Characterizing arbitrage-free Choquet pricing rules

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Abstract

The fundamental theory of asset pricing has been developed under the two main assumptions that markets are frictionless and have no arbitrage opportunities. In order to take into account general types of frictions in financial markets, market pricing rules have been studied in the subadditive framework. Recently, Cerreia-Vioglio et al. (2015a,b) have extended the Fundamental Theorem of Finance by showing that, with market frictions, requiring the put-call parity to hold is equivalent to the market pricing rule being represented as a discounted Choquet nonadditive expectation with respect to a risk-neutral nonadditive probability, providing a framework in which pricing is not subadditive.

This paper continues the study of Choquet pricing rules that are neither assumed to be subadditive nor monotone and provides characterizations of arbitrage-free market pricing rules, in terms of the existence of a linear stochastic discount factor below the pricing rule, or in terms of the nonvacuity of the core of the risk-neutral nonadditive probability associated with the pricing rule. Interestingly though, this approach allows us to provide an alternative proof to Schmeidler's theorem on the nonvacuity of the core in an infinite dimensional model of cooperative game theory.

Key words: Pricing Rule, Market Frictions, Arbitrage-free Pricing, Choquet Pricing, Balancedness, Core.

JEL Classification Number: G12, D81, C71.

Dedicated to Elyès Jouini on the occasion of his 60th birthday.

1. Introduction

The fundamental theory of asset pricing has been developed under the two main assumptions that markets are frictionless and have no arbitrage opportunities. In this case the market enforces that replicable assets are valued by a linear function of their payoffs, or as the discounted expectation with respect to the so-called risk-neutral probability as in the Fundamental Theorem of Asset Pricing as in by Harrison and Kreps (1979), Delbaen and Schachermayer (1994, 2006).

Important evidence of frictions in financial markets (transaction costs, taxes, bid-ask spreads, etc) has led to study sublinear pricing rules that can account for frictions, as in Garman and Ohlson (1981), Prisman (1986), Ross (1987), Bensaid et al. (1992), Jouini and Kallal (1995, 1999, 2001), Jouini and Napp (2006), Campi, Jouini, and Porte (2013), Castagnoli et al. (2002), Araujo et al. (2012, 2018), Chateauneuf and Cornet (2018, 2022a, 2025). A financial market can be defined by a family of securities that allows investors to generate payoffs by choosing adequately the quantities of the securities that are bought or sold and then the market price is derived from the prices of these underlying given securities. Another approach consists in taking the market price as the primitive concept, which is thus supposed to be exogenously given and axiomaticaly defined.

This paper will consider the axiomatic approach as in the literature that study markets with frictions generating Choquet pricing rules that are only subadditive Chateauneuf et al. (1996), Chateauneuf and Cornet (2018). More recently, the study has been extended to consider market pricing rules that are neither sublinear nor subadditive as in Cerreia-Vioglio et al. (2015a,b) who have extended the Fundamental Theorem of Finance by showing that, with market frictions, requiring the put-call parity to hold, together with the mild assumption of translation invariance, is equivalent to the market pricing rule being represented as a discounted Choquet non-additive expectation with respect to a risk-neutral nonadditive probability, which is not submodular (concave). We also refer to Bastianello et al. (2024) who observed that the violation of the call-put parity, a condition considered by Chateauneuf et al. (1996) similar to the put call parity in Cerreia-Vioglio et al. (2015a), is consistent with the existence of bid-ask spreads.

This paper extends previous work by Chateauneuf and Cornet (2022b), Bastianello et al. (2024), that was considering either the finite dimensional case and/or the monotone case, and characterizes two notions of arbitrage-free pricing rules f that are not assumed to be subadditive, that is, the absence of arbitrage opportunities that can be obtained by splitting payoffs in parts or in weighted parts. Our first characterization property is the existence of a linear continuous nonnegative function μ below the pricing rule f. Interpreting μ as a stochastic discount factor that allows to evaluate the present value of tomorrow payoffs x, then the pricing rule is arbitrage-free if and only if the present value $\mu(x)$ of every payoff is not greater than its cost $f(x)$ evaluated by the pricing rule f. Our second characterization property is the balancedness of the unique risk-neutral nonadditive probability v_f associated with the Choquet pricing rule f when f is assumed to be monotone, or equivalently the non-vacuity of the core of v_f, that is, the set of all *additive* probabilities below v_f. Then the associated (standard) discounted expectations of payoffs with respect to probabilities below v_f are all below the Choquet pricing rule f that values payoffs as their discounted nonadditive expectation with respect to v_f.

Our study will consider the case of monotone Choquet pricing rules f, thus allowing us to consider their associated capacities v_f as non-additive probabilities whenever they are normalized, but also the case of non monotone Choquet pricing rules by removing the nonnegative requirement on v_f and on the elements in the core of v_f. This latter case requires to consider weaker notions of arbitrage-free pricing rules that eliminate multiple buy & sell arbitrage opportunities having an aggregate payoff which is null in the future and having a gain today. Interestingly though the study of Choquet pricing rules in a nonmonotone setting allows us to deduce directly Schmeidler (1968)'s characterization of the nonvacuity of the core of a cooperative game v by the balancedness of v in an infinite-dimensional setting and provides an alternative proof. Our proof uses the pricing rule setting, that allows to extend every game v by a Choquet pricing rule f that is arbitrage-free whenever the game v is balanced, and thus deduce the existence of a stochastic discount factor below f, which can be easily proved to be an element in the core of v.

The paper is organized as follows. Section 2 introduces the model and our results. The model and the notion of pricing rules are presented in Section 2.1 in an infinite dimensional setting that allows to consider Choquet pricing rules. In Section 2.2 we define several notions of arbitrage-free pricing rules and show some equivalence properties (Proposition 1). Section 2.3 states our main results (Theorem 1 and Theorem 2) characterizing arbitrage-free pricing rules in terms of the existence of a stochastic discount factor below f. Section 2.4 defines the risk-neutral game (resp. capacity, resp. non-additive probability) together with the notion of core associated with a pricing rule. Section 2.5 states our main results (Theorem 3 and Theorem 4) characterizing arbitrage-free Choquet pricing rules in terms of the nonvacuity of their core or by the existence of a stochastic discount factor below f. Applications are then given to cooperative game theory and Section 2.6 provides a new proof of Schmeidler's result (Theorem 5) characterization of the nonvacuity of the core of a game by the balancedness of the game. Finally, Section 3 gathers proofs of our results.

2. The model and the results

2.1. The model and notations

This paper considers the simple framework of a stochastic two-date model: $t = 0$ (today) is known, $t = 1$ (tomorrow) is uncertain, and a single commodity (money) is available today and tomorrow at every state. Uncertainty is represented by a nonempty set Ω (finite or infinite) of *states of nature* and one, and only one, state of nature will be realized and known tomorrow. Throughout the paper, $(\Omega, \mathcal{A})$ will be a measurable space, that is, a pair of an arbitrary nonempty set Ω, and a σ-algebra $\mathcal{A} \subseteq 2^{\Omega}$ of subsets of Ω, that is, $\mathcal{A}$ contains the whole set Ω and is stable by countable union and complement. Note that we do not assume there is a reference probability defined on $(\Omega, \mathcal{A})$ as in Riedel (2015), Burzoni Riedel and Soner (2021).

A payoff is a random variable $x : \Omega \to \mathbb{R}$ that is measurable, i.e., $x^{-1}(I) \in \mathcal{A}$ for every interval $I \subseteq \mathbb{R}$ and bounded, i.e., $\|x\|_\infty := \sup\{|x(\omega)| : \omega \in \Omega\} < +\infty$. We denote by $X := B(\mathcal{A})$ the set of bounded measurable functions $x : \Omega \to \mathbb{R}$ and by $B_0(\mathcal{A})$ the subspace of $X := B(\mathcal{A})$ of the simple measurable functions, i.e., measurable functions $x : \Omega \to \mathbb{R}$ whose set of values $\{x(\omega) : \omega \in \Omega\}$ is finite. We recall that $X := B(\mathcal{A})$ is a Banach lattice for the coordinate-wise order $x \geq x'$, defined by $x(\omega) \geq x'(\omega)$ for all $\omega \in \Omega$, we also define the strict order $x > x'$ by $x \geq x'$ and $x \neq x'$, and we define $x \vee y$ by $(x \vee y)(\omega) := \max\{x(\omega), y(\omega)\}$ for all $\omega \in \Omega$ and $x \wedge y$ by $(x \wedge y)(\omega) := \min\{x(\omega), y(\omega)\}$ for all $\omega \in \Omega$. We let $X_+ :=$

$\{x \in B(\mathcal{A}) : x \geq 0\}$ (resp. $B_0^+(\mathcal{A}) := \{x \in B_0(\mathcal{A}) : x \geq 0\}$) be the set of non-negative bounded measurable functions (resp. non-negative simple measurable functions). We adopt the standard convention that if $x(\omega) < 0$ then $|x(\omega)|$ is the amount of money paid by the agent for buying the single commodity (money) if state ω prevails, and if $x(\omega) > 0$ then $|x(\omega)|$ is the gain received for selling the single commodity (money) if state ω prevails. Thus, if $x \geq 0$, the payoff x is buying $x(\omega) \geq 0$ at every state ω, and $-x$ is the payoff selling the quantity $|x(\omega)| \geq 0$ at every state; more generally, the same terminology will be used for arbitrary payoffs $x \in X$. For every $A \in \mathcal{A}$ we denote by $\mathbf{1}_A$ the payoff defined by $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ otherwise, and we set $\mathbf{1}_\emptyset = 0$.

Then X' denotes the topological dual space of X, that is, the space of continuous linear real-valued functions $\mu : X \to \mathbb{R}$. In X' the order $\mu \geq \mu'$ is defined by $\mu(x) \geq \mu'(x)$ for all $x \in X_+$ and we let $X'_+ := \{\mu \in X' : \mu(x) \geq 0$ for all $x \in X_+\}$. We also define the strict order $\mu > \mu'$ by $\mu \geq \mu'$ and $\mu \neq \mu'$. Clearly, the case Ω finite, is a particular case of the measurable one, taking $\mathcal{A} = 2^\Omega$; then every function $x : \Omega \to \mathbb{R}$ is measurable, bounded, and simple.

To every payoff $x \in X$, is assigned the market price $f(x)$ to be *paid* at $t = 0$, for the delivery of the random payoff x at $t = 1$, with the convention that the quantity $|f(x)|$ is *paid* if $f(x) > 0$ and is a *gain* if $f(x) < 0$. In other words, $f(x)$ is the *buying (ask) price*, that is the price *paid* today to *buy* x tomorrow, and $-f(-x)$ is the *(bid) selling price*, that is, the quantity of money *received* today to *sell* x tomorrow.

