g, t')$. In particular, $\Phi(f + h, t') \ngtr \Phi(g, t')$. As Φ satisfies x–single crossing property on $\mathbb{X}$ and basic x–single crossing property on $\mathbb{X} \times \mathbb{T}$ we have:

$$\Phi(f+h, t') \ngtr \Phi(g, t') \Longrightarrow \Phi(f, t') \ngtr \Phi(g-h, t') \Longrightarrow \Phi(f, t) \ngtr \Phi(g-h, t) \Longrightarrow \Phi(g-h, t) \geq \Phi(f, t)$$

Hence $\Phi(f, t) = \Phi(g - h, t)$ and consequently $g - h \in \operatorname{argmax}_F \Phi(., t)$. Since f, g are arbitrary $\operatorname{argmax}_F \Phi(\cdot, t) \sqsubseteq_x^{dso} \operatorname{argmax}_G \Phi(\cdot, t')$ as required.

($\Rightarrow$) We want to show that $\forall t \in \mathbb{T}$ $\Phi(\cdot, t)$ is x–quasisupermodular on $\mathbb{X}$. Fix arbitrary $t \in \mathbb{T}$ and arbitrary $f, g \in \mathbb{X}$ such that $f(x) > g(x)$. Form the sets $F = \{f, f \wedge g\}$ and $G = \{g, f \vee g\}$. Then $F \sqsubseteq_x^{dso} G$. To see this choose $f \in F$, $g \in G$ and let $h = g - (f \wedge g) = (f \vee g) - f$. Then $f + h = f \vee g \in G$ and $g - h = f \wedge g \in F$. The other cases are satisfied vacuously as

1. For $f \in F, f \vee g \in G$, $(f \vee g)(x) = f(x) \vee g(x) = f(x)$

2. For $f \wedge g \in F, g \in G$, $(f \wedge g)(x) = f(x) \wedge g(x) = g(x)$

3. For $f \wedge g \in F, f \vee g \in G$, $(f \wedge g)(x) = f(x) \wedge g(x) = g(x)$

Suppose $\Phi(f,t) \geq \Phi(f \wedge g, t)$ but $\Phi(g,t) > \Phi(f \vee g, t)$. Therefore $\{f\} \in \operatorname{argmax}_F \Phi(\cdot,t)$ and $\{g\} = \operatorname{argmax}_G \Phi(\cdot,t)$. As $\operatorname{argmax}_F \Phi(\cdot,t) \sqsubseteq_x^{dso} \operatorname{argmax}_G \Phi(\cdot,t)$, $\exists h = s[(f \vee g) - f] \in \mathbb{X}$ for some $s \in [0,1]$ such that $f + h \in \operatorname{argmax}_G \Phi(\cdot,t)$. Then it must be the case that $f + h = g$. Now $(f+h)(x) = f(x) + h(x) = f(x) + s[(f \vee g)(x) - f(x)] = f(x) + s[f(x) \vee g(x) - f(x)] = f(x) + s[f(x) - f(x)] = f(x)$. Hence $f + h \neq g$ and we arrive at a contradiction. Therefore $\Phi(f,t) \geq \Phi(f \wedge g, t) \Longrightarrow \Phi(f \vee g, t) \geq \Phi(g,t)$ as required. Now suppose $\Phi(f,t) > \Phi(f \wedge g, t)$ but $\Phi(g,t) \geq \Phi(f \vee g, t)$. Therefore $\{f\} = \operatorname{argmax}_F \Phi(\cdot,t)$ and $g \in \operatorname{argmax}_G \Phi(\cdot,t)$. As $\operatorname{argmax}_F \Phi(\cdot,t) \sqsubseteq_x^{dso} \operatorname{argmax}_G \Phi(\cdot,t)$, $\exists h = s[g - (f \wedge g)] \in \mathbb{X}$ for some $s \in [0,1]$ such that $g - h \in \operatorname{argmax}_F \Phi(\cdot,t)$. Then it must be the case that $g - h = f$. Since $h(x) = s[(f \vee g)(x) - f(x)] = s[f(x) \vee g(x) - f(x)] = s[f(x) - f(x)] = 0$, $(g-h)(x) = g(x) - h(x) = g(x) < f(x)$. Hence $g - h \neq f$ and we arrive at a contradiction. Therefore $\Phi(f,t) > \Phi(f \wedge g, t) \Longrightarrow \Phi(f \vee g, t) > \Phi(g,t)$ as required.

We want to show that $\forall t \in \mathbb{T}$ $\Phi(\cdot,t)$ satisfies the x–single crossing property on $\mathbb{X}$. Fix arbitrary $f, g \in \mathbb{X}$ with $f(x) > g(x)$ and $t \in \mathbb{T}$. Correspondingly fix $h = s[g - (f \wedge g)] = s[(f \vee g) - f] \in \mathbb{X}$ for $s \geq 0$ such that $f + h, g + h \in \mathbb{X}$. When $s = 0$, we have $h = 0$ and consequently $\Phi(\cdot,t)$ satisfies the x–single crossing property on $\mathbb{X}$ trivially. So now we consider the case when $s > 0$. Now let $w = g + h$ and $u = w - (f \wedge w) = (f \vee w) - f$. We claim that $f \wedge w = f \wedge g$. Without the loss of generality assume that $g(y) \geq f(y)$ $\forall y \in X, y \neq x$. Now $h(x) = s[g(x) - (f \wedge g)(x)] = s[g(x) - f(x) \wedge g(x)] = s[g(x) - g(x)] = 0$ and $h(y) = s[g(y) - (f \wedge g)(y)] = s[g(y) - (f(y) \wedge g(y))] = s[g(y) - f(y)]$. Therefore $w(x) = g(x) + h(x) = g(x)$ and $w(y) = g(y) + h(y) = g(y) + s[g(y) - f(y)] \geq f(y)$. Now $(f \wedge w)(x) = f(x) \wedge w(x) = f(x) \wedge g(x) = g(x) = (f \wedge g)(x)$ and $(f \wedge w)(y) = f(y) \wedge w(y) = f(y) = (f \wedge g)(y)$. Hence $f \wedge w = f \wedge g$. So $u(x) = w(x) - (f \wedge w)(x) = g(x) - g(x) = 0$ and $u(y) = w(y) - (f \wedge w)(y) = g(y) + s[g(y) - f(y)] - f(y) = (1+s)[g(y) - f(y)]$. Consequently $h(x) = u(x) = 0$ and $h(y) = \frac{s}{1+s} u(y)$. Define $s' = \frac{s}{1+s}$. Note that

