FINANCIAL MARKETS WITH HEDGING COMPLEMENTS

Alain Chateauneuf and Bernard Cornet

This paper considers financial markets with bid-ask spreads and studies the class of markets with hedging complements, a property formalized by the complementarity of its hedging price, in the same way as strategic complements is defined on agents' payoff functions in game theory.

The class of markets with hedging complements contains both markets with frictionless securities and the larger class markets with independent marketed securities together with the frictionless bond, assuming both to be arbitrage-free. Moreover, the hedging prices of the latter markets are proved to satisfy a tractable explicit formula, as the sum of a "generalized" convex Choquet integral and of a modular term. Finally this class of markets also satisfyg the put-call parity of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015).

Keywords: Financial markets, bid-ask spread, hedging complements, generalized Choquet integral, submodularity, decreasing difference property, put-call parity.

1. INTRODUCTION

Perfect complementarity has long been regarded by economists to be formally expressed by submodularity (resp. supermodularity) or by the decreasing (increasing) difference property of cost (utility) functions. The use of supermodularity for utility functions as a notion of complementarity, dates back to Fisher, Pareto, and Edgeworth. More recently, an important literature on games with strategic complements has opened new avenues in various problems in economics and game theory. Markets with hedging complements are formalized by the complementarity of the hedging price, in the same way as strategic complements are defined on agents' payoff functions in game theory. Such markets extend naturally the class of markets with frictionless securities, to take into account a large class of frictions, and behave similarly when using non-additive expectation (integration) models, developed recently in decision theory with ambiguity.

The study of market frictions is a very active research topic in economics and finance, both empirically and theoretically. Due to the important evidence for the existence of bid-ask spreads in the market, the finance literature has developed

Alain Chateauneuf, IPAG Business School and Paris School of Economics, Université Paris 1, Paris, France and
Bernard Cornet, University of Kansas, Lawrence, Kansas, USA, and Paris School of Economics, Paris France

models to study these types of frictions and others such as transactions costs, short sales, and taxes. The Fundamental Theorem of Asset Pricing has been extended to the case of financial markets with bid-ask spreads, assuming the bond is frictionless and such markets provide a natural rationale for the famous Gilboa-Schmeidler multi-prior model, extensively used in this paper in its dual form.[1] Actually, the Fundamental Theorem of Asset Pricing states that, in the basic two-date stochastic model, the absence of arbitrage opportunities in a financial market exhibits a risk-free interest rate, together with a family $\mathcal{P}$ of risk-neutral probability measures, consistent with the bid-ask prices of marketed securities and the prevalent uncertainty. Furthermore, valuing today a payoff, that will be delivered in the future, through the hedging or super-hedging price is exactly in line with the multi-prior model, since its price is the discounted supremum of the expected payoffs over the family $\mathcal{P}$ of risk-neutral probability measures.

The general framework of this paper considers an arbitrage-free financial market M with finitely many securities with bid-ask spread, together with a frictionless marketed security (e.g., the bond in most applications) and hereafter simply referred to as a market. The definition of hedging complements, with the decreasing difference properties of its hedging price, has a meaningful financial interpretation. That is, for every pair of marketed security payoffs the net cost of buying the first security (once the second security has been bought) is not greater than the net cost of buying the first security alone. Markets with hedging complements extend naturally the class of markets with frictionless securities. It appears also that frictionless markets can be characterized by the strengthening of the hedging complement property introduced in the study of cost games.

The class of markets with hedging complements includes the class of markets with independent marketed securities with bid-ask spread. This new class of markets is easily identified from the primitive features of the markets and it includes the class of markets with frictionless securities since, for such markets, redundant securities can always be eliminated at no cost. Our main contribution is to show that a market with independent securities satisfies an explicit two-part formula for its hedging price that is the sum of a "generalized" convex Choquet integral and a modular part. The two-part formula directly implies that such markets have hedging complements since their hedging prices are submodular,

[1] Here, the hedging and super-hedging prices are the supremum of expected payoffs for a given class of probabilities (see Theorem 2.1), instead of the infimum of expectation as in Gilboa and Schmeidler (1989).

but it makes also tractable both the computation of the hedging price and the way to find best strategies to obtain it.

Subadditive Choquet pricing rules have been introduced by Chateauneuf, Kast, and Lapied (1996), who provide an axiomatic characterization relying on comonotonic addity, and Cerreia-Vioglio, Maccheroni, and Marinacci (2015) characterize such pricing rules with the put-call parity that has been hugely studied in the literature. Thus, for such a market the multi-prior model by Gilboa and Schmeidler (1989) coincides with the Choquet expected utility model of Schmeidler (1989) in the sense that the risk-neutral *non-additive* probability associated with the generalized Choquet integral summarizes the information of the bundle of (standard) risk-neutral probabilities. Our main result considers a different non-axiomatic approach, based on the primitives of the market, that is, the payoffs and prices of its marketed securities. We show that a market with independent securities, together with a mild additional assumption (implying that the modular term in our formula is null) is a generalized Choquet integral. This allows us to characterize also the set of stochastic discount factors by the non-additive measure associated with the generalized integral.

Finally, the extended notion of put-call parity, as introduced by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015) is proved to be satisfied for markets with independent securities even in cases where the hedging price may not be a generalized Choquet integral. It happens that the notion of generalized Choquet integral used in the paper coincides with the notion of Choquet integral on Riesz space introduced by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015).

Relation to the literature

The study of market frictions in financial markets is a very active research topic in economics and finance both empirically and theoretically. Due to the important evidence for the existence of bid-ask spreads in financial market, Amihud and Mendelson (1986), the literature has developed models, which incorporate transactions costs, short sales, and taxes: Luttmer (1996), Prisman (1986), Ross (1987). The Fundamental Theorem of Asset Pricing has been proved by Jouini and Kallal (1995, 2001) for markets with bid-ask spreads, thus extending the theory developed in the frictionless case, first by Ross Ross (1976); **?**, Cox and Ross (1976), and in a dynamic setting by Harrison and Kreps (1979), Duffie and Huang (1986), Delbaen and Schachermayer (2006). For a general presentation of the topic, we refer to Dybvig and Ross (1986), Follmer and Schied, (2004), Ross (2005), Delbaen and Schachermayer (2006).

Perfect complementarity has long been regarded by economists to be formally expressed by submodularity (resp. supermodularity) or by the decreasing (increasing) difference property of cost (utility) functions. According to Samuelson (1974), the use of supermodularity for utility functions dates back to Fisher, Pareto, and Edgeworth. The seminal paper by Bulow, Geanakoplos, and Klemperer (1985) introduced the notion of strategic complements on agents' payoff functions in game theory. The complementarity property has been strenghened in the study of cost games by **?**, Sharkey (1982), Moulin (1992), and also by Marinacci and Montrucchio (2005) and Müller and Scarsini (2012). In our study, it appears that this stronger condition characterizes frictionless markets. Other important papers on supermodularity are**?**, Milgrom and Shannon (1994), Vives (1990), Milgrom and Roberts (1990). A general reference is the book by Topkis (1998) that also study the consequences of having complementarity properties in cost and utility functions.

More recent literature has studied the related concept of market pricing rules in an axiomatic framework, thus different from our approach that focuses on the primitives of the financial market, that is, the payoffs and (bid and ask) prices of its marketed securities. Araujo, Chateauneuf, and Faro (2012) characterize the super-hedging price of arbitrage-free frictionless markets on tradeable assets. In the seminal paper by Jouini and Kallal (2001), efficient trading strategies are characterized in the presence of market frictions, generalizing previous work by Dybvig and Ross (1986) for frictionless markets. In a more recent paper, using the efficiency notion of Jouini and Kallal (2001), Araujo, Chateauneuf, and Faro (2018) show that the class of efficient complete markets with bid-ask spreads is the prevalent case revealed by finitely generated pricing rules.

Seminal papers introducing and using Choquet integral in economics and decision theory are Schmeidler (1986), Schmeidler (1989), and Gilboa (1987). A presentation of the theory can be found in Denneberg (1994), Gilboa (2009), and Marinacci and Montrucchio (2004). Other related papers by Chateauneuf, Kast, and Lapied (1996), Cerreia-Vioglio, Maccheroni, and Marinacci (2015), Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015), and Chateauneuf and Cornet (2018) are studying the pricing of securities via Choquet integral (expectation) and provide an axiomatic characterization of such pricing rules.