This paper will take the real-valued function $f : X \to \mathbb{R}$ as a primitive concept and it is called pricing rule under the unique assumption that $f(0) = 0$, that is, there is no payment and no gain today in getting nothing tomorrow. Standard definitions are recalled below.

Definition 1. *Let $(\Omega, \mathcal{A})$ be a measurable space, let $X := B(\mathcal{A})$ be the payoff space of real-valued bounded measurable functions defined on Ω, a pricing rule is a real-valued function $f : X \to \mathbb{R}$ satisfying $f(0) = 0$. Then*

f is monotone if $x \leq x'$ implies $f(x) \leq f(x')$ for all x, x' in X;

f is subadditive if $f(x + x') \leq f(x) + f(x')$ for all x, x' in X;

f is positively homogeneous if $f(tx) = tf(x)$ for all $x \in X$ and all $t \in \mathbb{R}_+$;

f is sublinear if f is subadditive and positively homogeneous,

or equivalently if

$f(\alpha x + \beta x') \leq \alpha f(x) + \beta f(x')$ for all x, x' in X, all $\alpha \geq 0$, $\beta \geq 0$;

f is sublinear at $x \in X$ if

$\forall K \in \mathbb{N}, K \geq 1$, $\forall (x_1, \ldots, x_K) \in X^K, \forall (\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K$,

$\theta_1 x_1 + \cdots + \theta_K x_K = x \Rightarrow \theta_1 f(x_1) + \cdots + \theta_K f(x_K) \geq f(x)$*;*

and f is sublinear if and only if f is sublinear at every $x \in X$;

f is submodular if $f(x \vee x') + f(x \wedge x') \leq f(x) + f(x')$ for all x, x' in X.

2.2. Arbitrage-free pricing rules

Heuristically, a payoff $x \in X$ is an *arbitrage opportunity* for the pricing rule $f : X \to \mathbb{R}$ if "it allows to make money from nothing." When the pricing rule f is *subadditive*, the pricing rule $f : X \to \mathbb{R}$ is usually said to be arbitrage-free if $x \geq 0$ implies $f(x) \geq 0$, or equivalently, if there is no payoff $x \geq 0$ such that $f(x) < 0$, that is, having no payment at each state tomorrow and yielding a gain today (or equivalently having a negative cost).

The subadditivity assumption on f implies that there is no *"buy and sell" arbitrage opportunity,* that is, there is no gain of buying any payoff x and selling all of it (resp. part of it), that is, for all $x \in X$, $f(x) + f(-x) \geq 0$ (resp. $f(x) + f(-x') \geq 0$ for all $x' \leq x$). This amounts to say that the *aggregate cost*, $f(x)+f(-x)$, also called *bid-ask spread,* is not negative, thus does not yield a gain. Equivalently, there is no couple $(x, -x) \in X^2$ such that $f(x) < -f(-x)$, that is, $f(x)$, the *cost* of *buying* x (or the ask price of x) is smaller than $-f(-x)$, the *gain* of *selling* the same payoff x (the bid price of x).

In fact, the subadditivity assumption on f eliminates more general situations of *"multiple buy and sell"* arbitrage opportunities, involving K-list of payoffs $(x_1, \ldots, x_K) \in X^K$ $(K \in \mathbb{N}, K > 2)$ yielding an aggregate gain at $t = 0$, that is, $f(x_1) + \cdots + f(x_K) < 0$ and such that $x_1 + \cdots + x_K = 0$, thus yielding a zero aggregate payoff at $t = 1$, or more generally such that $x_1 + \cdots + x_K \geq 0$, thus yielding a nonnegative aggregate payoff at $t = 1$.

When the pricing rule f is no longer assumed to be subadditive, in order to eliminate the above arbitrage opportunities that are automatically eliminated by the subadditivity assumption, the pricing rule f is said to be arbitrage-free if there are no such *"multiple buy and sell" arbitrage opportunities,* with two different notions depending whether we assume the aggregate payoff to be zero or nonnegative. We now formally define the two notions of arbitrage-free pricing rules $f : X \to \mathbb{R}$ that were previously considered by Chateauneuf and Cornet (2022b) and Bastianello et al. (2024).

Arbitrage-free Condition AF₊: $\forall K \in \mathbb{N}, K \geq 1$, $(x_1, \ldots, x_K) \in X^K$,

$$x_1 + \cdots + x_K \geq 0 \Rightarrow f(x_1) + \cdots + f(x_K) \geq 0.$$

The following definition requires the aggregate payoff $x_1 + \cdots + x_K$ to be zero instead of being nonnegative as above.

Arbitrage-free Condition AF: $\forall K \in \mathbb{N}, K \geq 1$, $(x_1, \ldots, x_K) \in X^K$,

$$x_1 + \cdots + x_K = 0 \Rightarrow f(x_1) + \cdots + f(x_K) \geq 0.$$

We also introduce the following two notions of Arbitrage-free pricing rules that differ from the previous ones by considering list of payoffs to which nonnegative weights are associated.

Arbitrage-free Condition CAF₊: $\forall K \in \mathbb{N}, K \geq 1$, $(x_1, \ldots, x_K) \in X^K$, $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K, \theta_1 x_1 + \cdots + \theta_K x_K \geq 0 \Rightarrow \theta_1 f(x_1) + \cdots + \theta_K f(x_K) \geq 0$.

Arbitrage-free Condition CAF: $\forall K \in \mathbb{N}, K \geq 1$, $(x_1, \ldots, x_K) \in X^K$, $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K, \theta_1 x_1 + \cdots + \theta_K x_K = 0 \Rightarrow \theta_1 f(x_1) + \cdots + \theta_K f(x_K) \geq 0$.

The basic properties of the above Arbitrage-free Conditions are summarized in the following proposition. First CAF (resp. CAF₊) is stronger than AF (resp. CAF) and is stronger than AF₊ but in fact they are equivalent under a mild continuity assumption, and this equivalence property will be of fundamental use in the following. To illustrate this equivalence, consider a pair of payoffs (x_1, x_2) such that $\theta_1 x_1 + \theta_2 x_2 = 0$ with weights $(\theta_1, \theta_2) \in \mathbb{R}_+^2$, say $\theta_1 := \frac{2}{3}, \theta_2 = 1$. Then the Arbitrage-free Condition AF implies that $f(\frac{2}{3}x_1) + f(x_2) \geq 0$ but it also implies that $\frac{2}{3}f(x_1) + f(x_2) \geq 0$ since AF and $x_1 + x_1 + x_2 + x_2 + x_2 \geq 0$ imply $2f(x_1) + 3f(x_2) \geq 0$. This argument can be generalized for any K-list, with a limit argument to go from rational numbers to real numbers; see the complete argument in Section 3.1.

Proposition 1. *Let $f : X \to \mathbb{R}$ be a pricing rule.*

1. *f satisfies AF₊ (resp. CAF₊) $\Rightarrow$ f satisfies AF (resp. CAF);*
2. *AF $\Leftrightarrow$ CAF if f is bounded above on some neighborhood of 0;*
   AF₊ $\Leftrightarrow$ CAF₊;
3. *If f is monotone, then AF₊ $\Leftrightarrow$ AF and CAF₊ $\Leftrightarrow$ CAF;*
4. *If f is subadditive, then f satisfies AF,*
   AF₊ $\Leftrightarrow$ $x \geq 0$ implies $f(x) \geq 0$;
5. *If f is subadditive and monotone, then f satisfies AF₊.*

The proof of Proposition 1 is given in Section 3.1. Note that the bounded assumption in Assertion (2) holds under each of the following conditions that will be considered hereafter: (i) f is monotone, (ii) f is continuous at 0.

2.3. Characterizing arbitrage-free pricing rules

Our first result shows that the equivalent Arbitrage-Free Conditions AF and CAF are characterized by the existence of a continuous linear function $\mu \in X'$ below the pricing rule f. If such a $\mu \in X', \mu \leq f$ exists, then we can define the function $c^* : X \to \mathbb{R} \cup \{-\infty\}$ by

$$c^*(x) = \sup\{\mu(x) : \mu \in X', \mu \leq f\} \quad \text{for all } x \in X$$

and one can prove the Property (P) c^* is the greatest finite-valued, sublinear, continuous function below f, together with the nonvacuity of its subdifferential at 0, that is, $\partial c^*(0) := \{\mu \in X' : \mu \leq c^*\} = \{\mu \in X' : \mu \leq f\} \neq \emptyset$ that follows from Convex Analysis (e.g. Theorem 7 hereafter).

Of course the proof cannot be made with c^* since (P) relies on the existence of $\mu \in X', \mu \leq f$. The idea of the proof is thus to make a dual argument[1] and introduce the following function $c(f) : X \to \mathbb{R} \cup \{-\infty, \}$ that will be shown to satisfy the above property (P) (and to coincide with c^* as well). The function $c(f) : X \to \mathbb{R} \cup \{-\infty\}$, simply denoted c when there is no risk of confusion, is defined by:

$$c(x) := \begin{array}{l} \inf \sum_{k=1}^{K} \theta_k f(x_k) \\ \sum_{k=1}^{K} \theta_k x_k = x \\ (x_1, \ldots, x_K) \in X^K \\ (\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K \\ K \in \mathbb{N}, K \geq 1. \end{array}$$

Note that the function $c : X \to \mathbb{R} \cup \{-\infty\}$ can take the value $-\infty$, that is, c can have an infinite gain at some payoffs, but this situation will be excluded by the following theorem under the equivalent Arbitrage-free Conditions AF or CAF. We also point that for every payoff $x \in X$, $c(x)$ is the smallest cost that can be obtained by the agents when splitting the payoff x in parts, that is, $x = \sum_{k=1}^{K} \theta_k x_k$, and then evaluate the aggregate cost of the parts as $\sum_{k=1}^{K} \theta_k f(x_k)$. The function $c : X \to \mathbb{R}$ thus defines a new pricing rule, which is the "best" sublinear pricing rule below f.

[1]In fact, a primal argument if we consider that the previous one was a dual argument

We can now state our first result that characterizes the Arbitrage-free Conditions AF and CAF by the existence of a linear continuous function $\mu \in X'$ below f. Interpreting μ as a stochastic discount factor that allows to evaluate the present value of tomorrow payoffs x, then the pricing rule f is arbitrage-free if and only if the present value $\mu(x)$ of every payoff is not greater than its cost $f(x)$ evaluated by the pricing rule f. See also Section 2.4 and 2.5 for a probabilistic interpretation of μ and f.