$s' \in (0,1)$. Then $h = \frac{s}{1+s} u = s'u$. Now $w - s'(w - f \wedge w) = w - s'u = w - h$ and $f + s'(f \vee w - f) = f + s'u = f + h$. Form the sets $F = \{f, w - h\}$ and $G = \{f + h, w\}$. Then we claim that Then $F \sqsubseteq_x^{dso} G$. To see this consider $f \in F$ and $w \in G$. Then for $h = s'u = s'[w - (f \wedge w)] = s'[(f \vee w) - f]$ for some $s' \in (0,1)$ $f + h \in G$ and $w - h \in F$. The other cases are satisfied vacuously as

1. For $f \in F, f + h \in G$, $(f + h)(x) = f(x) + v(x) = f(x)$
2. For $w - h \in F, f + h \in G$, $(f + h)(x) = f(x)$ and $(w - h)(x) = g(x)$
3. For $w - h \in F, f \in G$, $(w - h)(x) = g(x) < f(x)$

Suppose $\Phi(f) \geq \Phi(g) = f(w-h)$ but $\Phi(w) = \Phi(g+h) > \Phi(f+h)$. This implies $f \in \mathrm{argmax}_F f(\cdot, t)$ and $\{w\} = \mathrm{argmax}_G f(\cdot, t)$. As Φ satisfies x-directional MCS on $\mathbb{X} \times \mathbb{T}$, $\exists \hat{s} \in [0,1]$ such that $f + \hat{s}[f \vee w - f] \in \mathrm{argmax}_G f(\cdot, t)$. Hence it must be the case that $f + \hat{s}[f \vee w - f] = w$. Note that $(f \vee w)(x) = f(x) \vee w(x) = f(x) \vee g(x) = f(x)$. Hence $(f + \hat{s}[f \vee w - f])(x) = f(x) + \hat{s}[(f \vee w)(x) - f(x)] = f(x) > g(x) = w(x)$. Hence $f + \hat{s}[f \vee w - f] \neq w$ and we arrive at a contradiction. Therefore $\Phi(f) \geq \Phi(g) \Longrightarrow \Phi(f+h) \geq \Phi(g+h)$ as required. Now suppose $\Phi(f) > \Phi(g) = \Phi(w-h)$ but $\Phi(w) = \Phi(g+h) \geq \Phi(f+h)$. This implies $\{f\} = \mathrm{argmax}_F f(\cdot, t)$. and $w \in \mathrm{argmax}_G f(\cdot, t)$. As Φ satisfies x-directional MCS on $\mathbb{X} \times \mathbb{T}$, $\exists \hat{s} \in [0,1]$ such that $w - \hat{s}[w - f \wedge w] \in \mathrm{argmax}_F f(\cdot, t)$. Hence it must be the case that $w - \hat{s}[w - f \wedge w] = f$. Note that $(f \wedge w)(x) = f(x) \wedge w(x) = f(x) \wedge g(x) = g(x)$. Hence $(w - \hat{s}[w - f \wedge w])(x) = w(x) - \hat{s}[w(x) - (f \wedge w)(x)] = w(x) - \hat{s}[g(x) - g(x)] = w(x) = g(x) < f(x)$. Hence $w - \hat{s}[w - f \wedge h] \neq f$ and we arrive at a contradiction. Therefore $\Phi(f) > \Phi(g) \Longrightarrow \Phi(f + h) > \Phi(g + h)$ as required.

Finally we want to show that Φ satisfies basic x–SCP on $\mathbb{X} \times \mathbb{T}$. Fix arbitrary $f, g \in \mathbb{X}$ with $f(x) > g(x)$ and $t, t' \in \mathbb{T}$ where $t' \geq t$. Let $F = \{f, g\}$. Then $F \sqsubseteq_x^{dso} F$. To see this choose $s = 0$ and hence $h = s[g - f \wedge g] = s[f \vee g - f] = 0$. Suppose $\Phi(f, t) \geq \Phi(g, t)$ but $\Phi(g, t') > \Phi(f, t')$. Hence $f \in \mathrm{argmax}_F \Phi(\cdot, t)$ and $\{g\} = \mathrm{argmax}_F \Phi(\cdot, t')$. As Φ satisfies x–directional MCS on $\mathbb{X} \times \mathbb{T}$ we have $f + h = f + s[g - (f \wedge g)] \in \mathrm{argmax}_F \Phi(\cdot, t')$ for some $s \in [0,1]$. Hence it must be the case that $f+h=g$. But since $(f+h)(x)=f(x)+h(x)=f(x)>g(x)$, $f+h\neq g$ and we arrive at a contradiction. Therefore $\Phi(f,t)\geq\Phi(g,t)\implies\Phi(f,t')\geq\Phi(g,t')$ as required. Now suppose $\Phi(f,t)>\Phi(g,t)$ but $\Phi(g,t')\geq\Phi(f,t')$. Then $\{f\}=\operatorname{argmax}_F\Phi(.,t)$ and $g\in\operatorname{argmax}_F\Phi(.,t')$. As Φ satisfies $x-$directional MCS on $\mathbb{X}\times\mathbb{T}$, we have $g-h=g-s[g-(f\wedge g)]\in\operatorname{argmax}_F\Phi(\cdot,t)$ for some $s\in[0,1]$. Hence it must be the case that $g-h=f$. But since $(g-h)(x)=g(x)-h(x)=g(x)<f(x)$, $g-h\neq f$ and we arrive at a contradiction. Therefore $\Phi(f,t)>\Phi(g,t)\implies\Phi(f,t')>\Phi(g,t')$ as required. □

Importantly, for non-vacuous comparisons, the set of optimizers should be nonempty. This is typically guaranteed by some form of upper semi-continuity over compact sets, for example as in Theorems 1 and 2 in Milgrom and Roberts (1990) or Theorem A4 in Milgrom and Shannon (1994), and the related results in Shannon (1995).

Theorem 2. The following statements are equivalent.

1. Φ satisfies directional monotone comparative statics on $\mathbb{X}\times\mathbb{T}$.

2. $\forall t\in\mathbb{T}\ \Phi(\cdot,t)$ is quasisupermodular on $\mathbb{X}$, satisfies directional single crossing property on $\mathbb{X}$ and exhibits basic directional single crossing property on $\mathbb{X}\times\mathbb{T}$.