The parity between call and put has long been tested by empirical research, see Stoll (1969); **?**, Gould and Galai (1974), Klemkosky and Resnick (1980), Sternberg (1984), Kamara and Miller (1995). Chateauneuf, Kast, and Lapied (1996) show that a pricing rule, which is a submodular Choquet integral allows to explain why, when introduced on the market, puts have a price smaller than the replicated payoff which can be obtained with a corresponding call and security, hence violating some version of the put-call parity for European Options. Cerreia-Vioglio, Maccheroni, and Marinacci (2015) show that another version of the put-call Parity for European options characterizes a pricing rule which turns out to be a discounting Choquet integral. This result is extended by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015) to the case of Riesz spaces and non-monotone pricing rules, a framework allowing to study hedging prices.

Organization

The paper is organized as follows. Section 2 is devoted to the study of markets with hedging complements. In Section 2.1 and 2.2 we present the model and recall the standard notions on financial markets used in the paper. In Section 2.3, markets with hedging complements are defined and their properties are summarized in Proposition 2.5. Strong hedging complements are defined in Section 2.4 and Proposition 2.8 shows this condition characterizes frictionless markets.

Section 3 is devoted to the study of markets with independent securities. In Section 3.2, our main result (Theorem 3.5) provides a two-part formula for the hedging price, as the sum of a submodular Choquet part and a modular part, thus shows that such markets have hedging complements. The two-part formula makes tractable the computation of the hedging price but also the way to find a strategy to obtain it, as stressed in Example 3.6. In Section 3.3 we show that the extended put-call parity property is satisfied (Theorem 3.5), even when the hedging price is not a Choquet integral (Example 3.7). Section 3.4 studies the case where the hedging price is a generalized Choquet integral (Corollary 1). Finally, in Section 3.5, we illustrate our general study with the case of Arrow securities and event securities (Theorem 3).

The proofs of the different results are given in the Appendix. [2]

[2]We recall some notations used throughout the paper. Let Ω be a finite set, we let $\mathbb{R}^\Omega$ be the vector space of functions $x : \Omega \to \mathbb{R}$, endowed with the point-wise order $x \geq x'$ defined by $x(\omega) \geq x'(\omega)$ for all $\omega \in \Omega$ and we define $x \gg x'$ by $x(\omega) > x'(\omega)$ for all $\omega \in \Omega$. We let $\mathbb{R}^\Omega_+ := \{x \in \mathbb{R}^\Omega : x \geq 0\}$ and $\mathbb{R}^\Omega_{++} := \{x \in \mathbb{R}^\Omega : x \gg 0\}$. The lattice operations $\wedge$ and $\vee$ are defined by $(x \wedge y)(\omega) := \min\{x(\omega), y(\omega)\}$, $(x \vee y)(\omega) := \max\{x(\omega), y(\omega)\}$ for all $\omega \in \Omega$.

2. MARKETS WITH HEDGING COMPLEMENTS

2.1. *Hedging and super-hedging prices of a market*

This paper considers a basic tractable two-date stochastic model, where $t = 0$ (today) is known and $t = 1$ (tomorrow) is uncertain. The set Ω, called the state space, represents tomorrow's uncertainty and lists the possible states that can prevail at $t = 1$, one and only one of which will be disclosed at $t = 1$.

A *financial market* M, or in short a *market*, is a collection of finitely many *securities with bid and ask prices*, all defined on the same state space Ω, indexed by $j \in \boldsymbol{J} := \{1, \dots, J\}$. Each security j is a contract that promises the *payoff* $V^j(\omega)$ (resp. $-V^j(\omega)$) at $t = 1$ if state ω prevails, for each unit bought (resp. sold) at $t = 0$, and it is paid $\overline{q}^j$ (resp. $-\underline{q}^j$) at $t = 0$. We adopt the standard sign convention for payoffs that $V^j(\omega)$ is a gain whenever positive and $|V^j(\omega)|$ is a payment whenever $V^j(\omega) < 0$. Similarly, if c is price, or later a cost, then c is a payment if $c > 0$, and $|c|$ is a gain if $c < 0$. For each security j, we denote by $V^j := (V^j(\omega)) \in \mathbb{R}^\Omega$ its vector of payoffs across states (or the random variable $V^j : \Omega \to \mathbb{R}$), simply called *payoff* of security j; $\underline{q}^j \in \mathbb{R}$ is called its *bid (selling) price*, and $\overline{q}^j \in \mathbb{R}$ its *ask (buying) price*. The market M can be summarized by:

$$M := \big((V^1, \underline{q}^1, \overline{q}^1), \dots, (V^J, \underline{q}^J, \overline{q}^J)\big).$$

Thus, choosing the *strategy* $(\alpha, \beta) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$, where $\alpha := (\alpha^1, \dots, \alpha^J)$ (resp. $\beta := (\beta^1, \dots, \beta^J)$) is the list of quantities bought (resp. sold) of each security:

- yields the payoff $\sum_{j=1}^J (\alpha^j - \beta^j) V^j \in \mathbb{R}^\Omega$ across states at $t = 1$,
- in exchange of the payment of $\sum_{j=1}^J \overline{q}^j \alpha^j - \underline{q}^j \beta^j$ at $t = 0$.

The market M is said to be *arbitrage-free* if it satisfies the following two conditions of present and future arbitrage-free, for all $(\alpha, \beta) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+$

$$\sum_{j=1}^J (\alpha^j - \beta^j) V^j \geq 0 \Rightarrow \sum_{j=1}^J \overline{q}^j \alpha^j - \underline{q}^j \beta^j \geq 0 \ (\textbf{PAF}),$$
$$\sum_{j=1}^J (\alpha^j - \beta^j) V^j > 0 \Rightarrow \sum_{j=1}^J \overline{q}^j \alpha^j - \underline{q}^j \beta^j > 0 \ (\textbf{FAF}).$$

The payoff $x \in \mathbb{R}^\Omega$ is said to be *replicable* (resp. *super-replicable*) if

$$\sum_{j=1}^J (\alpha^j - \beta^j) V^j = x \ \text{ (resp. } \geq x) \text{ for some } (\alpha, \beta) \in \mathbb{R}^J_+ \times \mathbb{R}^J_+.$$

The *hedging price* $c(x)$ and the *super-hedging price* $c_+(x)$ (often also referred to as replication and super-replication costs) associated with the market M are then defined as the infimum of the cost of all strategies $(\alpha, \beta) \geq 0$ that allow to replicate (resp. super-replicate) the payoff x, that is,

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We denote by $\mathbf{1}_A$ the indicator function of the subset $A \subseteq \Omega$, i.e., $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$, and $\mathbf{1}_A(\omega) = 0$ otherwise. We let $\mathbf{1}_\emptyset = 0$ by convention.

$c(x) := \inf \left\{ \sum_{j=1}^{J} \overline{q}^j \alpha^j - \underline{q}^j \beta^j : \sum_{j=1}^{J} (\alpha^j - \beta^j) V^j = x, (\alpha, \beta) \in \mathbb{R}_+^J \times \mathbb{R}_+^J \right\},$
$c_+(x) := \inf \left\{ \sum_{j=1}^{J} \overline{q}^j \alpha^j - \underline{q}^j \beta^j : \sum_{j=1}^{J} (\alpha^j - \beta^j) V^j \geq x, (\alpha, \beta) \in \mathbb{R}_+^J \times \mathbb{R}_+^J \right\}.$

We point out that $c(x)$ and $c_+(x)$ a priori belong to $[-\infty, +\infty]$, with the convention that $c(x) = +\infty$ if x is not replicable and $c_+(x) = +\infty$ if x is not super-replicable. Moreover, we will see later (see Theorem 2.1) that, whenever the market is arbitrage-free, both $c(x)$ and $c_+(x)$ will not take the value $-\infty$, i.e., will not generate an infinite gain $|c(x)|$ and $|c_+(x)|$.