Theorem 1. *Let $f : X \to \mathbb{R}$ be a pricing rule that is bounded above on some neighborhood of 0. Then the following assertions are equivalent.*

1. *f satisfies the Arbitrage-free Condition AF;*
2. *f satisfies the Arbitrage-free Condition CAF;*
3. *there exists $\mu \in X', \mu \leq f$.*

Moreover, under either Condition (1), (2), or (3), the following holds:

4. $\{\mu \in X' : \mu \leq f\} = \{\mu \in X' : \mu \leq c\}$ *and both sets are nonempty, convex, compact for the weak$*$ topology;*
5. *the function $c : X \to \mathbb{R}$ is the greatest finite-valued, continuous, sublinear function below f;*
6. $c(x) = \max_{\mu \leq c} \mu(x) = \max_{\mu \leq f} \mu(x) \leq f(x)$ *for all $x \in X$. and the equality holds at $x \in X$ if and only if f is sublinear at x;*
7. $\{\mu(x) : \mu \in X', \mu \leq f\} = [-c(-x), c(x)] \subseteq [-f(-x), f(x)]\ \forall x \in X.$

The proof of Theorem 1, given in Section 3.2, will show that (i) the function $c : X \to \mathbb{R}$ is finite-valued, sublinear, and continuous (Proposition 5), and thus (ii) the subdifferential of c at 0 is nonempty (Theorem 7), and $\emptyset \neq \partial c(0) = \{\mu \in X' : \mu \leq f\}$, together with the remaining statements of the theorem.

We now define $c_+(f) : X \to \mathbb{R} \cup \{-\infty\}$, simply denoted c_+ when there is no risk of confusion; see also Remark 1.

$$c_+(x) := \begin{array}{l} \inf \sum_{k=1}^{K} \theta_k f(x_k) \\ \sum_{k=1}^{K} \theta_k x_k \geq x \\ (x_1, \ldots, x_K) \in X^K \\ (\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K \\ K \in \mathbb{N}, K \geq 1 \end{array}$$

We can now state our second result that characterizes the equivalent Arbitrage-free Condition AF_+ and CAF_+ by the existence of a continuous linear function $\mu \in X'$ below the pricing rule f, with μ now assumed to be nonnegative. See also Section 2.4 and 2.5 for a probabilistic interpretation of μ and f.

Theorem 2. *Let $f : X \to \mathbb{R}$ be a pricing rule, then the following assertions are equivalent:*

1. *f satisfies the Arbitrage-free Condition AF_+;*
2. *f satisfies the Arbitrage-free Condition CAF_+;*
3. *there exists $\mu \in X', \mu \geq 0, \mu \leq f$.*

Moreover, under either Condition (1), (2), or (3), the following holds:

4. *$\{\mu \in X' : \mu \geq 0, \mu \leq f\} = \{\mu \in X' : \mu \geq 0, \mu \leq c\}$ and both sets are nonempty, convex, compact for the weak$*$ topology;*
5. *$c_+ : X \to \mathbb{R}$ is the greatest finite-valued, monotone, continuous, sublinear function below f;*
6. *$c_+(x) = \max\{\mu(x) : \mu \geq 0, \mu \leq f\} \leq c(x) \leq f(x)$ for all $x \in X$;*
7. *$[-c_+(-x), c_+(x)] \subseteq [-c(-x), c(x)] \subseteq [-f(-x), f(x)]\ \forall x \in X$.*

The proof is given in the Section 3.3 as a consequence of Theorem 1.

The next result considers the case where f is a monotone pricing rule. Then, the Arbitrage-free Conditions AF and AF$_+$ are equivalent and CAF and CAF$_+$ are also equivalent.

Corollary 1 (f monotone)**.** *Let $f : X \to \mathbb{R}$ be* **monotone** *pricing rule, then the following assertions are equivalent:* (1) *f satisfies AF;* (2) *f satisfies AF$_+$;* (3) *f satisfies CAF;* (4) *f satisfies CAF$_+$;* (5) *there exists $\mu \geq 0$ such that $\mu \leq f$.*

Proof. The proof of Corollary 1 is a direct consequence of Theorem 2 and the equivalences [AF $\Leftrightarrow$ AF$_+$] and [CAF $\Leftrightarrow$ CAF$_+$] proved by Proposition 1 when f is monotone. Moreover, when f is monotone one can easily prove that every stochastic discount factor $\mu \in X'$ below f satisfies $\mu \geq 0$. $\square$

2.4. The risk-neutral nonadditive probability

The stochastic model is represented by a measurable space $(\Omega, \mathcal{A})$ as previously and we now define a **game** as a real-valued set-function $v : \mathcal{A} \to \mathbb{R}$ satisfying the only requirement that $v(\emptyset) = 0$. Following Aumann and Shapley (1974), the variation norm of a game v is defined as follows:

$$\|v\| := \sup \big\{ \textstyle\sum_{k=1}^K |v(A_k) - v(A_{k-1})| \,:\, (A_k)_{k=0}^K \text{ finite chain in } \mathcal{A} \big\},$$

where, for finite chain $(A_k)_{k=0}^K$ in $\mathcal{A}$, $A_k \in \mathcal{A}$ for all $k = 1, \dots, K$ and $\emptyset = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_K = \Omega$. We let:

$$bv(\mathcal{A}) := \big\{ v : \mathcal{A} \to \mathbb{R} \,:\, v(\emptyset) = 0 \text{ and } \|v\| < +\infty \big\},$$

be the vector space of games having a finite variation norm, which is indeed a norm on $bv(\mathcal{A})$ that is a Banach space for this norm. It is worth pointing out that $bv(\mathcal{A})$ contains several important classes of games, namely the class of finite games, the class of capacities, i.e., monotone games (as defined below), and also the set $ba(\mathcal{A})$ of bounded signed charges, i.e., additive games of bounded variation.[2] We recall that the mapping $\hat{\mu} \to \mu$, from $X' := B(\Omega)'$ to $\mathrm{ba}(\mathcal{A})$, defined by $\mu(A) := \hat{\mu}(\mathbf{1}_A)$ is an isometry satisfying $\hat{\mu}(x) = \int_\Omega x \, d\mu$ for all $x \in X$. Throughout the paper the two spaces X' and $\mathrm{ba}(\mathcal{A})$ will be identified and $\hat{\mu}$ and μ will only be distinguished when there is risk of confusion. For all these properties we refer to Aumann and Shapley (1974) and Marinacci and Montrucchio (2004).

A *capacity* v on the measurable space $(\Omega, \mathcal{A})$ is a game $v : \mathcal{A} \mapsto \mathbb{R}$, which is monotone, that is, $A, B \in \mathcal{A}$ and $A \subseteq B$ imply $v(A) \leq v(B)$. The capacity is said to be normalized if $v(\Omega) = 1$ and a normalized capacity is also called a *non-additive probability*. Then a *probability* μ is a normalized capacity which is additive, that is, A, B in $\mathcal{A}$ and $A \cap B = \emptyset$ imply $\mu(A \cup B) = \mu(A) + \mu(B)$. The core of a game v is then defined as the set:

$$\mathrm{core}\,(v) := \big\{ \mu \in ba(\mathcal{A}) \,:\, \mu(A) \leq v(A) \, \forall A \in \mathcal{A} \text{ and } \mu(\Omega) = v(\Omega) \big\}.$$ [3]

We now introduce the following Balancedness Conditions BA and BA_+ for the set function $v : \mathcal{A} \to \mathbb{R}$:

Balancedness (BA) For all positive integer K, all $(\theta_1, \dots, \theta_K) \in \mathbb{R}_+^K$, and all $(A_1, \dots, A_K) \in (\mathcal{A})^K$, $\sum_{k=1}^K \theta_k \mathbf{1}_{A_k} = \mathbf{1}_\Omega \Rightarrow \sum_{k=1}^K \theta_k v(A_k) \geq v(\Omega)$.

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[2] Indeed $bv(\mathcal{A})$ contains the class of all finite games (i.e., Ω is finite) since there are finitely many finite chains. Moreover, $bv(\mathcal{A})$ contains all capacities $v : \mathcal{A} \mapsto \mathbb{R}$ since, using the monotonicity of v, for all finite chains (A_k), one gets $\sum_{k=1}^K |v(A_k) - v(A_{k-1})| = \sum_{k=1}^K v(A_k) - v(A_{k-1}) = v(\Omega) - v(\emptyset) = v(\Omega)$. Consequently, $\|v\| = v(\Omega) < \infty$. An additive game μ is also called a signed charge and we point out that the total variation norm of a signed charge $\|\mu\|$ is exactly the variation norm of the game μ defined above.

[3] This set is sometimes called anticore to distinguish it from the related concept of core when it is defined with the reverse inequality, as in cooperative game theory.

Balancedness (BA$_+$) For all positive integer $K \in \mathbb{N}$, all $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K$, and all $(A_1, \ldots, A_K) \in (\mathcal{A})^K$, $\sum_{k=1}^K \theta_k \mathbf{1}_{A_k} \geq \mathbf{1}_\Omega \Rightarrow \sum_{k=1}^K \theta_k v(A_k) \geq v(\Omega)$.

Let $f : X \to \mathbb{R}$ be a pricing rule, we associate the set function $v_f : \mathcal{A} \to \mathbb{R}$ defined by:

$$v_f(A) := f(\mathbf{1}_A) \text{ for all } A \in \mathcal{A},$$

with the convention that $\mathbf{1}_\emptyset = 0$. Thus v_f is a game since $v_f(\emptyset) := f(\mathbf{1}_\emptyset) = f(0) = 0$. We define the core of f as the core of the set function v_f:

$$\text{core}(f) := \text{core}(v_f).$$

We will say that the pricing rule f satisfies the Balancedness Condition BA (resp. BA$_+$) if the game $v_f : \mathcal{A} \to \mathbb{R}$ associated to f satisfies the Balancedness Condition BA (resp. BA$_+$) defined previously.

When the bond $\mathbf{1}_\Omega$ is frictionless, that is, $f(\mathbf{1}_\Omega) = -f(-\mathbf{1}_\Omega)$ (a property satisfied when f is a Choquet pricing rule as in Section 2.5), the following proposition shows a first relationship between (i) the Arbitrage-free Conditions and the Balancedness Conditions, (ii) the sets $\{\mu \in X' : \mu \leq f\}$ and core(f) (and Proposition 3 below will prove the equivalence when f is a Choquet pricing rule). Note that the above sets are not in the same space, but the two isometric spaces X' and ba$(\mathcal{A})$ have been identified.

Proposition 2. *Let $f : X \to \mathbb{R}$ be a pricing rule, whose bond $\mathbf{1}_\Omega$ is frictionless, that is, satisfying $f(\mathbf{1}_\Omega) = -f(-\mathbf{1}_\Omega)$.*
1. f satisfies CAF (resp. CAF$_+$) $\Rightarrow$ f satisfies BA (resp. BA$_+$);
2. $\{\mu \in X' : \mu \leq f\} \subseteq \text{core}(f)$;
3. $f \neq 0$ implies $f(\mathbf{1}_\Omega) > 0$ if f is monotone, satisfies CAF$_+$ and $f(k\mathbf{1}_\Omega) = kf(\mathbf{1}_\Omega)$ for all $k > 0$.