Proof. Φ satisfies directional monotone comparative statics on $\mathbb{X}\times\mathbb{T}$ if, and only if, Φ satisfies $x-$directional monotone comparative statics on $\mathbb{X}\times\mathbb{T}\ \forall x\in X$. By Theorem-1, this is equivalent to $\forall x\in X\ \forall t\in\mathbb{T}\ \Phi(\cdot,t)$ is $x-$quasisupermodular on $\mathbb{X}$, satisfies $x-$single crossing property on $\mathbb{X}$ and satisfies basic $x-$single crossing property on $\mathbb{X}\times\mathbb{T}$, which is equivalent to statement (2). □

The characterizations in Quah (2007), where the objective function does not have parameters, generalize as well, as follows. Suppose $\Phi:\mathbb{X}\to\mathbb{R}$. In this case, Φ satisfies x-directional monotone comparative statics on $\mathbb{X}$ if, and only if, Φ is x-quasisupermodular on $\mathbb{X}$ and satisfies x-single crossing property on $\mathbb{X}$. This is a special case of the theorem by setting $\mathbb{T} = \{t\}$. And similarly, Φ satisfies directional monotone comparative statics on $\mathbb{X}$ if, and only if, Φ is quasisupermodular on $\mathbb{X}$ and satisfies directional single crossing property on $\mathbb{X}$.

3 Differential Characterizations and Concavity Conditions

Differential and concavity conditions provide sufficient conditions that can be easier to use for directional monotone comparative statics. These are formalized as follows.

Consider the cardinal property suggested by the x-single crossing property, namely, x-increasing differences. For $x \in X$, Φ satisfies **x-increasing differences (or, x-ID) on** $\mathbb{X}$, if $\forall f, g \in \mathbb{X}$ with $f(x) > g(x)$ and for every $h = s[g - (f \wedge g)] \in \mathbb{X}$ with $s \geq 0$ such that $f + h, g + h \in \mathbb{X}$ we have $\Phi(f + h) - \Phi(g + h) \geq \Phi(f) - \Phi(g)$. Moreover, Φ satisfies **directional increasing differences (or, directional-ID) on** $\mathbb{X}$, if Φ satisfies x-increasing differences on $\mathbb{X}$ $\forall x \in X$. Analogous to Barthel and Sabarwal (2018), the x-increasing differences property can be characterized using directional derivatives as follows. Suppose $\mathbb{X}$ is an open set in $\mathbb{R}^X$ in the product topology and $\Phi : \mathbb{X} \to \mathbb{R}$ is a $\mathcal{C}^1$ function. For any $f \in \mathbb{X}$ and $u \in \mathbb{R}^X$ the directional derivative of Φ at f in the direction u is defined as $D_u\Phi(f) = \lim_{\sigma \to 0} \frac{\Phi(f+\sigma u) - \Phi(f)}{\sigma}$, whenever the limit exists.

Theorem 3. Suppose $\Phi : \mathbb{X} \to \mathbb{R}$ is a $\mathcal{C}^2$ function. The following are equivalent.

1. Φ satisfies x-increasing differences on $\mathbb{X}$.

2. $\forall g \in \mathbb{X}$, $\forall u \in \mathbb{R}^X$ with $u(x) > 0$, $D_{(-u)_+} D_u \Phi(g) \geq 0$.

Proof. It can be shown that Φ satisfies x-increasing differences on $\mathbb{X}$ is equivalent to the statement that $\forall g \in \mathbb{X}$, $\forall u \in \mathbb{R}^X$ with $u(x) > 0$, and $\forall s, \sigma \geq 0$ such that $g+u, g+s(-u)_+$ and $g+\sigma u+s(-u)_+ \in \mathbb{X}$ we have $\Phi\left(g+\sigma u+s(-u)_+\right) - \Phi\left(g+s(-u)_+\right)$ is weakly increasing in s. Combined with the definition of directional derivative, $D_u\Phi(g+s(-u)_+) = \lim_{\sigma\to 0}\frac{\Phi(g+\sigma u+s(-u)_+)-\Phi(g+s(-u)_+)}{\sigma}$, it follows that $\Phi(g+\sigma u+s(-u)_+) - \Phi(g+s(-u)_+)$ is weakly increasing in s is equivalent to $D_u\Phi(g+s(-u)_+)$ is weakly increasing in s, or equivalently $D_{(-u)_+}D_u\Phi(g) \geq 0$, $\forall g \in \mathbb{X}$ and $\forall u \in \mathbb{R}^X$ with $u(x) > 0$. □

For the parametric framework, suppose $(\mathbb{T}, \geq)$ is a poset of exogenous parameters and $\Phi : \mathbb{X}\times\mathbb{T} \to \mathbb{R}$ is a function. For $x \in X$ the function Φ satisfies **basic x-increasing differences on** $\mathbb{X}\times\mathbb{T}$, if $\forall f, g \in \mathbb{X}$ with $f(x) > g(x)$ and for every $t, t' \in \mathbb{T}$ with $t' \geq t$ we have $\Phi(f,t') - \Phi(g,t') \geq \Phi(f,t) - \Phi(g,t)$. The function Φ satisfies **basic directional increasing differences on** $\mathbb{X}\times\mathbb{T}$, if the function Φ satisfies basic x- increasing differences on $\mathbb{X}\times\mathbb{T}$ $\forall x \in X$. It is easy to check that if Φ satisfies basic x-increasing differences on $\mathbb{X}\times\mathbb{T}$ then Φ satisfies basic x- single crossing property on $\mathbb{X}\times\mathbb{T}$. Similarly if Φ satisfies basic directional increasing differences on $\mathbb{X}\times\mathbb{T}$ then Φ satisfies basic directional single crossing property on $\mathbb{X}\times\mathbb{T}$

Theorem 4. Suppose $\Phi : \mathbb{X}\times\mathbb{T} \to \mathbb{R}$ is a $\mathcal{C}^2$ function. For each $x \in X$, the following are equivalent.

1. Φ satisfies basic x-increasing differences on $\mathbb{X}\times\mathbb{T}$.

2. $\forall g \in \mathbb{X}, u \in \mathbb{R}^X$ with $u(x) > 0$, $D_t D_u \Phi(g) \geq 0$.

Proof. Φ satisfies basic x-increasing differences on $\mathbb{X}\times\mathbb{T}$ if, and only if, $\forall g \in \mathbb{X}$, $\forall u \in \mathbb{R}^X$ with $u(x) > 0$, $\Phi(g+u,t) - \Phi(g,t)$ is weakly increasing in t, or equivalently, $D_t D_u \Phi(g) \geq 0$. □

We show that x-supermodularity and a generalized notion of (relative) x-concavity provide convenient sufficient conditions for directional monotone comparative statics in function spaces.