The j-th security $(V^j, \underline{q}^j, \overline{q}^j)$ of M is said to be frictionless if $\underline{q}^j = \overline{q}^j$ and it will be simply denoted by (V^j, q^j) with $q^j := \underline{q}^j = \overline{q}^j$. The previous framework clearly encompasses the case of a market $M = \big((V^1, q^1), \ldots, (V^J, q^J)\big)$ whose securities are all frictionless, and in such a market the previous arbitrage-free notion coincides with the standard one, i.e., there does not exist $\theta \in \mathbb{R}^J$,

$(-\sum_{j=1}^{J} q^j \theta^j, \sum_{j=1}^{J} \theta^j V^j) > 0.$

2.2. *Stochastic discount factors and risk-neutral probabilities*

Given the market $M := \big((V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$, we let:

$\mathcal{M} = \left\{ \mu \in \mathbb{R}^\Omega : \underline{q}^j \leq V^j \cdot \mu \leq \overline{q}^j \;\; \forall j = 1, \ldots, J \right\}$
(resp. $\mathcal{M}_+ = \mathcal{M} \cap \mathbb{R}_+^\Omega$, and $\mathcal{M}_{++} = \mathcal{M} \cap \mathbb{R}_{++}^\Omega$),

be respectively the set of (resp. nonnegative, resp. strictly positive) *stochastic discount factors* μ, for which the discounted payoff $V^j \cdot \mu$ of each security j belongs to the price spread $[\underline{q}^j, \overline{q}^j]$ (hence is equal to its price $q^j := \underline{q}^j = \overline{q}^j$, whenever the security j is frictionless).

We now recall the *Fundamental Theorem of Asset Pricing* that characterizes an arbitrage-free market M by the existence of a strictly positive stochastic discount factor $\mu \in \mathcal{M}_{++}$ and also by the fact that the super-hedging price $c_+(x)$ is the supremum of the present discounted value $x \cdot \mu$ of x for all $\mu \in \mathcal{M}_{++}$ (the second part being sometimes called the *Duality Theorem of Asset Pricing*).

The following result synthesizes the two parts in a unique statement and also considers the cases of present and weakly arbitrage-free markets, extensively used throughout the paper. The proof of the theorem is standard.

Theorem 2.1 (Fundamental Theorem of Asset Pricing) *The following assertions are equivalent for the market $M := \big((V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$:*
(i) M is arbitrage-free (resp. present arbitrage-free, resp. weak arbitrage-free[3]);

[3]In the sense that, for all $(\alpha, \beta) \in \mathbb{R}_+^J \times \mathbb{R}_+^J$, one has:

(ii) $\mathcal{M}_{++} \neq \emptyset$ *(resp.* $\mathcal{M}_{+} \neq \emptyset$*, resp.* $\mathcal{M} \neq \emptyset$*);*

(iii) $\mathcal{M}_{++} \neq \emptyset$ *and* $\sup_{\mu \in \mathcal{M}_{++}} x \cdot \mu = c_+(x) \neq -\infty$ *for all* $x \in \mathbb{R}^\Omega$*,*
(resp. $\mathcal{M}_{+} \neq \emptyset$ *and* $\sup_{\mu \in \mathcal{M}_{+}} x \cdot \mu = c_+(x) \neq -\infty$ *for all* $x \in \mathbb{R}^\Omega$*,*
resp. $\mathcal{M} \neq \emptyset$ *and* $\sup_{\mu \in \mathcal{M}} x \cdot \mu = c(x) \neq -\infty$ *for all* $x \in \mathbb{R}^\Omega$*).*

Moreover, if the market M *is present arbitrage-free, then* $c_+ : \mathbb{R}^\Omega \to \mathbb{R} \cup \{+\infty\}$ *and* $c : \mathbb{R}^\Omega \to \mathbb{R} \cup \{+\infty\}$ *are sublinear*[4] *and the following hold:*

- *[Cost Spread]* $-c(-x) \leq -c_+(-x) \leq c_+(x) \leq c(x)$ *for all* $x \in \mathbb{R}^\Omega$*,*
- *[Frictionless]* $\underline{q}^j = \overline{q}^j$ *implies* V^j *frictionless, i.e.,* $-c(-V^j) = c(V^j)$*.*

We recall that if the frictionless bond $(\mathbf{1}_\Omega, q^0)$ is a marketed security of the arbitrage-free market M, then the stochastic discount factors $\mu \in \mathcal{M}_{++}$ can be uniformly normalized as *risk-neutral probabilities*, in a standard way, as follows.

Remark 2.2 (Risk-neutral Probabilities) Consider the arbitrage-free market:

$$M := \big((\mathbf{1}_\Omega, q^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big).$$

By Theorem 2.1, the present discounted value of the bond $\mathbf{1}_\Omega$ is independent of the choice of the stochastic discount factor μ in $\mathcal{M}_{++}$ since $c_+(\mathbf{1}_\Omega) = c(\mathbf{1}_\Omega) = q^0 = \mathbf{1}_\Omega \cdot \mu := \mu(\Omega) > 0$ for all $\mu \in \mathcal{M}_{++}$. Thus, $c_+(\mathbf{1}_\Omega)$ is interpreted as a *risk-free discount factor* and the *risk-free interest rate* r is (uniquely) defined by $c_+(\mathbf{1}_\Omega) = 1/1 + r$. So we can (uniformly) normalize each stochastic discount factor $\mu \in \mathcal{M}_{++}$ as a *risk-neutral probability*, that is, as an element of the set:

$$(1+r)\mathcal{M}_{++} = \big\{P \in \mathrm{Proba}_{++}(\Omega) : \underline{q}^j \leq \tfrac{1}{1+r}\, E_P(V^j) \leq \overline{q}^j \ \forall j = 0, \ldots, J\big\} \neq \emptyset.$$ [5]

Thus, for every risk-neutral probability P, the discounted expected payoff of every security j belongs to its price spread $[\underline{q}^j, \overline{q}^j]$ (hence is equal to the asset price $\underline{q}^j = \overline{q}^j$ whenever the security j is frictionless). □

Remark 2.3 (Multi-prior model of Gilboa and Schmeidler (1989)) It is worth noticing that the Fundamental Theorem of Asset Pricing (Theorem 2.1) provides a natural rationale for the famous multi-prior model of Gilboa and Schmeidler (1989), extensively used in this paper in its dual form, i.e., with a "sup" since we are dealing with cost functions, instead of an "inf" as in Gilboa and Schmeidler (1989) who are considering utility functions.

$\sum_{j \in \boldsymbol{J}} (\alpha^j - \beta^j) V^j = 0 \Rightarrow \sum_{j \in \boldsymbol{J}} \overline{q}^j \alpha^j - \underline{q}^j \beta^j \geq 0$ [*Weak Arbitrage-free*].

[4] $f : \mathbb{R}^\Omega \to \mathbb{R} \cup \{+\infty\}$ is sublinear if it is subadditive, i.e., $f(x + x') \leq f(x) + f(x')$ for all x, x', and positively homogeneous, i.e., $f(0) = 0$ and $f(tx) = tf(x)$ for all x and all $t > 0$.

[5] Where $\mathrm{Proba}_{++}(\Omega) := \{P \in \mathbb{R}^\Omega_{++} : \sum_{\omega \in \Omega} P(\omega) = 1\}$ is the set of strictly positive probability measures on Ω and $E_P(X) := X \cdot P$ is the expected payoff of $X : \Omega \to \mathbb{R}$.

Actually, since the market M is arbitrage-free, it exhibits a nonempty family of stochastic discount factors $\mathcal{M}_{++}$ (thus a family $\mathcal{P}$ of risk-neutral probability measures as in Remark 2.2 whenever the frictionless bond is part of the market M), consistent with the bid and ask security prices at $t = 0$ and the prevalent uncertainty at $t = 1$. Furthermore, the present value of the payoff $x : \Omega \to \mathbb{R}$ calculated by $c_+(x) = \sup_{\mu \in \mathcal{M}_{++}} x \cdot \mu$ (as in Theorem 2.1) is in the line with the multi-prior model of Gilboa and Schmeidler (1989). □

We end this section with the definition of an equivalence relation that is extensively used throughout the paper. Two markets M and M' are said to have the **same market structure** or to be **indistinguishable**, denoted $M \equiv M'$, whenever they have the same set of stochastic discount factors, that is, $\mathcal{M} = \mathcal{M}'$. From Theorem 2.1, if two markets are indistinguishable, they have the same hedging and super-hedging prices, i.e., $c = c'$ and $c_+ = c'_+$.[6] The marketed security $(V^j, \underline{q}^j, \overline{q}^j)$ of the market M is said to be **redundant**, if the market M_{-j}, obtained from M by removing its j-th security, is indistinguishable from M.