The proof of Proposition 2 is given in Section 3.4.

Whenever $f(\mathbf{1}_\Omega) > 0$, one can associate to every monotone pricing rule $f : X \to \mathbb{R}$ the risk-neutral non-additive probability $\tilde{v}_f : \mathcal{A} \to \mathbb{R}$ and the risk-free interest rate $r > -1$, each uniquely defined by

$$\tilde{v}_f(A) := f(\mathbf{1}_A)/f(\mathbf{1}_\Omega) \text{ with } r = (1/f(\mathbf{1}_\Omega)) - 1.$$

Thus, core$(v_f) = (1+r)$core$(\tilde{v}_f)$ and core$(\tilde{v}_f)$ is now a set of risk-neutral (additive) probabilities whenever v_f is monotone:

$$\text{core}(\tilde{v}_f) = \{P \in \text{ba}(\mathcal{A}) : P(\Omega) = 1, P(A) \leq \tilde{v}_f(A)\ \forall A \in \mathcal{A}\}.$$

The next section will show the relationship between a monotone Choquet pricing rule f and the associated set core$(\tilde{v}_f)$ of its risk-neutral probabilities. In the following, the games v_f and $\tilde{v}_f$ associated with f will be simply denoted v and $\tilde{v}$, respectively, when there is no risk of confusion.

2.5. Characterizing arbitrage-free Choquet pricing rules

The study of Choquet pricing rules is justified by their relationship with the Put-Call and Call-Put parities as demonstrated by Chateauneuf et al. (1996), Cerreia-Vioglio et al. (2015a), Bastianello et al. (2024) in the monotone case and by Cerreia-Vioglio et al. (2015b) for pricing rules that are not assumed to be monotone. This section will provide several characterization properties of arbitrage-free Choquet pricing rules, when they are neither assumed to be subadditive nor monotone. The function f is said to be a *Choquet pricing rule* with respect to the game $v \in bv(\mathcal{A})$ if, for all $x \in X$:

$$f(x) = \int_{\Omega}^{C} x\, dv := \int_{-\infty}^{0} (v(\{x \geq t\}) - v(\Omega))\, dt + \int_{0}^{+\infty} v(\{x \geq t\})\, dt,$$

where $\{x \geq t\} := \{\omega \in \Omega : x(\omega) \geq t\}$. The notation $\int_{\Omega}^{C}$ indicates that we consider a Choquet integral. We will use the notation $\int_{\Omega}$ for the standard integral with respect to a bounded signed charge $\mu \in ba(\mathcal{A})$, and in this case, the function $x \to \int_{\Omega} x\, d\mu$ is *linear*. Moreover, the game $v_f \in bv(\mathcal{A})$, called risk-neutral game, associated with the Choquet pricing rule f is uniquely defined by $v_f(A) := f(\mathbf{1}_A)$ for all $A \in \mathcal{A}$ and the Choquet pricing rule $f : X \to \mathbb{R}$ is continuous. For a general reference of Choquet integral we refer to Marinacci and Montrucchio (2004) and Denneberg (1994).

The next result shows that, for a Choquet pricing rule f, the equivalence between the Arbitrage-free Conditions AF and CAF and the existence of $\mu \in X'$ below f, proved in Theorem 1, can be extended to two additional conditions, the nonvacuity of the core of f and the balancedness of v_f.

Theorem 3 (Choquet Pricing Rule)**.** *Let $f : X \to \mathbb{R}$ be a Choquet pricing rule with respect to $v \in bv(\mathcal{A})$, then the following assertions are equivalent:*

1. *f satisfies the Arbitrage-free Condition AF;*
2. *f satisfies the Arbitrage-free Condition CAF;*
3. *there exists $\mu \in X', \mu \leq f$;*
4. core$(v) \neq \emptyset$*;*
5. *v satisfies the Balancedness Condition BA.*

Moreover, under one of the equivalent Assertions (1)-(5) one has

$$c(x) = \sup_{P \in \mathrm{core}\,(\tilde{v})} \frac{1}{1+r} \int_\Omega x\, dP \leq \frac{1}{1+r} \int_\Omega^C x\, d\tilde{v} =: f(x) \text{ for all } x \in X$$

and the equality holds at x whenever the pricing rule is subadditive at $x \in X$.

Recalling that, if f is a Choquet pricing rule, the following assertions are equivalent (i) f is subadditive, (ii) f is submodular, and (iii) v is submodular, e.g., (Marinacci and Montrucchio, 2004, Theorem 35), under one of these assumptions we immediately deduce that

$$\sup_{P \in \mathrm{core}\,(\tilde{v})} \frac{1}{1+r} \int_\Omega x\, dP = \frac{1}{1+r} \int_\Omega^C x\, d\tilde{v} =: f(x) \text{ for all } x \in X.$$

The above equality is due to Schmeidler (1986) for positive games and to De Waegenaere and Wakker (2001) for finite games; see also Delbaen (1974) and Marinacci and Montrucchio (2003).

Proof of Theorem 3. • First, the Choquet pricing rule $f : X \to \mathbb{R}$ with respect to $v \in bv(\mathcal{A})$ is continuous, hence f is bounded above on a neighborhood of 0. Thus, Theorem 1 implies the following equivalences:

(1) [f satisfies AF] $\Leftrightarrow$ (2) [f satisfies CAF] $\Leftrightarrow$ (3) [$\{\mu \in X' : \mu \leq f\} \neq \emptyset$].

Second, Proposition 3 below proves that

(3) $\Rightarrow$ (4) [$\mathrm{core}\,(v) \neq \emptyset$] $\Rightarrow$ (5) [v satisfies BA] $\Rightarrow$ (2) [f satisfies CAF].

• We now prove the last assertion of the theorem. First, from Assertion (6) of Theorem 1, we get $c(x) = \sup\{\mu(x) : \mu \in X', \mu \leq f\}$ and from Proposition 3 we have the equality $\{\mu \in X' : \mu \leq f\} = \mathrm{core}\,(v)$. Thus the first part of the statement follows.

Moreover, the Choquet pricing rule f is subadditive at $x \in X$ if and only if it is sublinear at x (since f is positively homogeneous), thus if and only if

$$f(x) \leq c(x) := \begin{array}{l} \inf \sum_{k=1}^K \theta_k f(x_k) \\ \sum_{k=1}^K \theta_k x_k = x \\ (x_1, \ldots, x_K) \in X^K \\ (\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K \\ K \in \mathbb{N}^* \end{array}$$

which is equivalent to the equality $c(x) = f(x)$ (since $c(x) \leq f(x)$). $\square$

Proposition 3. *Let v be a game belonging to $bv(\mathcal{A})$ and let $f : X \to \mathbb{R}$ be the Choquet pricing rule with respect to v.*
1. $\{\mu \in X' : \mu \leq f\} = \mathrm{core}(v)$*;*
2. If $\mathrm{core}(v) \neq \emptyset$*, then v satisfies BA;*
If $\mathrm{core}(v) \cap \{\mu : \mu \geq 0\} \neq \emptyset$*, then v satisfies BA_+;*
3. v satisfies BA $\Rightarrow$ f satisfies CAF.
v satisfies BA_+ $\Rightarrow$ f satisfies CAF_+.

The proof of the proposition is given in Section 3.4.

We now state a second result that characterizes the Arbitrage-free Condition AF_+ and CAF_+ by the existence of $\mu \in X'$ below f or μ in the core of f, but μ is now additionally assumed to be nonnegative.

Theorem 4. *[$\mu \geq 0$] Let $f : X \to \mathbb{R}$ be a Choquet pricing rule with respect to a game $v \in bv(\mathcal{A})$, then the following assertions are equivalent:*
1. f satisfies the Arbitrage-free Condition AF_+;
2. f satisfies the Arbitrage-free Condition CAF_+;
3. $\exists \mu \in X', \mu \geq 0, \mu \leq f$*;*
4. $\mathrm{core}(v) \cap \{\mu \geq 0\} \neq \emptyset$*;*
5. v satisfies the Balancedness Condition BA_+.
Moreover if f is additionally assumed to be monotone (or equivalently, v to be a capacity), the above assertions are also equivalent to the following ones
1'. f satisfies Arbitrage-free Condition AF;
2'. f satisfies Arbitrage-free Condition CAF;
3'. $\emptyset \neq \{\mu \in X', \mu \leq f\} \subseteq \{\mu \geq 0\}$*;*
4'. $\emptyset \neq \mathrm{core}(v) \subseteq \{\mu \geq 0\}$*;*
5'. v satisfies the Balancedness Condition BA.

Proof. First, Theorem 2 implies the following equivalences hold:

$$(1)\ [f \text{ satisfies } \mathrm{AF}_+] \Leftrightarrow (2)\ [f \text{ satisfies } \mathrm{CAF}_+] \Leftrightarrow (3)\ [\exists \mu \geq 0, \mu \leq f].$$

Second, Proposition 3 proves that $\{\mu \in X' : \mu \leq f\} = \mathrm{core}(v)$ thus

$$(3) \Rightarrow (4)\ \exists \mu \geq 0, \mu \in \mathrm{core}(v) \Rightarrow (5)\ v \text{ satisfies } \mathrm{BA}_+ \Rightarrow (2).$$ □

2.6. On a theorem by Schmeidler on the nonvacuity of the core

In this section we show, that the Choquet pricing rule approach allows to provide a new proof of the following classical result of Cooperative Game

Theory on the nonvacuity of the core due to Bondareva (1963) and Shapley (1967) for finite games and extended by Schmeidler (1968) for infinite games. In the infinite setting, we also refer to (Marinacci and Montrucchio, 2004, Theorem 5) and to Kannai (1969) for positive games where the result is proved as a consequence of a result of Fan (1956) on systems of linear inequalities in normed spaces.

In all the paper except this section, the pricing rule f is given and the game v is associated to f to capture or characterize properties of f such as the arbitrage-free property. The following standard Bondareva-Shapley-Schmeidler result from cooperative game theory is taking the game v as the primitive concept and the proof below is using the Choquet pricing rule $f : X \to \mathbb{R}$ as a tool to extend $v : \mathcal{A} \to \mathbb{R}$ to the space $X := B(\mathcal{A})$. Then Theorem 3 guarantees that the Choquet extension f admits an element in the core of v if and only if f is arbitrage-free and if and only if v is balanced.