We start by stating several definitions that capture the directional aspect of concavity in $(\mathbb{R}^X, \geq)$. Let $\mathbb{X}$ be a sublattice of $(\mathbb{R}^X, \geq)$, $\Phi : \mathbb{X} \to \mathbb{R}$, and $u \in \mathbb{R}^X$ be a non-zero function. The function Φ is **(Relatively) concave in direction** u, if $\Psi(\lambda) = \Phi(f + \lambda u)$ is a concave function, $\forall f \in \mathbb{X}$. That is, $\Psi[\alpha\lambda_1 + (1-\alpha)\lambda_2] \geq \alpha\Psi(\lambda_1) + (1-\alpha)\Psi(\lambda_2)$ for every $f + \lambda_1 u, f + \lambda_2 u \in \mathbb{X}$ and $\alpha \in [0,1]$. Moreover, Φ is **(Relatively) concave on** $\mathbb{X}$, if Φ is (Relatively) concave in direction u for every non zero $u \in \mathbb{R}^X$. Alternatively, we can say Φ is (Relatively) concave in the direction u if $\Phi[\alpha(f + \lambda_1 u) + (1-\alpha)(f + \lambda_2 u)] \geq \alpha\Phi(f + \lambda_1 u) + (1-\alpha)\Phi(f + \lambda_2 u)$ $\forall f \in \mathbb{X}$ and $\forall \alpha \in [0,1]$. Notice that if $\Phi : \mathbb{X} \to \mathbb{R}$ is a C^2 function, then Φ is (Relatively) concave in the direction u if $\Psi''(\lambda) = \frac{\partial^2\Phi(f+\lambda u)}{\partial\lambda^2} \leq 0$. This can be reformulated in terms of directional derivatives: Φ is (Relatively) concave in the direction u on $\mathbb{X}$ if $D_u\Phi(f)$ is non-increasing in the direction u, that is, $D_u^2\Phi(f) \leq 0$.

Finally, for $x \in X$, Φ is **(Relatively)** ***x*****-concave on** $\mathbb{X}$, if for every non-zero $u \in \mathbb{R}^X$ with $u(x) = 0$, Φ is (Relatively) concave in the direction u on $\mathbb{X}$ and Φ is **(Relatively) directionally concave on** $\mathbb{X}$, if Φ is (Relatively) $x-$ concave on $\mathbb{X}$, $\forall x \in X$. If $\Phi : \mathbb{X} \to \mathbb{R}$ is (Relatively) concave on $\mathbb{X}$ then it is also (Relatively) directionally concave on $\mathbb{X}$. To see this, suppose $\Phi : \mathbb{X} \to \mathbb{R}$ (Relatively) concave on $\mathbb{X}$. Fix any $x \in X$ and consider any non-zero $u \in \mathbb{R}^X$ with $u(x) = 0$. Then Φ is (Relatively) directionally concave on $\mathbb{X}$. Also if a function $\Phi : \mathbb{X} \to \mathbb{R}$ is concave over $\mathbb{X}$ then it is (Relatively) concave and consequently (Relatively) directionally concave. However, the converse is not true. In this context consider the following examples.

Example 1. Let $\mathbb{X} = \mathbb{R}^2_{++}$ and $\Phi(x_1, x_2) = x_1 x_2$. Then Φ is (Relatively) concave on $\mathbb{R}^2_{++}$ but not concave on $\mathbb{R}^2_{++}$.

Example 2. Let $\mathbb{X} = \mathbb{R}^2_{++}$, $\Phi(x_1, x_2) = x_1^{\frac{1}{2}}$ and $u = (u_1, u_2) \in \mathbb{R}^2_{++}$ be a non-zero vector. Choose any $\lambda \in \mathbb{R}$ such that $x + \lambda u = (x_1 + \lambda u_1, x_2 + \lambda u_2) \in \mathbb{R}^2_{++}$. Then $\frac{d^2\Phi(x+\lambda u)}{d\lambda^2} = \frac{-1}{4} u_1^2 (x_1 + \lambda u_1)^{\frac{-3}{2}} \leq 0$. Thus Φ is (Relatively) concave in the direction

u and (Relatively) concave on $\mathbb{R}^2_{++}$.

Example 3. Let $\mathbb{X} = \mathbb{R}^\infty_+$ and $\Phi(x) = \sum_{i=1}^{+\infty} \beta^{i-1}(\alpha_i \log x_i)\ \forall i \in \mathbb{N}$ with $0 < \alpha_i, \beta < 1$. Fix any $x = (x_i)_{i\in\mathbb{N}}$, any non-zero $u = (u_i)_{i\in\mathbb{N}}$ and a scalar $\lambda > 0$ such that $x+\lambda u \in \mathbb{X}$. Hence $\Phi(x+\lambda u) = \sum_{i=1}^{+\infty} \beta^{i-1}[\alpha_i \log(x_i + \lambda u_i)]$ where $x + \lambda u = (x_i + \lambda u_i)_{i\in\mathbb{N}}$. This implies $\frac{\partial^2\Phi(x+\lambda u)}{\partial\lambda^2} = \sum_{i=1}^{+\infty} \frac{-\beta^{i-1}\alpha_i {u_i}^2}{(x_i+\lambda u_i)^2}$. Provided the following infinite series on the right-hand side converges, we have $\frac{\partial^2\Phi(x+\lambda u)}{\partial\lambda^2} \le 0$. In this case, Φ is (Relatively) concave on $\mathbb{R}^\infty_+$ and consequently, (Relatively) directionally concave on $\mathbb{R}^\infty_+$.

The following propositions connect x-supermodularity, (relative) x-concavity, and directional MCS. Both propositions are proved by following the proof in Barthel and Sabarwal (2018).

Proposition 1. For $x \in X$, if $\forall t \in \mathbb{T}$, $\Phi(\cdot, t)$ is x-supermodular on $\mathbb{X}$ and (Relatively) x-concave on $\mathbb{X}$, then Φ satisfies x-single crossing property on $\mathbb{X}$.

Proposition 2. For $x \in X$, if $\forall t \in \mathbb{T}$, $\Phi(\cdot, t)$ is supermodular on $\mathbb{X}$ and (Relatively) directionally concave on $\mathbb{X}$, then Φ satisfies directional single crossing property on $\mathbb{X}$.

4 Examples

Let's apply these results to three classes of applications in which the underlying spaces are infinite-dimensional.