2.3. *Hedging complements*

The formal expression of perfect complementarity, which dates back to Fisher, Pareto, and Edgeworth, according to Samuelson (1974), is usually formulated by a *submodularity* assumption on cost functions and by a *supermodularity* assumption when dealing with utility and payoff functions. Submodularity (supermodularity) can then be equivalently reformulated as a decreasing (increasing) difference property (see, e.g. Topkis (1998)) and this property will be taken as the primitive definition of complementarity in our paper since it has a natural interpretation when dealing with hedging prices.

Two payoffs X^1, X^2 in $\mathbb{R}^\Omega$ are said to be **hedging complements** for the market M if its hedging price $c : \mathbb{R}^\Omega \to \mathbb{R}$ satisfies the following decreasing difference property for all $x \in \operatorname{span}\{X^1, X^2\}$ and for all $\alpha_1 \geq 0$, $\alpha_2 \geq 0$:

$$c(x + \alpha_1 X_1 + \alpha_2 X_2) - c(x + \alpha_2 X_2) \leq c(x + \alpha_1 X_1) - c(x),$$

that is, *the net cost of buying the first security (once the second one is bought) is not greater than the net cost of buying the same amount of the first security.*

[6] Conversely, if $c = c'$, then M and M' are indistinguishable since from Theorem 2.1, one can prove that $\mathcal{M} = \partial c(0)$. The converse is not true in general with super-hedging prices.

In particular, if $X^1 := \mathbf{1}_{\omega_1}$, $X^2 := \mathbf{1}_{\omega_2}$ are two Arrow securities on states $\omega_1 \neq \omega_2$, say (tomorrow) "my house is burnt" and "my car is stolen" the decreasing difference property reads now as : *the net cost for insuring the house (once the car is insured) is not greater than the net cost for insuring the house.*

We now formally define a market with hedging complements.

Definition 2.4 Let $c : \mathbb{R}^\Omega \to \mathbb{R} \cup \{+\infty\}$, the family $(X^1, \ldots, X^K)$ of vectors in $\mathbb{R}^\Omega$ is said to be *complements* with respect to c if:

[**Complements**] c is finite on $\langle X \rangle := \operatorname{span}\{X^1, \ldots, X^K\}$ and for all $x \in \langle X \rangle$

$c(x + \alpha X^j + \beta X^k) - c(x + \beta X^k) \leq c(x + \alpha X^j) - c(x)$ for all $j \neq k$, all $\alpha \geq 0, \beta \geq 0$.

- *Consider now a market* $M := \big((V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$.

[**Hedging Complements**] The payoffs $X^1, \ldots, X^K$ in $\mathbb{R}^\Omega$ are *hedging complements* for the market M if they are complements for its hedging price c.

The market M has *hedging complements* if $M \equiv \bar{M} := \big((V^1, \underline{q}^1, \bar{q}^1), \ldots, (V^J, \underline{q}^J, \bar{q}^J)\big)$ and the payoffs $V^1, \ldots, V^J$ are complements of the hedging price c of M.[7]

The hedging complement property of a market is thus formalized in the same way as strategic complements of agents' payoff functions is formulated in game theory (see Bulow, Geanakoplos, and Klemperer (1985)).

The following result summarizes the main properties of hedging complements specific to financial markets. The first assertion is the standard characterization of complements by different types of submodularity properties. Second, if a market with hedging complements is merged with frictionless securities, the merger will still have hedging complements. Thus the class of markets with hedging complements contains markets with frictionless securities. The last property is used in the next section to characterize a stronger notion of hedging complements.

In the following, we let $\langle V \rangle$ be the vector space spanned by the vectors $V^1, \ldots, V^J$. The linear mapping $V : \mathbb{R}^J \to \langle V \rangle$ is defined by $V\theta = \sum_{j=1}^J \theta^j V^j$. When the vectors $V^1, \ldots, V^J$ are independent, then V is bijective, its inverse is denoted by V^{-1}, and the **order** $\geq_V$ on $\langle V \rangle$ is defined by $x' \geq_V x$ whenever $V^{-1}(x') \geq V^{-1}(x)$ (denoting $\geq$ the coordinatewise order of $\mathbb{R}^J$).

Proposition 2.5 *Let* $c : \mathbb{R}^\Omega \to \mathbb{R} \cup \{+\infty\}$ *be sublinear, let* $V^j \in \mathbb{R}^J$ *(*$j = 1, \ldots, J$*).*

- *The following assertions (1) and (2) are equivalent:*

[7] Note that c is also the hedging price of $\bar{M}$ since $M \equiv \bar{M}$.

(1) *the vectors* $V^1, \dots, V^J$ *in* $\mathbb{R}^\Omega$ *are complements for* c*;*

(2) $f := c \circ V$ *is submodular (for the coordinate-wise order* $\geq$ *on* $\mathbb{R}^J$*), i.e.:*

$f(\theta \vee \theta') + f(\theta \wedge \theta') \leq f(\theta) + f(\theta')$ *for all* θ, θ' *in* $\mathbb{R}^J$*;*

- *and also equivalent to* (3) *whenever the vectors* $V^1, \dots, V^J$ *are independent:*

(3) c *is submodular on* $\langle V \rangle$ *for the order* $\geq_V$*, i.e.:*

$c(x \vee_V x') + c(x \wedge_V x') \leq c(x) + c(x')$ *for all* x, x' *in* $\langle V \rangle$ *;*[8]

- *and also equivalent to* (4) *for every frictionless vector* $V^0 \in \mathbb{R}^\Omega$

(4) *the vectors* $V^0, V^1, \dots, V^J$ *are complements for* c*.*[9]

- *If each* V^j $(j = 1, \dots, J)$ *is frictionless, then* $V^1, \dots, V^J$ *are complements for* c*.*
- *Every market* M *with frictionless securities has hedging complements.*
- *If* $V^1, \dots, V^J$ *in* $\mathbb{R}^\Omega$ *are complements for* c *and* $V^1 = V\bar{\theta}$ *for some* $\bar{\theta} := (0, \theta^2, \dots, \theta^J) \geq 0$ *(e.g.,* $V^1 = V^j$ *for some* $j \neq 1$*), then* V^1 *is frictionless.*

The proof is given in the Supplementary Material Appendix A.

2.4. *Strong hedging complements*

The previous definition of complements has an equivalent formulation, when the couples (V^j, V^k) of payoffs are replaced by more general bundles of vectors.

Remark 2.6 The market M has hedging complements if and only if its hedging price c satisfies for all $x \in \langle V \rangle$, for all α, β in $\mathbb{R}^J_+$ such that $\alpha \wedge \beta = 0$:

$$c(x + V\alpha + V\beta) - c(x + V\beta) \leq c(x + V\alpha) - c(x).$$

This is a standard result, whose proof can be found in Topkis (1998), considering the function $f := c \circ V : \langle V \rangle \to \mathbb{R}$. □

Thus, the hedging complements property can be strengthened by removing the condition $\alpha \wedge \beta = 0$ as follows.

Definition 2.7 The market M has *strong hedging complements* if $M \equiv \bar{M} := \left((V^1, \underline{q}^1, \bar{q}^1), \dots, (V^J, \underline{q}^J, \bar{q}^J)\right)$ and the payoffs $V^1, \dots, V^J$ are strong complements of the hedging price c of M:[10]

[Strong Hedging Complements] for all $x \in \langle V \rangle$, for all α, β in $\mathbb{R}^J_+$

$$c(x + V\alpha + V\beta) - c(x + V\beta) \leq c(x + V\alpha) - c(x).$$

[8]The definition of submodularity of c on $\langle V \rangle$ has a clear meaning since $(\langle V \rangle, \geq_V)$ is a Riesz space. See the Supplementary Material Appendix C.