Theorem 5. *Let $(\Omega, \mathcal{A})$ be a measurable space and let $v : \mathcal{A} \to \mathbb{R}$ be a game such that $v \in bv(\mathcal{A})$, then the following two conditions are equivalent:*
1. $\text{core}(v) \neq \emptyset$*;*
2. v satisfies the Balancedness Condition BA.

Proof. We let $f : X \to \mathbb{R}$ be the Choquet pricing rule associated to $v \in bv(\mathcal{A})$ and the result follows from Theorem 3. □

Similarly, we can deduce the characterization of the existence of a non-negative element in the core of v by the Balancedness Condition BA_+.

Theorem 6. *Let $(\Omega, \mathcal{A})$ be measurable space and let $v : \mathcal{A} \to \mathbb{R}$ be a game such that $v \in bv(\mathcal{A})$, then the following two conditions are equivalent:*
1. $\text{core}(v) \cap \{\mu \geq 0\} \neq \emptyset$*;*
2. v satisfies the Balancedness Condition BA_+.

Proof. We let $f : X \to \mathbb{R}$ be the Choquet pricing rule associated to $v \in bv(\mathcal{A})$ and the result follows from Theorem 4. □

3. Proofs

3.1. Proof of Proposition 1

Proof. • The proof of (1), (3), (4), and (5) is left to the reader. □

• (2) [CAF $\Rightarrow$ AF], [CAF$_+$ $\Rightarrow$ AF$_+$]. Take $\theta_k = 1$ for all k.

• [AF $\Rightarrow$ CAF], [AF$_+$ $\Rightarrow$ CAF$_+$]. Let $K \in \mathbb{N}, K \geq 1$, $(x_1, \dots, x_K) \in X^K$, and $(\theta_1, \dots, \theta_K) \in \mathbb{R}_+^K$ such that $\theta_1 x_1 + \cdots + \theta_K x_K = 0$ (resp. ≥ 0).
$\star\star$ First, we prove the result when the coefficient $\theta_k \geq 0$ are **rational numbers**. Indeed, suppose that $\theta_k = p_k/q_k$ for some integers p_k, q_k with $q_k > 0$ for all k. Then $n_1 x_1 + \cdots + n_K x_K = 0$ (resp. ≥ 0) where $n_k := p_k q_1 \dots q_K / q_k$ is an integer for all k. Thus AF (resp. AF$_+$) implies that $n_1 f(x_1) + \cdots + n_K f(x_K) = 0$ (resp. ≥ 0), noticing that $n_k x_k = x_k + \cdots + x_k$ (n_k times when $n_k > 0$ and 0 when $n_k = 0$) and similarly $n_k f(x_k) = f(x_k) + \cdots + f(x_k)$. Dividing by $q_1 \dots q_K > 0$ we get $\theta_1 f(x_1) + \cdots + \theta_K f(x_K) = 0$ (resp. ≥ 0). $\square$

$\star\star$ We now prove the result when the coefficient $\theta_k \geq 0$ are **real numbers**.

$\star$ [AF $\Rightarrow$ CAF] (assuming f is bounded above on the ball $B_\infty(0,1)$). Assume that $\theta_1 x_1 + \cdots + \theta_K x_K = 0$. Then there exist sequences of nonnegative rational numbers $(\theta_k^n)_{n \in \mathbb{N}} \to \theta_k$ $(k = 1, \dots, K)$. We have

$$\theta_1^n x_1 + \cdots + \theta_K^n x_K + \varepsilon^n = 0 \text{ with } \varepsilon^n := -\textstyle\sum_k \theta_k^n x_k \to -\sum_k \theta_k x_k = 0.$$

From the first part, taking $\theta_{K+1} = 1$, since θ_k^n $(k = 1, \dots, K)$ and θ_{K+1} are rational numbers, from AF we get

$$\theta_1^n f(x_1) + \cdots + \theta_K^n f(x_K) + f(\varepsilon^n) \geq 0.$$

Either, (i) there exists $N \in \mathbb{N}^*$ such that $\varepsilon^n \neq 0$ for all $n \geq N$, or (ii) there exists a subsequence (denoted identically) such that $\varepsilon^n = 0$ for all n. In the second case, we have $\theta_1^n f(x_1) + \cdots + \theta_K^n f(x_K) \geq 0$ since $f(0) = 0$ and at the limit, when $n \to +\infty$, we get $\theta_1 f(x_1) + \cdots + \theta_K f(x_K) \geq 0$.

In the first case, since f is bounded above on some neighborhood of 0, there exist a positive integer m and $\rho > 0$ such that $f(x) \leq \rho$ for all $x \in B_\infty(0, 1/m)$. Moreover, since for n large enough, $\varepsilon^n \neq 0$, we can choose $\theta_{K+1}^n \in \mathbb{Q}, \|\varepsilon^n\|_\infty < \theta_{K+1}^n < 2\|\varepsilon^n\|_\infty$. Thus, $\varepsilon^n / m\theta_{K+1}^n \in B_\infty(0, (1/m))$ implies $f(\varepsilon^n / m\theta_{K+1}^n) \leq \rho$. Hence, we have

$$\textstyle\sum_{k=1}^K \theta_k^n x_k + m\theta_{K+1}^n (\varepsilon^n / m\theta_{K+1}^n) = \sum_{k=1}^K \theta_k^n x_k + \varepsilon^n = 0.$$

From the above equality and the first part, since the coefficient θ_k^n $(k = 1, \dots, K+1)$ are nonnegative rational numbers, we get

$$\textstyle\sum_{k=1}^K \theta_k^n f(x_k) + m\theta_{K+1}^n \rho \geq \sum_{k=1}^K \theta_k^n f(x_k) + m\theta_{K+1}^n f(\varepsilon^n / m\theta_{K+1}^n) \geq 0.$$

When $n \to +\infty$, $(\theta_k^n)_{n\in\mathbb{N}} \to \theta_k$ $(k = 1, \ldots, K)$ and $\theta_{K+1}^n \to 0$ since $\varepsilon^n \to 0$. Thus, at the limit in the above inequality we get $\sum_{k=1}^K \theta_k f(x_k) \geq 0$. $\square$

$\star$ [$\mathrm{AF}_+ \Rightarrow \mathrm{CAF}_+$] (without any assumption on f). Assume $\sum_{k=1}^K \theta_k x_k \geq 0$. Then there exist sequences of positive rational numbers $(\theta_k^n)_{n\in\mathbb{N}} \to \theta_k$ $(k = 1, \ldots, K)$. For every integer m, we have $\sum_{k=1}^K \theta_k x_k + (1/m)\mathbf{1}_\Omega \gg 0$. Thus

for all m, for n large enough, $\sum_{k=1}^K \theta_k^n x_k + (1/m)\mathbf{1}_\Omega \gg 0$.

Thus, from the first part of the proof (with rational coefficients), taking $\theta_{K+1} = 1/m$, we deduce from AF_+ that :

for all m, for n large enough, $\sum_{k=1}^K \theta_k^n f(x_k) + (1/m)f(\mathbf{1}_\Omega) \geq 0$.

Letting $n \to +\infty$, and then $m \to +\infty$, at the limit we get

$$\theta_1 f(x_1) + \cdots + \theta_K f(x_K) \geq 0. \qquad \square$$

3.2. Proof of the Characterization Theorem 1

We prepare the proof of the theorem with a proposition.

Proposition 4. *Let $f : X \to \mathbb{R}$ be a pricing rule, bounded above on some neighborhood of* 0.
1. If f satisfies CAF, then $-f(-x) \leq c(x) \leq f(x)$ for all $x \in X$.
2. f satisfies CAF $\Leftrightarrow$ c is finite-valued $\Leftrightarrow$ $c(0) \geq 0$.
3. If f satisfies CAF, then c is finite-valued, continuous, and sublinear.

Proof of Proposition 4. • (1) From the definition of the function c, for all $x \in X$, for all $\varepsilon > 0$, there exist $K \in \mathbb{N}, K \geq 1$, $(x_1, \ldots, x_K) \in X^K$, $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K$ such that $\sum_k \theta_k x_k = x$ and $\sum_k \theta_k f(x_k) < c(x) + \varepsilon$. Since f satisfies CAF, $\sum_k \theta_k x_k + (-x) = 0$ implies $\sum_k \theta_k f(x_k) + f(-x) \geq 0$. Hence, $c(x) + \varepsilon > \sum_k \theta_k f(x_k) \geq -f(-x)$. Taking the limit, when $\varepsilon \to 0$, we get $-f(-x) \leq c(x)$ and clearly $c(x) \leq f(x)$. $\square$

• (2) $\star$ [f satisfies CAF $\Rightarrow$ c is finite-valued] Since f satisfies CAF, from (1) one has $-f(-x) \leq c(x) \leq f(x)$ for all $x \in X$. Thus, c is finite-valued since f is finite-valued. $\square$

$\star$ [c is finite-valued $\Rightarrow$ $c(0) \geq 0$] By contradiction. Suppose c is finite-valued and $c(0) < 0$. Then there exist $K \in \mathbb{N}, K \geq 1$, $(x_1, \ldots, x_K) \in X^K$, $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K$ such that $\sum_k \theta_k x_k = 0$ and $\alpha := \sum_k \theta_k f(x_k) < 0$. Thus, for all $t > 0$, $\sum_k (t\theta_k) x_k = 0$, which implies that $c(0) \leq \sum_k (t\theta_k) f(x_k) =$

$t\alpha \to -\infty$ when $t \to +\infty$ (since $\alpha < 0$). This proves that $c(0) = -\infty$, thus contradicting that c is finite-valued. $\square$

$\star$ [$c(0) \geq 0 \Rightarrow f$ satisfies CAF] The proof is immediate. $\square$

• (3) Assume that f satisfies CAF. We know from (2) that c is finite-valued.

$\star$ We now prove that c is sublinear. Let x, x' in X, let $\alpha \geq 0, \alpha' \geq 0$, for all $\varepsilon > 0$, there exist $K \in \mathbb{N}, K \geq 1$, $(x_1, \ldots, x_K) \in X^K$, $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K$, and $K' \in \mathbb{N}, K' \geq 1$, $(x'_1, \ldots, x'_{K'}) \in X^n$, $(\theta'_1, \ldots, \theta'_{K'}) \in \mathbb{R}_+^{K'}$ such that:

$$\textstyle\sum_k \theta_k f(x_k) < c(x) + \varepsilon \text{ and } \sum_k \theta_k x_k = x,$$

$$\textstyle\sum_{k'} \theta'_{k'} f(x'_{k'}) < c(x') + \varepsilon \text{ and } \sum_{k'} \theta'_{k'} x'_{k'} = x'.$$

From CAF and $\sum_k \alpha\theta_k x_k + \sum_{k'} \alpha'\theta'_{k'} x'_{k'} = \alpha x + \alpha' x'$, one gets

$$\begin{aligned} c(\alpha x + \alpha' x') &\leq \textstyle\sum_k \alpha\theta_k f(x_k) + \sum_{k'} \alpha'\theta_{k'} f(x'_{k'}) \\ &\leq \alpha c(x) + \alpha' c(x') + (\alpha + \alpha')\varepsilon. \end{aligned}$$

Letting $\varepsilon \to 0$, at the limit we get $c(\alpha x + \alpha' x') \leq \alpha c(x) + \alpha' c(x')$. $\square$

$\star$ We now prove that c is Lipschitzian, which proves that c is continuous.