4.1 *Neoclassical growth model with infinite horizon*

Consider a discrete time infinite horizon economy with time indexed by $t \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. The economy is a representative agent economy with a large number of identical households and population at time t is $L_t = (1+n)^t L_0$, where the initial population L_0 and the population growth rate n are exogenously given. At time t, a representative household supplies one unit of labor inelastically. The firms are owned by households. The CRS production function is $Y_t = A_t K_t^{\alpha}(Z_t L_t)^{1-\alpha}$ where Y_t is the produced output, A_t is the Hicks-neutral technological progress, K_t is the capital stock, Z_t is the labor augmenting technology at time t and $\alpha \in (0,1)$. We assume A_t follows a stationary stochastic process and $Z_t = (1+z)^t Z_0$ where Z_0 is exogenously given, z is the rate of labor augmenting technological progress. The national income identity is $Y_t = C_t + I_t$ where C_t is the consumption of a representative household and I_t is the amount of investment at time t. The capital accumulation equation is $K_{t+1} = I_t + (1-\delta)K_t$ where $\delta \in (0,1)$ is the depreciation rate of capital. Substituting the values of I_t and Y_t, the capital accumulation equation can be written as $K_{t+1} = A_t K_t^{\alpha}(Z_t L_t)^{1-\alpha} - C_t + (1-\delta)K_t$. Given discount factor $\beta \in (0,1)$ and instantaneous utility function $u(C_t)$ of the representative household, the social planner solves the following optimization problem:

$$\begin{gathered}\max_{C_t} \mathbb{E}_0 \textstyle\sum_{t=0}^{\infty} \beta^t u(C_t) \\ \text{s.t. } K_{t+1} = A_t K_t^{\alpha}(Z_t L_t)^{1-\alpha} - C_t + (1-\delta)K_t \quad \forall t \in \mathbb{Z}_+\end{gathered}$$

For simplicity, we normalize the initial conditions to $Z_0 = L_0 = 1$ and set $\gamma = (1+n)(1+z)$. This implies $Z_t L_t = \gamma^t$. Define $c_t = \frac{C_t}{Z_t L_t}$ and $k_t = \frac{K_t}{Z_t L_t}$. Then we can rewrite the optimization problem as:

$$\begin{gathered}\max_{c_t} \mathbb{E}_0 \textstyle\sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } \gamma k_{t+1} = A_t k_t^{\alpha} - c_t + (1-\delta)k_t \quad \forall t \in \mathbb{Z}_+\end{gathered}$$

The Euler equation for this problem is used to study equilibrium behavior of the economy. It is a first order difference equation in c_t with A_{t+1} and parameters $\alpha, \beta, \gamma, \sigma, \delta$. In general, it does not yield a closed form solution. Therefore, particular functional forms for utility are typically proposed to study equilibrium and its comparative statics.

One common example is constant elasticity of substitution: $u(C_t) = \beta^t \frac{C_t^{1-\sigma}-1}{1-\sigma}$ where $\sigma \geq 0, \sigma \neq 1$ in which case the Euler equation is given as $c_t^{-\sigma} =$

$\beta\gamma^{-\sigma}\mathbb{E}_t c_{t+1}^{-\sigma}\left[\alpha A_{t+1}k_{t+1}^{\alpha-1}+(1-\delta)\right]$. The steady state value of consumption per unit of effective labor c^* can be obtained in terms of the steady-state value of the Hicks neutral technological progress A^* by solving the Euler equation. Since A_t follows some stationary stochastic process, A^* is set equal to its (unconditional) mean $\bar{A}$. Then the steady-state value of consumption per unit of effective labor c^* can be solved in terms of $\bar{A}$ and the parameters $\alpha, \beta, \gamma, \sigma, \delta$. If $\delta = 1$, this derivation yields $c^* = \bar{A}\left(\frac{\alpha\beta\bar{A}}{\gamma^{\sigma}}\right)^{\frac{\alpha}{1-\alpha}} - \left(\frac{\alpha\beta\bar{A}}{\gamma^{\sigma-1}}\right)^{\frac{1}{1-\alpha}}$. From this closed-form expression of c^*, it follows that c^* increases in $\bar{A}$, that is, consumption rises with (average) technological progress.

We provide comparative statics for a broader class of preferences, without deriving or solving an explicit Euler equation, and our results can provide distributional comparative statics (that is, for each time t) as compared to comparative statics based on the mean level of technology.

Consider a positive technological shock in period t, that is, A_t increases from $\hat{A}_t$ to $\tilde{A}_t$. Then capital accumulation equation shifts from $\gamma k_{t+1} = \hat{A}_t k_t^{\alpha} - c_t + (1-\delta)k_t$ to $\gamma k_{t+1} = \tilde{A}_t k_t^{\alpha} - c_t + (1-\delta)k_t$. Define the constraint sets $\hat{C}$ and $\tilde{C}$ as follows:

$$\hat{C} = \{(c_t)_{t\in\mathbb{Z}_+} : \frac{c_t + \gamma k_{t+1} - (1-\delta)k_t}{k_t^{\alpha}} \leq \hat{A}_t \quad (\forall t \in \mathbb{Z}_+)\}$$

$$\tilde{C} = \{(c_t)_{t\in\mathbb{Z}_+} : \frac{c_t + \gamma k_{t+1} - (1-\delta)k_t}{k_t^{\alpha}} \leq \tilde{A}_t \quad (\forall t \in \mathbb{Z}_+)\}$$

In the appendix, we show that $\hat{C} \sqsubseteq_t^{dso} \tilde{C}$, that is, in the t-directional set order.

For arbitrary consumption stream $c = (c_t)_{t=0}^{\infty}$, let $\upsilon(c) = \sum_{t=0}^{\infty}\beta^t u(c_t)$, where $u(\cdot)$ is an arbitrary (weakly) increasing, twice continuously differentiable, concave function ($\frac{\partial^2 u(c_t)}{\partial {c_t}^2} \leq 0$). Assuming the series converges absolutely on its (perhaps restricted) domain, it follows that $\upsilon(c)$ is supermodular (and hence quasisupermodular) because for every $i, j \in \mathbb{Z}_+$ with $i \neq j$, the second cross partial derivative of the function $\upsilon(c)$ with respect to c_i and c_j is zero. Moreover, $\upsilon(c)$ is a concave function and hence, (Relatively) directionally concave. Therefore, by Proposition 2, $\upsilon(c)$ satisfies directional single crossing property. Finally, for every direction vector $d \in \mathbb{R}^{\mathbb{N}}$ with $d_t > 0$, $D_d\upsilon(c) = \sum_{j=0}^{\infty} \beta^j u'(c_j)d_j$ and hence, $D_{A_t}D_d(\upsilon) = 0$. Therefore, $\upsilon(c)$ satisfies the basic directional single crossing property on the product set of consumption per unit of effective labor and Hicks-neutral technological progress.