[9]and, by a simple induction argument, for every family of vectors $(V^{01}, \dots, V^{0K})$ in $\mathbb{R}^\Omega$, it is also equivalent to $(4')$ the vectors $V^{01}, \dots, V^{0K}, V^1, \dots, V^J$ are complements for c.

[10]Note that c is also the hedging price of $\bar{M}$ since $M \equiv \bar{M}$.

The strengthened property of complements has been introduced for the study of cost games. See Sharkey (1982) and Moulin (1992), and also Marinacci and Montrucchio (2005) and Müller and Scarsini (2012) for a general study of functions satisfying the strong complementarity property.

The next result shows that Strong Hedging Complements characterizes frictionless markets, that is, which are indistinguishable from markets with frictionless securities (Assertion (iv) hereafter).

Proposition 2.8 *Let M be a present arbitrage-free market, with hedging price c. Then the following assertions are equivalent:*

(i) the market M has strong hedging complements;

(ii) $M \equiv \big((V^1, \underline{q}^1, \overline{q}^1), \dots, (V^J, \underline{q}^J, \overline{q}^J)\big)$ and $-c(-V^j) = c(V^j)$ for all j;

(iii) c is (finite-valued and) linear on the space of replicable payoffs;

(iv) M is a frictionless market, that is, $M \equiv \big((V^1, q^1), \dots, (V^J, q^J)\big)$.

If (iv) holds, then $q^j := c(V^j)$ for all j.

The equivalence $[(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)]$ is a standard property of frictionless markets. The linearity of the hedging price c clearly implies that M has strong hedging complements, thus $[(iii) \Rightarrow (i)]$. We now prove $[(i) \Rightarrow (ii)]$. Assume $M \equiv \bar{M} := \big((V^1, \underline{q}^1, \overline{q}^1), \dots, (V^J, \underline{q}^J, \overline{q}^J)\big)$ and the payoffs $V^1, \dots, V^J$ are strong complements of c. Notice that the two markets $\bar{M}$ and $(\bar{M}, \bar{M})$[11] are indistinguishable, hence they have the same hedging price c. One checks that the security payoffs $(V^1, \dots, V^J, V^1, \dots, V^J)$ of $(\bar{M}, \bar{M})$ are also strong complements of c, and in particular complements of c. But every V^j appears twice in the list of complements, thus V^j is frictionless by the last assertion of Proposition 2.5. A detailed proof is given in the Supplementary Material Appendix A.

3. MARKETS WITH INDEPENDENT SECURITIES

The last part of the paper studies markets with independent securities, which will be proved to be a subclass of markets with hedging complements, but whose definition depends now only on the primitive features of the market.

Definition 3.1 The market M is said to have **independent securities** if

$M \equiv \bar{M} := \big((V^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \dots, (V^J, \underline{q}^J, \overline{q}^J)\big)$ satisfying:

[11] That is, the merger (or the concatenation) of $\bar{M}$ with itself. Formally: $(\bar{M}, \bar{M}) := \big((V^1, \underline{q}^1, \overline{q}^1), \dots, (V^J, \underline{q}^J, \overline{q}^J), (V^1, \underline{q}^1, \overline{q}^1), \dots, (V^J, \underline{q}^J, \overline{q}^J)\big)$.

PAF: *the market M is present arbitrage-free,*

IND: *the marketed security payoffs $V^1, \ldots, V^J$ are independent vectors.*

Note that, the market $\bar{M}$ contains the frictionless security (V^0, q^0), usually taken to be the frictionless bond $(\mathbf{1}_\Omega, q^0)$ in finance, and the independence assumption IND is made only on securities $j = 1, \ldots J$ (excluding $j = 0$). The relationship between the classes of markets we consider in this paper can be summarized as follows, considering all markets hereafter to be present arbitrage-free:

Frictionless $\Leftrightarrow$ **Strong Hedging Complements**

$\Rightarrow$ **Independent Securities** $\Rightarrow$ **Hedging Complements**.

Add **Hedging Complements** and ?? $\Rightarrow$ Generalized Choquet

Indeed, the equivalence property is prove din Proposition 2.8. Second, when all securities of M are frictionless, every security whose payoff is a linear combination of the other marketed security payoffs is redundant, that is, it can be eliminated without changing the market structure. Thus by eliminating all its redundant assets, every frictionless market M is indistinguishable from a market $\bar{M}$ satisfying **IND**. Finally, the second implication is proved in the next sections (Theorem 3.5 hereafter), together with a tractable formula for the hedging price that depends only on the primitive features of the market.

Other examples of markets with **independent securities** are given in the next section (see Example 3.3).

3.1. *Normalized markets with independent securities*

Consider the following market with independent securities:

$M := ((\mathbf{1}_{12}, 1), (2\mathbf{1}_1, 1, 1.2), (-\mathbf{1}_2, -.6, -.4))$ with $\Omega = \{1, 2\}$.

Then, the market M is indistinguishable from:

$\bar{M} := ((\mathbf{1}_{12}, 1), (\mathbf{1}_1, .5, .6), (\mathbf{1}_2, .4, .6))$ that satisfies $\mathbf{1}_{12} = \mathbf{1}_1 + \mathbf{1}_2$.

It appears that the above simple transformation of the market M in $\bar{M}$, by rescaling securities and prices, or by changing the signs of securities and prices accordingly, works in the general case of markets with independent securities, and allows to write M in a ("better") normal form that will be of fundamental use hereafter (see Theorem 3.5). In other words, *without any loss of generality*, markets with independent securities can always be normalized as follows.

PROPOSITION 3.2 *If M is a market with independent securities, then*

$M \equiv \bar{M} := \big((V^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$
with $\bar{M}$ satisfying **PAF**, **IND**, *together with:*

NF: *either* $V^0 = \sum_{j \in \bar{\boldsymbol{J}}} V^j$ *for some* $\bar{\boldsymbol{J}} \subseteq \boldsymbol{J}$, $\bar{\boldsymbol{J}} \neq \emptyset$, *or* $V^0 = 0$.[12]

A market $\bar{M}$ satisfying **NF** *is said to be normalized, or to be written in Normal Form. We will also say that that $\bar{M}$ is a normalization of M.*

Proof: We distinguish the following three cases.

- $V^0 = 0$, thus, M is already written in normal form (and in this case $q^0 = 0$).
- $V^0 \notin \langle V \rangle$ implies that the vectors $V^0, V^1, \ldots, V^J$ are independent. Let $\bar{M} := ((0,0), M)$ be the merger of M and the frictionless security $(0,0)$, then $M \equiv \bar{M}$ and $\bar{M}$ satisfies PAF, IND, and NF since its first security payoff is null.
- $V^0 \in \langle V \rangle$ and $V^0 \neq 0$. Then, $V^0 = \sum_{j \in \boldsymbol{J}} \theta^j V^j$ for some $\theta^j \in \mathbb{R}$ $(j \in \boldsymbol{J})$ not all zero and one defines the market $\bar{M}$ as follows:

$$\bar{M} := \big((V^0, q^0), (\bar{V}^1, \underline{\underline{q}}^1, \overline{\overline{q}}^1), \ldots, (\bar{V}^J, \underline{\underline{q}}^J, \overline{\overline{q}}^J)\big), \text{ and for } j \in \boldsymbol{J}:$$

$$\big(\overline{V}^j, \underline{\underline{q}}^j, \overline{\overline{q}}^j\big) = \begin{cases} \big(V^j, \underline{q}^j, \bar{q}^j\big) & \text{if } \theta^j = 0, \\ \big(\theta^j V^j, \theta^j \underline{q}^j, \theta^j \bar{q}^j\big) & \text{if } \theta^j > 0, \\ \big(\theta^j V^j, \theta^j \bar{q}^j, \theta^j \underline{q}^j\big) & \text{if } \theta^j < 0. \end{cases}$$

Then one checks that $M \equiv \bar{M}$, $\bar{M}$ satisfies PAF and IND, and $V^0 = \sum_{j \in \bar{\boldsymbol{J}}} \bar{V}^j$ where $\bar{\boldsymbol{J}} = \{j \in \boldsymbol{J} : \theta^j \neq 0\} \neq \emptyset$. *Q.E.D.*

We end this section with several examples of markets that help to better understand the class of normalized markets with independent securities.