We first prove that $\|c\|_\infty := \sup\{|c(x)| : x \in X, \|x\| \leq 1\} < +\infty$. Indeed, by assumption, f is bounded above on some neighborhood $\mathcal{N}$ of 0, hence there exist $\rho \in \mathbb{R}$ and $\varepsilon > 0$ such that, for all $x \in B_\infty(0, \varepsilon) \subseteq \mathcal{N}$, $f(x) \leq \rho$ and $f(-x) \leq \rho$. But $c(x) \leq f(x) \leq \rho$ and $-c(x) \leq c(-x) \leq f(-x) \leq \rho$ since $c \leq f$ and c is sublinear. Thus $-\rho \leq c(x) \leq \rho$ for all $x \in B_\infty(0, \varepsilon)$. This proves that $\|c\|_\infty := \sup\{|c(x)| : x \in B(0,1)\} \leq \rho/\varepsilon < +\infty$.

We now prove that c is Lipschitzian. From above, we notice first that $c(x) \leq |c(x)| \leq \|c\|_\infty \|x\|$ for all $x \in X$. Thus, for all x, y in X, $c(x) = c(x - y + y) \leq c(x - y) + c(y)$ hence

$$c(x) - c(y) \leq c(x - y) \leq \|c\|_\infty \|x - y\| \text{ and } c(y) - c(x) \leq \|c\|_\infty \|y - x\|.$$

Thus $|c(x) - c(y)| \leq \|c\|_\infty \|x - y\|$, which proves that c is Lipschitzian. $\square$

We state the following result for which we refer to (Aliprantis and Border, 1999, Theorem 7.52).

Theorem 7. *Let $c : X \to \mathbb{R}$, the following assertions are equivalent:*
(i) c is continuous and sublinear;
(ii)$\big\{\mu \in X' : \mu \leq c\big\}$ is nonempty, convex, compact for the weak$*$ topology and $c(x) = \max\big\{\mu(x) : \mu \in X', \mu \leq c\big\}$ for all $x \in X$.

We now give the proof of Theorem 1

Proof of Theorem 1. • [(1) AF $\Leftrightarrow$ (2) CAF] Follows from Proposition 1. $\square$

• [(3) $\exists \mu, \mu \leq f \Rightarrow f$ (2) satisfies CAF] Let $K \in \mathbb{N}, K \geq 1$, let $(x_1, \ldots, x_K) \in X^K$, and let $(\theta_1, \ldots, \theta_K) \geq 0$ such that $\sum_{k=1}^K \theta_k x_k = 0$. Then $\mu(x_k) \leq f(x_k)$ for all k. Multiplying each inequality by $\theta_k \geq 0$ and summing up we get:

$$0 = \mu(0) = \mu(\textstyle\sum_{k=1}^K \theta_k x_k) = \sum_{k=1}^K \theta_k \mu(x_k) \leq \sum_{k=1}^K \theta_k f(x_k).$$

This shows that f satisfies CAF. $\square$

• [(2) f satisfies CAF $\Rightarrow$ (3) $\exists \mu, \mu \leq f$] Since f satisfies CAF, from Proposition 4, the function c is finite-valued, continuous, and sublinear. Thus, by Theorem 7, there is $\mu \in X', \mu \leq c$. Since $c \leq f$ we deduce that $\mu \leq f$. $\square$

• (4) $\star$ [$\{\mu : \mu \leq c\} = \{\mu : \mu \leq f\}$] The inclusion $\{\mu : \mu \leq c\} \subseteq \{\mu : \mu \leq f\}$ follows directly from the fact that $c \leq f$. We now show the converse inclusion. Let $\mu \leq f$ and let $x \in X$. For all $K \in \mathbb{N}, K \geq 1$, for all $(x_1, \ldots, x_K) \in X^K$, all $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K$ such that $\sum_{k=1}^K \theta_k x_k = x$, then, $\mu(x_k) \leq f(x_k)$ for all k since $\mu \leq f$. Multiplying each inequality by $\theta_k \geq 0$ and summing up we get $\mu(x) = \sum_{k=1}^K \theta_k \mu(x_k) \leq \sum_{k=1}^K \theta_k f(x_k)$. Thus

$$\mu(x) \leq c(x) := \begin{array}{l} \inf \sum_{k=1}^K \theta_k f(x_k) \\ (x_1, \ldots, x_K) \in X^K \\ (\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K \\ K \in \mathbb{N}, K \geq 1 \end{array}$$

This proves that $\mu \leq c$. $\square$

$\star$ We proved previously that $\{\mu : \mu \leq c\} = \{\mu : \mu \leq f\}$, and these two sets are nonempty, convex, compact for the weak$*$ topology by Theorem 7. $\square$

• (5) The function $c : X \to \mathbb{R}$ is a finite-valued, continuous, sublinear function below f from Proposition 4. Consider now a function $g : X \to \mathbb{R}$ that is finite-valued, continuous, sublinear and below f. Then, for all $x \in X$, all $K \in \mathbb{N}, K \geq 1$, all $(x_1, \ldots, x_K) \in X^K$, and all $(\theta_1, \ldots, \theta_K) \geq 0$ such that $\sum_{k=1}^K \theta_k x_k = x$, one has $g(x) \leq \sum_{k=1}^K \theta_k g(x_k) \leq \sum_{k=1}^K \theta_k f(x_k)$. Thus, for all $x \in X$, $g(x) \leq c(x)$ from the definition of the function c. $\square$

• (6) Since f satisfies CAF, from Proposition 4, the function c is finite-valued, sublinear, and continuous. Thus, from Theorem 7 and Assertion (4)

$$c(x) = \max_{\mu \leq c} \mu(x) = \max_{\mu \leq f} \mu(x) \leq f(x) \text{ for all } x \in X. \quad \square$$

Moreover, the fact that the above inequality is an equality for all x if and only if f is sublinear follows directly from Assertion (5). $\square$

• (7) $\star$ $[\{\mu(x) : \mu \in X', \mu \leq f\} = [-c(-x), c(x)]]$ It is sufficient to show that (i) the set $I := \{\mu(x) : \mu \in X', \mu \leq f\}$ is a nonempty bounded interval of $\mathbb{R}$ and (ii) the bounds of I are $-c(-x)$ and $c(x)$. Indeed, the first assertion (i) follows from the fact that $I \subseteq \mathbb{R}$ is nonempty convex compact as the image of the nonempty convex compact (for the weak$*$ topology) set $\{\mu \in X' : \mu \leq f\}$ by the continuous linear function $\mu \to \mu(x)$ from X' to $\mathbb{R}$. The second assertion (ii) follows from (6). $\square$

$\star$ $[[-c(-x), c(x)] \subseteq [-f(-x), f(x)]$ for all $x \in X]$ For all $x \in X$ we have $c(x) \leq f(x)$ and $c(-x) \leq f(x)$. Moreover, since the function c is finite-valued and sublinear we have $0 = f(0) = f(x + (-x)) \leq f(x) + f(-x)$. Consequently, $-f(-x) \leq -c(-x) \leq c(x) \leq f(x)$. $\square$

3.3. Proof of the Characterization Theorem 2

The proof of Theorem 2 is a direct consequence of Theorem 1 that will be applied to the lower monotone envelope $f_+ : X \to \mathbb{R} \cup \{-\infty\}$ of the pricing rule $f : X \to \mathbb{R}$. We recall that f_+ is defined by:

$$f_+(x) = \inf\{f(x') : x' \in X, x' \geq x\} \text{ for all } x \in X,$$

and that f_+ is the greatest monotone function below f, that is, $f_+ \leq f$, and $f_+ \geq g$ for all monotone function $g : X \to \mathbb{R} \cup \{-\infty\}$ such that $g \leq f$. Moreover, f is monotone if and only if $f = f_+$. Note that f_+ may take the value $-\infty$, that is, there may exist payoffs $x \in X$ such that $f(x) = -\infty$, thus yielding an infinite gain, a situation that will typically be eliminated under the arbitrage-free equivalent conditions AF$_+$ or CAF$_+$.

Remark 1. *[$c_+(f) = c(f)_+ = c(f_+)$]* We used the notation $c_+(f)$, or in short c_+, in Section 2.3 and in Theorem 2. If the notation $c_+(f)$ is not ambiguous, the use of the short c_+ we use previously is clearly ambiguous since it may be the short of three different meanings, namely $c_+(f)$ (the function defined in Section 2.3), $c(f)_+$ (the lower monotone envelope of the function $c(f)$), and $c(f_+)$ (the function $c(g)$ defined is Section 2.3 associated to $g = f_+$). However, it is straightforward to prove that $c_+(f) = c(f)_+ = c(f_+)$, thus justifying the use of the short c_+, unless the context needs to be more precise. $\square$

When $v : \mathcal{A} \to \mathbb{R}$ is a game, we also define $v_+ : \mathcal{A} \to \mathbb{R} \cup -\infty$ by:

$$v_+(A) = \inf\{v(A') : A' \supseteq A, A' \in \mathcal{A}\} \text{ for all } A \in \mathcal{A}.$$

The following proposition summarizes properties of f_+ and v_+ that are needed in the following. We also recall that assuming f_+ to be a pricing rule requires f_+ to be finite-valued and to satisfy $f_+(0) = 0$, and assuming v_+ to be a game requires v_+ to satisfy $v_+(\emptyset) = 0$ or equivalently that $v \geq 0$.

Proposition 5 (f_+)**.** *Let $f : X \to \mathbb{R}$ be a pricing rule.*
1. f satisfies AF₊ $\Leftrightarrow$ f_+ is a pricing rule satisfying AF;
2. f satisfies CAF₊ $\Leftrightarrow$ f_+ is a pricing rule satisfying CAF;
3. $\{\mu \in X' : \mu \geq 0, \mu \leq f\} = \{\mu \in X' : \mu \leq f_+\}$ hence
$\exists \mu \geq 0, \mu \leq f \quad \Leftrightarrow \quad f_+$ is a pricing rule and $\exists \mu \in X', \mu \leq f_+$;
4. f_+ is bounded above on a neighborhood of 0, more precisely
$f_+(x) \leq f(\mathbf{1}_\Omega)$ for all $x \in B_\infty(0, 1)$.
5. If f is monotone, then $f = f_+$,
f satisfies AF₊ $\Leftrightarrow$ f satisfies AF
f satisfies CAF₊ $\Leftrightarrow$ f satisfies CAF,
$\{\mu \in X' : \mu \leq f\} \subseteq X'_+$.