By Theorem 1, the optimal level of consumption per unit of effective labor (weakly) increases in period t as a result of an increase in the Hicks-neutral technological progress in period t. Theorem 2 yields the result for every t, leading to an optimal policy that is (weakly) higher in every period t. This provides richer comparative statics results as compared to those that depend on the mean level. To our knowledge, this is the first application of *directional* monotone comparative statics (using directional set orders) to infinite horizon macroeconomic models.

Notably, Amir (1996b) provides monotone comparative dynamics results in Ramsey-type multidimensional dynamic optimization problems using the foundational methods from Topkis (1978), which uses the strong set order. Those results yield higher optimal policies (in every direction x) under the assumption that constraints sets are (weakly) increasing in the strong set order. The results here use the weaker x-directional set order which can directly rank budget-type sets that involve tradeoffs of the form: at any time t, more consumption is available only with less savings, and conversely.

4.2 *Infinite-dimensional consumer demand*

Infinite-dimensional consumption spaces arise naturally in commodity differentiation or uncertainty or time horizon. Modeling commodity differentiation in terms of quality differential or spatial characteristics is studied in Mas-Colell (1975). Differentiated commodities can be described by specifying characteristics in a compact metric space (K, d_K) called the characteristics space, which may be infinite-dimensional. The points in K are referred to as commodities. Mas-Colell (1975) showed that this characterization of the characteristics space is consistent with the earlier works of Houthakker (1952), Lancaster (1966) and Rosen (1974). One real world analogy of characteristics space is continuous scale rating.

Suppose the commodity space $\mathbb{X}$ is a sublattice of $(\mathbb{R}_{+}^{\mathbb{N}}, \geq)$ and is isometric to a compact subset of $(\mathbb{R}_{+}^{\mathbb{N}}, \geq)$. A bundle of commodities $x \in \mathbb{X}$ is an infinite sequence $(x_1, x_2, \ldots)$ of non-negative real numbers. Suppose $(\mathbb{T}, \geq)$ is a poset of exogenous parmaters and $u : \mathbb{X} \times \mathbb{T} \to \mathbb{R}$ is a continuous real valued utility function. The demand correspondence is $D(A,t) = \operatorname{argmax}_A u(x,t)$ where A is a non-empty compact subset of $\mathbb{X}$. Theorem-1 (Theorem-2) provides a set of sufficient conditions under which $D(A,t)$ is increasing in (A,t) in the n -directional set order (the directional set order) where $n \in \mathbb{N}$.

Conditions for *normal demand* in this model follow from our results. Let the price space $\mathbb{P}$ be a compact sublattice of $(\mathbb{R}_{++}^{\mathbb{N}}, \geq)$ with prices uniformly bounded away from zero, that is, $\inf\{p_n : n \in \mathbb{N}, p \in \mathbb{P}\} > 0$. The budget set is $B(p,w) = \{x \in \mathbb{X} | p \cdot x \leq w\}$, where $w \in \mathbb{R}_{++}$ is the wealth level, and Walrasian Demand is $D(p,w) = \operatorname{argmax}_{B(p,w)} u(x)$. When $w < w'$, $B(p,w)$ and $B(p,w')$ are not always comparable in the standard lattice set order, but they are comparable in the n-directional set order following the reasoning in Barthel and Sabarwal (2018). In other words, for every $n \in \mathbb{N}$, $B(p,w) \sqsubseteq_n^{dso} B(p,w')$. Therefore, using Theorem-1, if u satisfies n-QSPM, n-SCP and basic n-SCP, then the demand of good n is defined normal. A similar application of Theorem-2 gives sufficient conditions for demand for all goods to be normal.

Conditions for the *law of demand* also follow naturally. Consider a price system $p \in \mathbb{P}$ and consider good $n \in \mathbb{N}$. Suppose price of good n falls from p_n to $p'_n < p_n$. Let $p' \in \mathbb{P}$ be the price system obtained by replacing p_n by p'_n in price system p. The law of demand holds for good n, if $p'_n < p_n \Rightarrow D(p,w) \sqsubseteq_n^{dso} D(p',w)$. Following the reasoning in Barthel and Sabarwal (2018), it can be shown that $B(p, w) \sqsubseteq_n^{dso} B(p', w)$. As above, using Theorem-1, if u satisfies n-QSPM, n-SCP and basic n-SCP, then good n satisfies the law of demand. Similarly, Theorem-2 gives sufficient conditions for all goods to satisfy the law of demand.

To our knowledge these are the first results proving normality of demand and the law of demand in infinite-dimensional spaces using lattice based methods.

4.3 *Directional and global Le Chatelier principle in infinite dimensions*

Le Chatelier principle in economics formalizes the idea that long run input demand is more responsive than short run input demand to a change in its price. Samuelson (1947) formalized this for the case of smooth functions and small changes in price. Global Le Chatelier principle is formalized by Milgrom and Roberts (1996) using lattice-based methods and arbitrary change in prices. For the finite dimensional case, Quah (2007) provides a version that allows for more general constraint sets for the firm. We extend this to infinite dimensional spaces of inputs. As earlier, these arise naturally when considering decision-making over time or under uncertainty or with commodity differentiation.

Consider a production function φ with infinitely many inputs indexed by $n \in \mathbb{N}$ and a single output. Inputs are partitioned into those that are variable in the short run, denoted $V \subseteq \mathbb{N}$ and those that are fixed in the short run, denoted $F \subseteq \mathbb{N}$, with $\{V, F\}$ forming a partition of $\mathbb{N}$. All inputs are variable in the long run. Let $\mathbb{X}_V \subseteq \mathbb{R}_+^V$ be the space of variable inputs, assumed to be a compact sublattice of $\mathbb{R}_+^V$, and $\mathbb{X}_F \subseteq \mathbb{R}_+^V$ be the space of fixed inputs, assumed to be a compact sublattice of $\mathbb{R}_+^F$. Typical elements of $\mathbb{X}_V$ and $\mathbb{X}_F$ are denoted $z^V = \left(z_k^V\right)_{k \in V}$ and $z^F = \left(z_k^F\right)_{k \in F}$, respectively. Let $w^F \in \mathbb{R}_{++}^F$ denote fixed input prices and $w^V \in -\mathbb{R}_{++}^V$ denote negative of the variable input prices, so that $-w^V$ are the prices of variable inputs. Given variable input prices $-w^V$ and a bundle of variable inputs z^V, the cost of z^V is $(-w^V) \cdot z^V = \sum_{k \in V} -w_k^V z_k^V$, and given fixed input prices w^F and a bundle of fixed inputs z^V, the cost of z^F is $w^F \cdot z^F = \sum_{k \in F} w_k^F z_k^F$, restricting attention to cases when each series converges. Let $p \in \mathbb{R}_{++}$ be the price of firm output. The firm's profit is given by $p\varphi(z^V, z^F) - w^F \cdot z^F + w^V \cdot z^V$. For this application, we keep the price of output as fixed, and the prices of fixed inputs as fixed, and suppressing this notation, write the profit function as $\Pi(z^V, z^F, w^V) = p\varphi(z^V, z^F) - w^F \cdot z^F + w^V \cdot z^V$.