Example 3.3 Examples of normalized markets with independent securities are given hereafter, assuming all markets are present arbitrage-free.

- $M = \big((V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$, with $V^1, \ldots, V^J$ independent vectors. [13]
- *Markets with frictionless securities:* $M = \big((V^1, q^1), \ldots, (V^J, q^J)\big)$.

We saw previously that M has independent securities, and the market $\bar{M}$ obtained after eliminating from M all its redundant assets, is normalized.

- *Markets with* **all** *Arrow securities or* **partition** *event securities:* (see Sec. 3.5)

$M_E := \big((\mathbf{1}_\Omega, q^0), (\mathbf{1}_{A^1}, \underline{q}^1, \overline{q}^1), \ldots, (\mathbf{1}_{A^J}, \underline{q}^J, \overline{q}^J)\big)$, with $A_j \subseteq \Omega$, partition of Ω

[12] This assumption can be equivalently rewritten in the more concise way as follows: $V^0 = \sum_{j \in \bar{\boldsymbol{J}}} V^j$ for some $\bar{\boldsymbol{J}} \subseteq \boldsymbol{J}$, with the convention that $\sum_{j \in \emptyset} V^j = 0$.

[13] Indeed, $M \equiv \bar{M} := ((V^0, q^0), M)$ with $(V^0, q^0) = (0, 0)$. Thus, $\bar{M}$ has independent securities and $\bar{M}$ is normalized since $V^0 = 0$).

$M_A := \big((\mathbf{1}_\Omega, q^0), (\mathbf{1}_1, \underline{q}^1, \overline{q}^1), \ldots, (\mathbf{1}_J, \underline{q}^J, \overline{q}^J)\big)$ with $\Omega = \boldsymbol{J} = \{1, \ldots, J\}$.

• *Markets with Arrow securities or disjoint event securities:* (see Section 3.5)

If the $A_j \subseteq \Omega$ $(j = 1, \ldots, J)$ are only pairwise disjoint and $\cup_j A^j \neq \Omega$ (hence also when M_A does not have all Arrow securities), the market M_E has independent securities since the vectors $\mathbf{1}_\Omega, \mathbf{1}_{A_1}, \ldots, \mathbf{1}_{A_J}$ are independent. □

EXAMPLE 3.4 • $[\bar{\boldsymbol{J}} \subsetneq \boldsymbol{J}]$ M is normalized with independent securities:

$M := ((\mathbf{1}_{12}, 1), (\mathbf{1}_1, .5)(\mathbf{1}_2, .5)(\mathbf{1}_3, .4, .6))$ with $\bar{\boldsymbol{J}} = \{1, 2\} \subsetneq \boldsymbol{J} = \{1, 2, 3\}$.

Moreover the hedging price of M is not a Choquet integral (Example 3.7).

3.2. *A tractable two-part formula for the hedging price*

We introduce the main result of this section with two particular cases. First, we consider the present arbitrage-free market:

$M = \big((V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$, with $V^1, \ldots, V^J$ independent vectors.

Then, we prove (Theorem 3.5) that the hedging price c satisfies:

$c(x) = \sum_{j=1}^J \overline{q}^j (\theta^j)^+ - \underline{q}^j (\theta^j)^-$ for all replicable payoff $x \in \langle V \rangle$,

that is, whenever $x := V\theta$ for some $\theta = (\theta_1, \ldots, \theta_J) \in \mathbb{R}^J$.

Then the market M has hedging complements (by Proposition 2.5) since $c \circ V$ is submodular (and is in fact modular). This provides a natural generalization of the standard frictionless case for which $c : \langle V \rangle \to \mathbb{R}$ is linear and satisfies $c(x) = q \cdot \theta$ whenever $x := V\theta$.

Second, we consider the present arbitrage-free market M_A with all Arrow securities, together with the frictionless bond:

$M_A := \big((\mathbf{1}_\Omega, q^0), (\mathbf{1}_{\omega_1}, \underline{q}^1, \overline{q}^1), \ldots, (\mathbf{1}_{\omega_J}, \underline{q}^J, \overline{q}^J)\big)$ with $\Omega = \boldsymbol{J} = \{1, \ldots, J\}$.

Then the market M_A has hedging complements since its hedging price c_A is a submodular Choquet integral (Theorem 3.5):

$c_A(x) = \int_\Omega x dv_A$ for all $x \in \mathbb{R}^\Omega$ [$\int_\Omega$ denoting the Choquet integral].

where v_A is submodular and defined by the explicit formula:

$v_A(K) = \min\big\{ \sum_{j \in K} \overline{q}^j \ , \ q^0 - \sum_{j \notin K} \underline{q}^j \big\}$ for $K \subseteq \boldsymbol{J} = \Omega$.

Moreover, both formulas are meaningful (Remark 3.9) and make tractable the computation of the hedging price $c(x)$ and $c_A(x)$ but also to find a strategy to obtain the hedging price (see Example 3.6).

More generally, the two above results are consequences of the following theorem that considers markets with independent securities. The hedging price is proved to be the sum of two parts. The first part is a Choquet integral similar to the one in our second example with Arrow securities[14] and the second part is a modular function of the portfolios as in our first example.

Theorem 3.5 *Consider a market M with independent securities:*

$$M \equiv \bar{M} := \big((V^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big) \text{ and } \bar{M} \text{ is normalized.}$$

Then the hedging price c of M satisfies the following properties:

- $c(V\theta) = \int_{\bar{\boldsymbol{J}}} (\theta_1, \ldots, \theta_{\bar{J}})\, dv_A \; + \sum_{j \in \boldsymbol{J} \setminus \bar{\boldsymbol{J}}} \overline{q}^j (\theta^j)^+ - \underline{q}^j (\theta^j)^- \;\; \forall \theta = (\theta_1, \ldots, \theta_J) \in \mathbb{R}^J$ *where $v_A : 2^{\bar{\boldsymbol{J}}} \to \mathbb{R}$ is submodular and defined by:* $v_A(K) = \min \big\{ \sum_{j \in K} \overline{q}^j \; , \; q^0 - \sum_{j \notin K} \underline{q}^j \big\}$ *if $K \subseteq \bar{\boldsymbol{J}}$.*
- *M has hedging complements, the function $c \circ V$ is submodular, and c is submodular on $\langle V \rangle$ for the order $\geq_V$.*

The proof of Theorem 3.5 is given in the Supplementary Material Appendix B. The two-part formula is obtained by separating the Choquet and the modular parts of c and show they are the hedging prices respectively of the markets:

$$M_c := \big((V^0, q^0), (V^j, \underline{q}^j, \overline{q}^j)_{j \in \bar{\boldsymbol{J}}}\big) \text{ and } M_m := \big(V^j, \underline{q}^j, \overline{q}^j\big)_{j \in \bar{\boldsymbol{J}} \setminus \bar{\boldsymbol{J}}}.$$

In other words, it is proved that M can be decomposed in the two markets M_c and M_m, which can thus be studied independently. The first part of the formula is null whenever $V^0 = 0$ (or equivalently if $\bar{\boldsymbol{J}} = \emptyset$), in which case $c \circ V$ is modular (and linear whenever the market M is frictionless). The second part is null whenever $\bar{\boldsymbol{J}} = \boldsymbol{J}$, in which case c is a generalized Choquet integral (see Section 3.4 for a more precise result). These two cases correspond respectively to the two above examples.

For the second part of the proof, we recall that M has hedging complements if $M \equiv \bar{M}$ and the payoffs $V^0, V^1, \ldots, V^J$ of $\bar{M}$ are complements for the hedging price c (Definition 2.4), which is equivalent to $V^1, \ldots, V^J$ complements for c (since V^0 is frictionless by (1) $\Leftrightarrow$ (4) in Proposition 2.5), and is also equivalent to $c \circ V$ submodular ((1) $\Leftrightarrow$ (2) in Proposition 2.5). Then $c \circ V$ is submodular from the two part formula since the first part is submodular and the second part is modular.