Proof. • (1) The proof is similar to the proof of (2) given hereafter. □

• (2) ⋆ [⇒] First, we prove that $f_+(0) = 0$. From CAF+, $f(x) \geq 0$ for all $x \geq 0$. Thus, $0 \leq f_+(0) := \inf\{f(x') : x' \geq 0\} \leq f(0) = 0$.

Second, we prove that f_+ is finite-valued and this is sufficient to show that $f(x) \geq f_+(x) \geq -f(-x)$ for all x. Indeed, for all $x \in X$, $x' \geq x$ implies $x' + (-x) \geq 0$, hence $f(x') + f(-x) \geq 0$ from Condition CAF+. Thus, $f(x) \geq f_+(x) := \inf\{f(x') : x' \geq x, x' \in X\} \geq -f(-x)$.

Finally, we prove that f_+ satisfies CAF. Let $K \in \mathbb{N}, K \geq 1$, $(x_1, \ldots, x_K) \in X^K$, $(\theta_1, \ldots, \theta_K) \in \mathbb{R}^K_+$ such that $\sum_{k=1}^K \theta_k x_k = 0$. Then

$$\begin{aligned} \textstyle\sum_{k=1}^K \theta_k f_+(x_k) \quad &:= \textstyle\sum_{k=1}^K \theta_k \inf\{f(x'_k) : x'_k \geq x_k\} \\ &= \inf\{\textstyle\sum_{k=1}^K \theta_k f(x'_k) : x'_k \geq x_k \ (k = 1, \ldots, K)\} \\ &\geq \inf\{\textstyle\sum_{k=1}^K \theta_k f(x'_k) : \sum_{k=1}^K \theta_k x'_k \geq 0\} \\ &\geq 0. \end{aligned}$$

The first above inequality holds since $(x'_1, \ldots, x'_K) \in X^K$ and $x'_k \geq x_k$ for all k imply $\sum_{k=1}^K \theta_k x'_k \geq \sum_{k=1}^K \theta_k x_k = 0$ and the second above inequality holds since f satisfies CAF₊.

We have thus proved that f_+ is a pricing rule satisfying CAF. $\square$

$\star$ [$\Leftarrow$] We prove that f satisfies CAF$_+$. For all $K \in \mathbb{N}, K \geq 1$, for all $(x_1, \ldots, x_K) \in X^K$, all $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K$ such that $\sum_{k=1}^K \theta_k x_k \geq 0$ we define $x_{K+1} := -\sum_{k=1}^K \theta_k x_k \leq 0$ so that $\sum_{k=1}^K \theta_k x_k + x_{K+1} = 0$. Then $\sum_{k=1}^K \theta_k f_+(x_k) + f_+(x_{K+1}) \geq 0$ since f_+ satisfies CAF. But $f_+(x_{K+1}) \leq f_+(0) = 0$ since $x_{K+1} \leq 0$ and f_+ is a monotone pricing rule. Thus

$$\sum_{k=1}^K \theta_k f(x_k) \geq \sum_{k=1}^K \theta_k f_+(x_k) \geq \sum_{k=1}^K \theta_k f_+(x_k) + f_+(x_{K+1}) \geq 0.$$

Thus, f satisfies CAF$_+$. $\square$

• (3) [$\{\mu \in X' : \mu \leq f_+\} = \{\mu \in X' : \mu \geq 0, \mu \leq f\}$]

$\star$ [$\subseteq$] Let $\mu \leq f_+$, then $\mu \leq f_+ \leq f$. Moreover, $\mu \geq 0$ as a consequence of the following standard claim since $f_+ : X \to \mathbb{R} \cup \{-\infty\}$ is monotone.

Claim 3.1. *If $g : X \to \mathbb{R} \cup \{-\infty\}$ be monotone, then*

$$\{\mu \in X' : \mu \leq g\} \subseteq X'_+.$$

The proof is given for the sake of completeness. Indeed, let $\mu \in X', \mu \leq g$, let $p \in X_+$. Since g is monotone, for all $t > 0$, $\mu(x - tp) \leq g(x - tp) \leq g(x)$ (hence $g(x)$ is finite). Thus, $\mu(p) \geq (1/t)[\mu(x) - g(x)]$. Letting $t \to +\infty$, at the limit we get $\mu(p) \geq 0$ for all $p \in X_+$. Hence $\mu \geq 0$. $\square$

$\star$ [$\supseteq$] Let $\mu \geq 0$, $\mu \leq f$. For all x, all $x' \geq x$, since $\mu \geq 0$ one has $\mu(x) \leq \mu(x') \leq f(x')$. Thus $\mu(x) \leq \inf\{f(x') : x' \in X, x' \geq x\} := f_+(x)$. We have thus proved that $\mu \leq f_+$. $\square$

• (4) Since f_+ is monotone, for $x \in B_\infty(0,1)$, i.e., $-\mathbf{1}_\Omega \leq x \leq \mathbf{1}_\Omega$, one has $f_+(x) \leq f(\mathbf{1}_\Omega)$. $\square$

• (5) The proof follows directly from Assertions (1), (2), and (3). $\square$

We now give the proof of Theorem 2.

Proof of Theorem 2. • We show first that each Assertion (1), (2) and (3) of Theorem 2 has an equivalent formulation in term of f_+, recalling that for f_+ to be a pricing rule requires f_+ to be finite-valued and $f_+(0) = 0$:

(1) f satisfies AF$_+$ $\Leftrightarrow$ (1$_+$) f_+ is a pricing rule satisfying AF,
(2) f satisfies CAF$_+$ $\Leftrightarrow$ (2$_+$) f_+ is a pricing rule satisfying CAF,
(3) $\exists \mu \geq 0, \mu \leq f$ $\Leftrightarrow$ (3$_+$) f_+ is a pricing rule and $\exists \mu \in X', \mu \leq f_+$.

The three above equivalences follow from Proposition 5. From Theorem 1 the equivalence $(1_+) \Leftrightarrow (2_+) \Leftrightarrow (3_+)$ hold for the pricing rule f_+ since f_+ is bounded above on $B_\infty(0,1)$, a neighborhood of 0, by Proposition 5. Consequently, the equivalence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follows.

• The proofs of Conditions (4)-(7) are similar to the ones in Theorem 1 $\square$

3.4. Proofs of Proposition 2 and Proposition 3

Proof of Proposition 2. Let $f : X \to \mathbb{R}$ be a pricing rule, $f(\mathbf{1}_\Omega) = -f(-\mathbf{1}_\Omega)$.

• (1) [CAF $\Rightarrow$ BA and CAF$_+$ $\Rightarrow$ BA$_+$] Let K be a positive integer, let $(\theta_1, \dots, \theta_K) \in \mathbb{R}_+^K$, $(A_1, \dots, A_K) \in (\mathcal{A})^K$ satisfying $\sum_{k=1}^K \theta_k \mathbf{1}_{A_k} = \mathbf{1}_\Omega$. Then $\sum_{k=1}^K \theta_k \mathbf{1}_{A_k} - \mathbf{1}_\Omega = 0$ and CAF imply $\sum_{k=1}^K \theta_k f(\mathbf{1}_{A_k}) \geq -f(-\mathbf{1}_\Omega) = f(\mathbf{1}_\Omega)$.

Thus, f satisfies the Balancedness Condition BA.

The proof of [CAF$_+$ $\Rightarrow$ BA$_+$] is similar. $\square$

• (2) $[\{\mu \in X' : \mu \leq f\} \subseteq \mathrm{core}(f)]$ We recall that the mapping $\hat{\mu} \to \mu$, from X' to $\mathrm{ba}(\mathcal{A})$, defined by $\mu(A) = \hat{\mu}(\mathbf{1}_A)$ is an isometry that allowed us to identify the two spaces X' to $\mathrm{ba}(\mathcal{A})$ throughout the paper. Let $\hat{\mu} \in X'$ such that $\hat{\mu} \leq f$. Then, $\mu(A) = \hat{\mu}(\mathbf{1}_A) \leq f(\mathbf{1}_A) = v_f(A)$ for all $A \in \mathcal{A}$. In particular, $\hat{\mu}(\mathbf{1}_\Omega) \leq f(\mathbf{1}_\Omega)$ and $\hat{\mu}(-\mathbf{1}_\Omega) \leq f(-\mathbf{1}_\Omega) = -f(\mathbf{1}_\Omega)$, thus $\mu(\Omega) = \hat{\mu}(\mathbf{1}_\Omega) = f(\mathbf{1}_\Omega) = v_f(\Omega)$. This proves that $\mu \in \mathrm{core}(f)$. $\square$

• (3) $[f \neq 0 \Rightarrow f(\mathbf{1}_\Omega) > 0]$ By contradiction. Assume $f \neq 0$ and $f(\mathbf{1}_\Omega) \leq 0$. For every $x \in X$, there exists a positive integer k such that $-k\mathbf{1}_\Omega \leq x \leq k\mathbf{1}_\Omega$. But Condition CAF+ and $x + k\mathbf{1}_\Omega \geq 0$ imply $f(x) + kf(\mathbf{1}_\Omega) \geq 0$, thus $0 \leq -kf(\mathbf{1}_\Omega) \leq f(x)$. Since f is monotone, $f(x) \leq f(k\mathbf{1}_\Omega) \leq kf(\mathbf{1}_\Omega) \leq 0$. We have thus proved that $f(x) = 0$. Therefore $f = 0$, a contradiction. $\square$

Proof of Proposition 3. • (1) $[\{\mu \in X' : \mu \leq f\} = \mathrm{core}(v)]$ The inclusion $[\subseteq]$ follows from Proposition 2. We now prove the converse inclusion. Indeed, let $\mu \in \mathrm{core}(v) \subseteq \mathrm{ba}\,(\mathcal{A})$, then $\mu(A) \leq v(A)$ for all $A \in \mathcal{A}$ and $\mu(\Omega) = v(\Omega)$. Let $\hat{\mu} \in X'$ be defined by $\hat{\mu}(x) = \int_\Omega x\, d\mu$ for all $x \in X$.