Consider a price system for variable inputs $-\hat{w}^V$ and suppose the price of a particular variable input $k \in V$ goes down from $-\hat{w}_k^V$ to $-\tilde{w}_k^V$. Let $-\tilde{w}^V$ be the price system for variable inputs derived from $\hat{w}^V$ by replacing $-\hat{w}_k^V$ with $-\tilde{w}_k^V$. Therefore, $-\tilde{w}^V \leq -\hat{w}^V$, or equivalently, $\hat{w}^V \leq \tilde{w}^V$. Define the full input bundle as Z, the full input price system as $\hat{W}$ and the new input price system as $\tilde{W}$, where

$$Z = \begin{bmatrix} z^V \\ z^F \end{bmatrix}, \hat{W} = \begin{bmatrix} -\hat{w}^V \\ w^F \end{bmatrix}, \tilde{W} = \begin{bmatrix} -\tilde{w}^V \\ w^F \end{bmatrix}.$$

Then $\tilde{W} \leq \hat{W}$, because $-\tilde{w}^V \leq -\hat{w}^V$. Let $\mathbb{S}$ be a sublattice of the product lattice $\mathbb{X}_V \times \mathbb{X}_F$, consider a constraint set $C \subseteq \mathbb{S}$, and define

$$(z^{*V}(w^V),\ z^{*F}(w^V)) = \sup \operatorname*{argmax}_{(z^V, z^F) \in C} \Pi(z^V, z^F, w^V)$$

$$z^{*V}(w^V, z^F) = \sup \operatorname*{argmax}_{\{z^V \in \mathbb{X}_V :\ (z^V, z^F) \in C\}} \Pi(z^V, z^F, w^V)$$

Note that $z^{*V}(\cdot) = (z_k^{*V}(\cdot))_{k \in V}$, $z^{*F}(\cdot) = (z_k^{*F}(\cdot))_{k \in F}$ and $z^{*V}(\cdot,\cdot) = (z_k^{*V}(\cdot,\cdot))_{k \in V}$. The constraint set is arbitrary in the problem definition. The theorem below considers changes in constraint sets in some directional set order. For example, consider two constraint sets $\hat{C} \subseteq \mathbb{S}$ and $\tilde{C} \subseteq \mathbb{S}$ such that $\hat{C} \sqsubseteq_j^{dso} C$ for some (subset of) inputs $j \in \mathbb{N}$. This occurs, for example, if there is a total cost budget $c > 0$ for a given planning horizon, that is, $\hat{C} = \{(z^V, z^F) \in \mathbb{S} : \hat{W} \cdot Z \leq c\}$ and $\tilde{C} = \{(z^V, z^F) \in \mathbb{S} : \tilde{W} \cdot Z \leq c\}$. It also occurs in the setup in Milgrom and Roberts

(1996). Moreover, the solution proposed above selects the largest maximizer. A similar result holds with the smallest optimizer as well.

Theorem 5. Suppose $\forall j \in F$, $\hat{C} \sqsubseteq_j^{dso} \tilde{C}$ and Π satisfies j-QSPM, j-SCP in $\mathbb{X}_F$, and basic j-SCP in (z^F, w^V).
For each $k \in V$, if $\hat{C} \sqsubseteq_k^{dso} \tilde{C}$, and Π satisfies k-QSPM and k-SCP in $\mathbb{X}_V$, and basic k-SCP in (z^V, z^F, w^V), then

$$z_k^{*V}(z^{*F}(\tilde{w}^V), \tilde{w}^V) \geq z_k^{*V}(z^{*F}(\hat{w}^V), \tilde{w}^V) \geq z_k^{*V}(z^{*F}(\hat{w}^V), \hat{w}^V).$$

Proof. The assumptions for fixed inputs imply that $\forall j \in F$, Π satisfies j-directional MCS in (z^F, w^V), and therefore, $z_j^{*F}(\cdot)$ is non-decreasing. Consequently, $\tilde{w}^V \geq \hat{w}^V \Rightarrow (\forall j \in F)\ z_j^{*F}(\tilde{w}^V) \geq z_j^{*F}(\hat{w}^V) \Rightarrow z^{*F}(\tilde{w}^V) \geq z^{*F}(\hat{w}^V)$. Moreover, $\hat{C} \sqsubseteq_k^{dso} \tilde{C}$ and Π satisfies k-QSPM, k-SCP in $\mathbb{X}_V$, and basic k-SCP in (z^V, z^F, w^V) implies that Π satisfies k-directional MCS in (z^V, z^F, w^V), and therefore, $z_k^{*V}(\cdot,\cdot)$ is non-decreasing. It follows that $z_k^{*V}(z^{*F}(\tilde{w}^V), \tilde{w}^V) \geq z_k^{*V}(z^{*F}(\hat{w}^V), \tilde{w}^V) \geq z_k^{*V}(z^{*F}(\hat{w}^V), \hat{w}^V)$, as required. □

This shows that the long run demand for variable input k is more responsive to a decrease in its price than the short run demand. This is a k-directional monotone comparatives statics for input k. The result is global in the sense that it is not restricted to small local changes in the price of the input. To our knowledge this is the first result to show global and directional Le Chatelier principle in models with infinitely many goods.

5 Appendix

Theorem 6. Suppose X is a partially ordered set, $(\mathbb{R}^X, \geq)$ is a lattice and $\mathbb{X}$ is a sub-lattice of $(\mathbb{R}^X, \geq)$. Let F, G be two non-empty subsets of $\mathbb{X}$. If $F \sqsubseteq_s G$ then $F \sqsubseteq_x^{dso} G$ for every $x \in X$.