We stress the fact that the two-part formula of Theorem 3.5 makes tractable the computation of the hedging price $c_A(x)$ but also the way to find a strategy

[14]In the theorem it is formulated as a Choquet integral in the portfolio space and itwill be later interpreted as a generalized Choquet integral in the payoff space (see Remark 3.10)

to obtain $c_A(x)$ (see Example 3.6 hereafter). Choquet integral calculus provides powerful tools for computation (e.g., see Denneberg (1994), Gilboa (2009), and Marinacci and Montrucchio (2004)) but especially when we have also (as in Theorem 3.5) an explicit and tractable formula for the set function v_A.

EXAMPLE 3.6 Consider the market with independent securities:

$M := \big((\mathbf{1}_\Omega, 1), (V^1, 0, 1), (V^2, \frac{5}{18}, \frac{5}{18}), (V^3, 0, 1)\big)$, with $\Omega = \{1, 2, 3\}$ and

$V^1 := (1, \frac{1}{2}, \frac{1}{3})$, $V^2 := (0, \frac{1}{2}, \frac{1}{3})$, $V^3 := (0, 0, \frac{1}{3})$.

Let $x := (1, 0, 1) = 3\mathbf{1}_\Omega - 2V^1 - 4V^2$, then

$c(x) = 3c(\mathbf{1}_\Omega) + 2c(-V^1) + 4c(-V^2) = 17/9$

provides a strategy to get the hedging price $c(x)$, that is buying 3 units of the bond, selling 2 units of the first security and 4 units of the second security. [15]

3.3. *Put-call Parity à la Cerreia-Vioglio, Maccheroni, and Marinacci (2015)*

Following Cerreia-Vioglio, Maccheroni, and Marinacci (2015), the put-call Parity holds for the hedging price $c : \mathbb{R}^\Omega \to \mathbb{R}$ (without specifying the market it is derived from) whenever the following equality holds:

PCP $c([x - k\mathbf{1}_\Omega]^+) + c(-[-x + k\mathbf{1}_\Omega]^+) = c(x) - kc(\mathbf{1}_\Omega)$ $\forall x \in \mathbb{R}^\Omega$, $\forall k \geq 0$,

where k is the strike price, $[x - k\mathbf{1}_\Omega]^+$ is a call option on x, and $[-x + k\mathbf{1}_\Omega]^+$ is a put option on x (where $[\cdot]^+$ is associated with the pointwise order of $\mathbb{R}^\Omega$).

Thus the basic financial identity:

Call - Put = Stock - (Strike Price) $\times$ Bond

is formulated mathematically as:

$$[x - k\mathbf{1}_\Omega]^+ - [-x + k\mathbf{1}_\Omega]^+ = x - k\mathbf{1}_\Omega.$$

The above property **PCP** is similar to the call-put parity CPP introduced by Chateauneuf, Kast, and Lapied (1996). Both properties PCP and CPP coincide

---

[15] Clearly M satisfies Assumptions **PAF**, **IND**, and **NF** since $\mu := (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \in \mathcal{M}_+$, the vectors V^1, V^2, V^3 are independent, and $\mathbf{1}_\Omega = V^1 + V^2 + V^3$.

Moreover, from Theorem 3.5, since $c \circ V$ is a Choquet integral we get $x = 3V^3 + V^1 - V^2 = V(3\mathbf{1}_3 + \mathbf{1}_1 - \mathbf{1}_2)$. Thus, from the definition of the Choquet integral:

$$\begin{aligned} c(x) &= c \circ V(3\mathbf{1}_3 + \mathbf{1}_1 - \mathbf{1}_2) = (3-1)v(\{3\}) + (1-(-1))v(\{13\}) - v(\{123\}) \\ &= 2\min\{1, 1 - 0 - \tfrac{5}{18}\} + 2\min\{1+1, 1 - \tfrac{5}{18}\} - \min\{1 + \tfrac{5}{18} + 1, 1\} = \tfrac{17}{9}. \end{aligned}$$

Note that the above formula can be rewritten

$$\begin{aligned} c(x) &= 2(q^0 - \underline{q}^1 - \underline{q}^2) + 2(q^0 - \underline{q}^2) - q^0 = 3q^0 - 2\underline{q}^1 - 4\underline{q}^2, \\ &= 3c(\mathbf{1}_\Omega) + 2c(-V^1) + 4c(-V^2) \text{ [since } \underline{q}^j = -c(-V^j) \text{ and } \overline{q}^j = c(V^j)]. \end{aligned}$$ *Q.E.D.*

and hold when $c : \mathbb{R}^\Omega \to \mathbb{R}$ is linear. Then the fundamental result by Cerreia-Vioglio, Maccheroni, and Marinacci (2015) (when c is monotone) and Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015) (in the non-monotone case, which allows to consider hedging prices) proves that the put-call parity characterizes pricing rules $c : \mathbb{R}^\Omega \to \mathbb{R}$ that are Choquet integrals, assuming the bond is a frictionless security.

The following example shows that the put-call property **PCP** may still hold when the frictionless bond $\mathbf{1}_\Omega$ is slightly perturbed, and the hedging price may not be represented by a Choquet functional. Consider for example the case for which the bond pays 1 in all states but one, that is, more generally the bond is replaced by $\mathbf{1}_A$ for an event $A \subsetneq \Omega$ distinct from Ω.

Example 3.7 Consider $\Omega := \{1, 2, 3\}$, and the market with 4 securities:

$$M := \big((\mathbf{1}_{12}, 1), (\mathbf{1}_1, .5), (\mathbf{1}_2, .5), (\mathbf{1}_3, .4, .6)\big).$$

Simple calculation shows that its hedging price $c : \mathbb{R}^3 \to \mathbb{R}$ is given by:

$$c(x_1, x_2, x_3) = (.5)x_1 + (.5)x_2 + (.6)[x_3]^+ - (.4)[x_3]^-.$$

Moreover, c satisfies the put-call parity **PCP**, when the bond $\mathbf{1}_{123}$ in PCP is replaced by the frictionless security $\mathbf{1}_A$, with $A = \{1, 2\}$. That is, for all $x \in \mathbb{R}^\Omega$, all $k \geq 0$, noticing that $c(\mathbf{1}_A) = 1$, one has

$$c([x - k\mathbf{1}_A]^+) + c(-[-x + k\mathbf{1}_A]^+) = c(x) - k.$$[16]

However, c is not a Choquet integral since $c(\mathbf{1}_\Omega) = 1.6 \neq -c(-\mathbf{1}_\Omega) = 1.4$. □

The previous example extends naturally to the case of a present arbitrage-free market with all Arrow securities, together with a frictionless security $\mathbf{1}_A$ ($A \subseteq \Omega$)

$$M_A := \big((\mathbf{1}_A, q^0), (\mathbf{1}_1, \underline{q}^1, \overline{q}^1), \ldots, (\mathbf{1}_J, \underline{q}^J, \overline{q}^J)\big), \text{ with } \boldsymbol{J} = \Omega$$

and our next result will show that the market M_A satisfies the put-call parity PCP (with the bond replaced by $\mathbf{1}_A$) since M_A has independent securities (even in cases, e.g., $A \neq \Omega$, where its hedging price may not be a Choquet integral). The next result is stated for markets with independent securities and its marketed payoff space $\langle V \rangle$ is now endowed with the order $\geq_V$, which coincides with the coordinate-wise order of $\mathbb{R}^\Omega$ for the market M_A with Arrow securities.

Proposition 3.8 *Consider a market with independent securities:*

[16] In other words, in this modified put-call parity, the call would specify the reservation price to be of amount k if event A occurs and to be free otherwise. Symmetrically, the put would allow to sell the underlying asset for free if event $\{3\}$ occurs and at price k if event A occurs.

$$M \equiv \bar{M} := \big((V^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \dots, (V^J, \underline{q}^J, \overline{q}^J)\big) \text{ and } \bar{M} \text{ is normalized.}$$

Then its hedging price $c : \langle V \rangle \to \mathbb{R}$ satisfies

PCP $c([x - kV^0]^+) + c(-[-x + kV^0]^+) = c(x) - kc(V^0)\ \forall x \in \langle V \rangle\ \ \forall k \geq 0$

where $[\cdot]^+$ is associated with the order $\geq_V$.