$\star$ We first show that $f(x) \geq \hat{\mu}(x)$ for all $x : \Omega \to \mathbb{R}$ measurable and *simple* function, that is, its set of values $\{x(\omega) : \omega \in \Omega\} = \{x_1, \dots, x_m\}$ is finite and is ranked in decreasing order, that is, $x_1 > \cdots > x_i > \cdots > x_m$. We let $A_i := \{\omega \in \Omega \ : \ x(\omega) = x_i\}$. From the definition of the Choquet integral and $\mu \in \mathrm{core}(v)$, we get:

$$\begin{array}{rl} f(x) & = \int_\Omega^C x\, dv := \sum_{k=1}^{m-1}(x_i - x_{i+1})v(A_1 \cup \cdots \cup A_i) + x_m v(\Omega) \\ & \geq \sum_{i=1}^{m-1}(x_i - x_{i+1})\mu(A_1 \cup \cdots \cup A_i) + x_m \mu(\Omega) \\ & = \int_\Omega x\, d\mu = \hat{\mu}(x). \end{array}$$

$\star$ We now show that $f(x) \geq \hat{\mu}(x)$ for all $x : \Omega \to \mathbb{R}$ is measurable and *bounded.* Indeed, x can be approximated by a sequence of simple measurable functions (x^n) such that $\lim_{n\to+\infty} \|x^n - x\|_\infty = 0$. From the first part, one gets $f(x^n) \geq \hat{\mu}(x^n)$ for all n. Since the Choquet pricing rule is continuous for the sup-norm and $\hat{\mu} \in X'$ is continuous, one gets:

$$f(x) = \lim_{n\to+\infty} f(x^n) \geq \lim_{n\to+\infty} \hat{\mu}(x^n) = \hat{\mu}(x). \qquad \square$$

• (2) The proof is standard and is left to the reader. $\square$

• (3) [BA $\Rightarrow$ CAF] We prepare the proof with a claim.

Claim 3.2. *If v is a game satisfying BA, for all positive integer K, for all $(\theta_1, \ldots, \theta_K, \theta_{K+1}) \in \mathbb{R}_+^K \times \mathbb{R}$, and for all $(A_1, \ldots, A_K) \in (\mathcal{A})^K$*

$$\sum_{k=1}^K \theta_k \mathbf{1}_{A_k} + \theta_{K+1}\mathbf{1}_\Omega = 0 \Rightarrow \sum_{k=1}^K \theta_k v(A_k) + \theta_{K+1} v(\Omega) \geq 0.$$

Proof of the claim. Let K be a positive integer, $(\theta_1, \ldots, \theta_K, \theta_{K+1}) \in \mathbb{R}_+^K \times \mathbb{R}$, and $(A_1, \ldots, A_K) \in (\mathcal{A})^K$ satisfy $\sum_{k=1}^K \theta_k \mathbf{1}_{A_k} + \theta_{K+1}\mathbf{1}_\Omega = 0$.

$\star$ First, assume $\theta_{K+1} \geq 0$, then we prove that $\theta_k = 0$ for all $k = 1, \ldots, K, K+1$ and thus the result follows. Clearly $\theta_{K+1} \geq 0$ implies that for every $k = 1, \ldots, K$, one has $\theta_k \mathbf{1}_{A_k} = 0$, thus $\theta_k v(A_k) = 0$ and $\theta_{K+1} = 0$. Indeed, if $A_k = \emptyset$, then $v(A_k) = v(\emptyset) = 0$, and if $A_k \neq \emptyset$, taking $\omega \in A_k$ we get $0 = \sum_{k'=1}^K \theta_{k'} \mathbf{1}_{A_{k'}}(\omega) + \theta_{K+1}\mathbf{1}_\Omega(\omega) \geq \theta_k + \theta_{K+1} \geq \theta_k$. Thus $\theta_k = 0$ and $\theta_{K+1} = 0$.

$\star$ Now assume $\theta_{K+1} < 0$. Then $\sum_{k=1}^K -(\theta_k/\theta_{K+1})\mathbf{1}_{A_k} = \mathbf{1}_\Omega$ and BA imply $\sum_{k=1}^K -(\theta_k/\theta_{K+1})v(A_k) \geq v(\Omega)$. Thus $\sum_{k=1}^K \theta_k v(A_k) + \theta_{K+1}v(\Omega) \geq 0$. $\square$

• [BA $\Rightarrow$ CAF]. Let K be a positive integer, let $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K$, let $(x_1, \ldots, x_K) \in X^K$ such that $\sum_{k=1}^K \theta_k x_k = 0$.
$\star$ Assume first that all the functions $x_k : \Omega \to \mathbb{R}$ are measurable, **simple** functions, that is, each x_k has finitely many values that are ranked in decreasing order as follows for $k = 1, \ldots, K$

$$\{x_k(\omega) : \omega \in \Omega\} = \{a_k^1, \ldots, a_k^{m_k}\} \text{ and } a_k^1 > \cdots > a_k^i > \cdots > a_k^{m_k}.$$

Then, if $A_k^i := \{\omega \in \Omega \ : \ x_k(\omega) = a_k^i\}$, we recall that:

$$x_k = \sum_{i=1}^{m_k} a_k^i \mathbf{1}_{A_k^i} = \sum_{i=1}^{m_k-1}(a_k^i - a_k^{i+1})\mathbf{1}_{A_k^1\cup\cdots\cup A_k^i} + a_k^{m_k}\mathbf{1}_\Omega$$

$$f(x_k) = \int_\Omega^C x_k\, dv := \sum_{i=1}^{m_k-1}(a_k^i - a_k^{i+1})v(A_k^1\cup\cdots\cup A_k^i) + a_k^{m_k}v(\Omega).$$

Thus, $\sum_{k=1}^K \theta_k x_k = 0$ implies that

$$\sum_{k=1}^K \sum_{i=1}^{m_k-1} \theta_k(a_k^i - a_k^{i+1})\mathbf{1}_{A_k^1\cup\cdots\cup A_k^i} + \sum_{k=1}^K \theta_k a_k^{m_k}\mathbf{1}_\Omega = 0$$

From the Balancedness Condition BA, and the above claim we get:

$$\begin{aligned} 0 \quad &\leq \sum_{k=1}^K \theta_k \Big[\sum_{k=1}^{m_k-1}(a_k^i - a_k^{i+1})v(A_k^1\cup\cdots\cup A_k^i) + a_k^{m_k}v(\Omega)\Big] \\ &= \sum_{k=1}^K \theta_k \int_\Omega^C x_k\, dv = \sum_{k=1}^K \theta_k f(x_k). \end{aligned}$$

Thus, f satisfies the Arbitrage-free Condition CAF. □

⋆ Assume now the general case, with $(x_1,\ldots,x_K)\in X^K$. Since the space X_0 of measurable simple function is dense in X for the sup-norm, there exists sequences $(x_k^n)_{n\in\mathbb{N}} \subseteq X_0$, $(x_k^n)\to x_k$ $(k=1,\ldots,K)$. We have

$$\sum_{k=1}^K \theta_k x_k^n + \varepsilon^n = 0 \text{ with } \varepsilon^n := -\sum_{k=1}^K \theta_k x_k^n \to -\sum_k \theta_k x_k = 0.$$

Clearly, $\varepsilon^n \in X_0$ since X_0 is a vector space, hence from the first part

$$\theta_1 f(x_1^n) + \cdots + \theta_K f(x_K^n) + f(\varepsilon^n) \geq 0.$$

Passing to the limit, when $n\to+\infty$, since the Choquet integral is continuous for the sup-norm, one gets $\theta_1 f(x_1)+\cdots+\theta_K f(x_K)\geq 0$.

Thus, f satisfies the Condition CAF. □

• [BA$_+$ ⇒ CAF$_+$] ⋆ We first prove that v_+ is a game satisfying BA and $v_+ \in \mathrm{bv}(\mathcal{A})$. For v_+ to be a game we need to prove that $v_+(\emptyset)=0$. Indeed, since v satisfies BA$_+$, for all $B\in\mathcal{A}$, $\mathbf{1}_B+\mathbf{1}_\Omega \geq \mathbf{1}_\Omega$ implies $v(B)+v(\Omega)\geq v(\Omega)$, hence $v(B)\geq 0$. Thus, $v_+(\emptyset) := \inf\{v(B) : B\supseteq\emptyset\}\geq 0$ but $v_+(\emptyset)\leq v(\emptyset)=0$. Hence $v_+(\emptyset)=0$.

Second $v_+\in\mathrm{bv}(\mathcal{A})$ since v_+ is monotone.

Finally, we show that v_+ satisfies BA. Let $K\in\mathbb{N}, K\geq 1$, let $(A_1,\ldots,A_K)\in\mathcal{A}^K$, and let $(\theta_1,\ldots,\theta_K)\in\mathbb{R}_+^K$ such that $\sum_{k=1}^K \theta_k\mathbf{1}_{A_k} = \mathbf{1}_\Omega$. Then

$$\begin{aligned} \sum_{k=1}^K \theta_k v_+(A_k) \quad &= \sum_{k=1}^K \theta_k \inf\{v(B_k) : \forall k\, B_k\supseteq A_k, B_k\in\mathcal{A}\} \\ &= \inf\{\textstyle\sum_{k=1}^K \theta_k v(B_k) : \forall k\, B_k\supseteq A_k, B_k\in\mathcal{A}\} \\ &\geq \inf\{\textstyle\sum_{k=1}^K \theta_k v(B_k) : \sum_{k=1}^K \theta_k \mathbf{1}_{B_k}\geq \mathbf{1}_\Omega\} \\ &\geq v(\Omega). \end{aligned}$$

The first inequality holds since, $(B_1,\ldots,B_K)\in\mathcal{A}^K$ and, for all k, $B_k\supseteq$

A_k, (that is, $\mathbf{1}_{B_k} \geq \mathbf{1}_{A_k}$) imply $\sum_{k=1}^{K} \theta_k \mathbf{1}_{B_k} \geq \sum_{k=1}^{K} \theta_k \mathbf{1}_{A_k} = \mathbf{1}_\Omega$, thus implying the second inequality since v satisfies BA$_+$.

We have proved that v_+ is a game satisfying BA and $v_+ \in \mathrm{bv}(\mathcal{A})$. $\square$

$\star$ [f satisfies CAF$_+$] Since v_+ is a game satisfying BA, and $v_+ \in \mathrm{bv}(\mathcal{A})$, the Choquet pricing rule g with respect to v_+ satisfies CAF (from the first statement of (3) [BA $\Rightarrow$ CAF], proved previously) and also CAF$_+$ since g is monotone (by Proposition 1). But, $g(x) := \int_\Omega^C x\, dv_+ \leq f(x) := \int_\Omega^C x\, dv$ for all x, since $v_+ \leq v$ and $v_+(\Omega) = v(\Omega)$. Since g satisfies CAF$_+$ and $g \leq f$ one deduces that f satisfies also CAF$_+$. $\square$

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