Proof. Suppose $F, G \subset \mathbb{X}$ are two non-empty sets and $F \sqsubseteq_s G$. For any $x \in X$ fix $f \in F$ and $g \in G$ such that $f(x) > g(x)$. Since $F \sqsubseteq_s G$ we have $f \wedge g \in F$ and $f \vee g \in G$. Define $h = s[g - (f \wedge g)] = s[(f \vee g) - f] \in \mathbb{X}$ for $s \in [0,1]$. Choose $s = 1$. Then $g - h = g - g + f \wedge g = f \wedge g \in F$ and $f + h = f + f \vee g - f = f \vee g \in G$. Hence $F \sqsubseteq_x^{dso} G$. Since $x \in X$ is arbitrary, we have $F \sqsubseteq^{dso} G$ as required. $\square$

Theorem 7. $\hat{C} \sqsubseteq_t^{dso} \tilde{C}$ where $\hat{C} = \{(c_t)_{t\in\mathbb{Z}_+} : \frac{c_t + \gamma k_{t+1} - (1-\delta)k_t}{k_t^\alpha} \leq \hat{A}_t\}$ and $\tilde{C} = \{(c_t)_{t\in\mathbb{Z}_+} : \frac{c_t + \gamma k_{t+1} - (1-\delta)k_t}{k_t^\alpha} \leq \tilde{A}_t\}$.

Proof. Denote any $(c_t)_{t\in\mathbb{Z}_+} \in \hat{C}$, $(c_t)_{t\in\mathbb{Z}_+} \in \tilde{C}$ by $(c_t^{\hat{C}})_{t\in\mathbb{Z}_+}$,$(c_t^{\tilde{C}})_{t\in\mathbb{Z}_+}$. In particular $(c_t^{\hat{C}})_{t\in\mathbb{Z}_+}$ and $(c_t^{\tilde{C}})_{t\in\mathbb{Z}_+}$ can be written explicitly as $(c_0^{\hat{C}}, c_1^{\hat{C}}, ..., c_{t-1}^{\hat{C}}, c_t^{\hat{C}}, c_{t+1}^{\hat{C}}, ...)$ and $(c_0^{\tilde{C}}, c_1^{\tilde{C}}, ..., c_{t-1}^{\tilde{C}}, c_t^{\tilde{C}}, c_{t+1}^{\tilde{C}}, ...)$. Fix t and choose any $(c_t^{\hat{C}})_{t\in\mathbb{Z}_+} \in \hat{C}$ and $(c_t^{\tilde{C}})_{t\in\mathbb{Z}_+} \in \tilde{C}$ such that $c_t^{\hat{C}} > c_t^{\tilde{C}}$. Without the loss of generality assume that $c_j^{\tilde{C}} \geq c_j^{\hat{C}}$ $\forall j \in \mathbb{Z}_+, j \neq t$. Now we can write:

$$(c_0^{\hat{C}}, c_1^{\hat{C}}, ..., c_{t-1}^{\hat{C}}, c_t^{\hat{C}}, c_{t+1}^{\hat{C}}, ...) \wedge (c_0^{\tilde{C}}, c_1^{\tilde{C}}, ..., c_{t-1}^{\tilde{C}}, c_t^{\tilde{C}}, c_{t+1}^{\tilde{C}}, ...) = (c_0^{\hat{C}}, c_1^{\hat{C}}, ..., c_{t-1}^{\hat{C}}, c_t^{\tilde{C}}, c_{t+1}^{\hat{C}}, ...)$$

For any $s \geq 0$ define:

$$v = s\left[(c_t^{\tilde{C}})_{t\in\mathbb{Z}_+} - (c_0^{\hat{C}}, c_1^{\hat{C}}, ..., c_{t-1}^{\hat{C}}, c_t^{\tilde{C}}, c_{t+1}^{\hat{C}}, ...)\right] = \left(s\left[c_0^{\tilde{C}} - c_0^{\hat{C}}\right], ..., s\left[c_{t-1}^{\tilde{C}} - c_{t-1}^{\hat{C}}\right], 0, s\left[c_{t+1}^{\tilde{C}} - c_{t+1}^{\hat{C}}\right], ...\right)$$

Consequently

$$(c_t^{\hat{C}})_{t\in\mathbb{Z}_+} + v = \left(c_0^{\hat{C}} + s\left[c_0^{\tilde{C}} - c_0^{\hat{C}}\right], ..., c_{t-1}^{\hat{C}} + s\left[c_{t-1}^{\tilde{C}} - c_{t-1}^{\hat{C}}\right], c_t^{\hat{C}}, c_{t+1}^{\hat{C}} + s\left[c_{t+1}^{\tilde{C}} - c_{t+1}^{\hat{C}}\right], ...\right)$$

$$(c_t^{\tilde{C}})_{t\in\mathbb{Z}_+} - v = \left(c_0^{\tilde{C}} - s\left[c_0^{\tilde{C}} - c_0^{\hat{C}}\right], ..., c_{t-1}^{\tilde{C}} - s\left[c_{t-1}^{\tilde{C}} - c_{t-1}^{\hat{C}}\right], c_t^{\tilde{C}}, c_{t+1}^{\tilde{C}} - s\left[c_{t+1}^{\tilde{C}} - c_{t+1}^{\hat{C}}\right], ...\right)$$

Now at t, $\frac{c_t^{\hat{C}} + \gamma k_{t+1} - (1-\delta)k_t}{k_t^\alpha} \leq \tilde{A}_t$ as $\frac{c_t^{\hat{C}} + \gamma k_{t+1} - (1-\delta)k_t}{k_t^\alpha} \leq \hat{A}_t$ and $\hat{A}_t \uparrow \tilde{A}_t$. To show $\frac{\left(c_j^{\hat{C}} + s\left[c_j^{\tilde{C}} - c_j^{\hat{C}}\right]\right) + \gamma k_{j+1} - (1-\delta)k_j}{k_j^\alpha} \leq \tilde{A}_j$ $\forall j \in \mathbb{Z}_+, j \neq t$ choose $s = 1$. This implies $(c_t^{\hat{C}})_{t\in\mathbb{Z}_+} + v \in \tilde{C}$. Similarly at t, $\frac{c_t^{\tilde{C}} + \gamma k_{t+1} - (1-\delta)k_t}{k_t^\alpha} < \frac{c_t^{\hat{C}} + \gamma k_{t+1} - (1-\delta)k_t}{k_t^\alpha} \leq \hat{A}_t$ as $c_t^{\hat{C}} > c_t^{\tilde{C}}$. To show $\frac{\left(c_j^{\tilde{C}} - s\left[c_j^{\tilde{C}} - c_j^{\hat{C}}\right]\right) + \gamma k_{j+1} - (1-\delta)k_j}{k_j^\alpha} \leq \hat{A}_j$ $\forall j \in \mathbb{Z}_+, j \neq t$ choose $s = 1$. Therefore $(c_t^{\tilde{C}})_{t\in\mathbb{Z}_+} - v \in \hat{C}$. Hence $\hat{C} \sqsubseteq_t^{dso} \tilde{C}$ as required. $\square$

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