The proof follows from routine arguments, once we know the submodularity property of the hedging price given by Theorem 3.5. Thus the put-call parity à la Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015), formulated in the case of a Riesz spaces (in the above proposition, the space $(\langle V \rangle, \geq_V)$) is an important consequence of the hedging complement property of markets with independent securities and may hold even when the hedging price is not a Choquet functional as noticed in Example 3.7.

Proof: • First we notice that c is constant additive, that is:

$$c(x + kV^0) = c(x) + kc(V^0)\ \forall x \in \langle V \rangle, \forall k \in \mathbb{R}.$$

This follows from the fact that (V^0, q^0) is frictionless, thus $c(-V^0) = -c(V^0)$ by Theorem 2.1, together with the sublinearity of c (Theorem 2.1).

• We now prove that **PCP** holds. By Theorem 3.5, the hedging price c is submodular on $\langle V \rangle$ for the order $\geq_V$. Thus for all $x \in \langle V \rangle$ and $k \geq 0$:

$$\begin{aligned} c([x-kV^0]^+)+c(-[-x+kV^0]^+) &= c((x-kV^0)\vee_V 0)+c((x-kV^0)\wedge_V 0) \\ &\leq c(x-kV^0)+c(0) \text{ [since } c \text{ is submodular on } \langle V \rangle \text{ for } \geq_V] \\ &= c(x)-kc(V^0) \text{ [since } c \text{ is constant additive and } c(0)=0]. \end{aligned}$$

Moreover, since c is subadditive by Theorem 2.1, one gets:

$$\begin{aligned} c([x-kV^0]^+)+c(-[-x+kV^0]^+) &= c((x-kV^0)\vee_V 0)+c((x-kV^0)\wedge_V 0) \\ &\geq c((x-kV^0)\vee_V 0+(x-kV^0)\wedge_V 0) = c(x-kV^0) \\ &= c(x)-kc(V^0) \text{ [since } c \text{ is constant additive].} \end{aligned}$$

Hence c satisfies **PCP**. *Q.E.D.*

3.4. *The hedging price as a generalized Choquet integral*

From Theorem 3.5, $c \circ V$ is a Choquet functional if we additional assume that:

(*) $V^0 = \sum_{j \in \boldsymbol{J}} V^j$, that is, $\bar{\boldsymbol{J}} = \boldsymbol{J}$,[17]

[17]Note the difference between Assumption (*) and the normalization property, that is, $V^0 = \sum_{j \in \bar{\boldsymbol{J}}} V^j$ for some $\bar{\boldsymbol{J}} \subseteq \boldsymbol{J}$, which can be assumed without any loss of generality, thanks to Proposition 3.2.

since the modular term of the two-part formula is null. This remark can be improved by showing that $c \circ V$ is a Choquet functional if and only if its modular term is linear, or equivalently if each of its corresponding security is frictionless.

COROLLARY 1 *Consider a market with independent securities:*

$M \equiv \bar{M} := \big((V^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$ *and* $\bar{M}$ *is normalized.*

- *Then* $c \circ V$ *is a Choquet integral if and only if* $\overline{q}^j = \underline{q}^j$ *for all* $j \in \boldsymbol{J} \setminus \bar{\boldsymbol{J}}$ *and in this case:*

  $c(V\theta) = \int_{\boldsymbol{J}} \theta \, dv$ *for all* $\theta \in \mathbb{R}^J$,

  where $v : 2^{\boldsymbol{J}} \to \mathbb{R}$ *is submodular and defined by:*

  $v(K) = \min\big\{ \sum_{j \in K} \overline{q}^j \ , \ q^0 + \sum_{j \in \boldsymbol{J} \setminus \bar{\boldsymbol{J}}} q^j - \sum_{j \notin K} \underline{q}^j \big\}$ *if* $K \subseteq \boldsymbol{J}$.

- *Moreover, if all securities are frictionless, then* v *is additive, that is,* $v(K) = \sum_{j \in K} q^j$, *and the Choquet integral coincides with the standard integral:*

  $c(V\theta) = \int_{\boldsymbol{J}} \theta dv = \sum_{j \in \boldsymbol{J}} \theta^j q^j$ *for all* $\theta \in \mathbb{R}^J$,

  $c(x) = \int_{\boldsymbol{J}} V^{-1}(x) dv = V^{-1}(x) \cdot q$ *for all* $x \in \langle V \rangle$.

The proof is given in the Supplementary Material Appendix B and amounts to recover Condition (*), by replacing the frictionless security (V^0, q^0) by $(V^0 + \sum_{j \in \boldsymbol{J} \setminus \bar{\boldsymbol{J}}} V^j, q^0 + \sum_{j \in \boldsymbol{J} \setminus \bar{\boldsymbol{J}}} q^j)$, without changing the market structure of $\bar{M}$.

The Choquet integral in Theorem 3.5 and Corollary 1 comes with an explicit and tractable formula for v_A, whose intuition is now presented in financial terms.

REMARK 3.9 Assume $V^0 = \sum_{j \in \boldsymbol{J}} V^j$. From Theorem 3.5 we get:

$c(\sum_{j \in K} V^j) = v_A(K) := \min\big\{ \sum_{j \in K} \overline{q}^j \ , \ q^0 - \sum_{j \notin K} \underline{q}^j \big\}$ for all $K \subseteq \boldsymbol{J}$. [18]

It is worth noticing that the **inequality** $c(\sum_{j \in K} V^j) \leq v_A(K)$ always holds and can be interpreted in financial terms and proved directly as follows.

There are at least two strategies to replicate the payoff $\sum_{j \in K} V^j$. The first one is to buy one unit of each security $j \in K$, whose cost is $\sum_{j \in K} \overline{q}^j$. The second strategy is to buy one unit of security 0 and sell one unit of each security $j \notin K$; thus $V^0 - \sum_{j \notin K} V^j = \sum_{j \in K} V^j$ and its cost is $q^0 - \sum_{j \notin K} \underline{q}^j$. Thus $c(\sum_{j \in K} V^j) \leq \min\big\{ \sum_{j \in K} \overline{q}^j \ , \ q^0 - \sum_{j \notin K} \underline{q}^j \big\} = v_A(K)$. In other words, the equality guarantees no other strategy yields a smaller cost than the "min". □

We now interpret c as a "generalized" Choquet integral in the payoff space.

[18] Indeed, $c(\sum_{j \in K} V^j) = c(V \mathbf{1}_K) = \int_{\boldsymbol{J}} \mathbf{1}_K \, dv_A = v_A(K)$.

Remark 3.10 (Generalized Choquet) If $V^0 = \sum_{j\in \boldsymbol{J}} V^j$, then

$$c(V\theta) = \int_{\boldsymbol{J}} (\theta_1, \ldots, \theta_J)\, dv_A \;\text{ for all } \theta = (\theta_1, \ldots, \theta_J) \in \mathbb{R}^J,$$
$$c(x) = \int_{\boldsymbol{J}} V^{-1}(x)\, dv_A \;\text{ for all } x \in \langle V \rangle \,.$$

Thus the hedging price c is a generalized Choquet integral, in the sense that, up to the isomorphism $V : \mathbb{R}^J \to \langle V \rangle$, $c \circ V$ is a standard Choquet integral. It happens that this notion coincides also with the Choquet integral on Riesz space (here $(\langle V \rangle, \geq_V)$) introduced by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015). See the Supplementary Material Appendix C. □

The assumption $V^0 = \sum_{j=1}^{J} V^j$ is essential for the Choquet representation of the hedging price (Example 3.7). It happens that it can be formulated in terms of order unit of the Riesz space $(\langle V \rangle, \geq_V)$ (see Appendix C for the definitions).

Remark 3.11 (Order Unit) Consider a normalized market M with independent securities, then $V^0 = \sum_{j\in \bar{\boldsymbol{J}}} V^j$ for some $\bar{\boldsymbol{J}} \subseteq \boldsymbol{J}$. Thus V^0 belongs to the positive cone of the Riesz space $(\langle V \rangle, \geq_V)$ and the following assertions are equivalent:

(i) $V^0 = \sum_{j\in \boldsymbol{J}} V^j$, that is, $\bar{\boldsymbol{J}} = \boldsymbol{J}$,

(ii) V^0 is an order unit of the Riesz space $(\langle V \rangle, \geq_V)$.

The order unit assumption is essential in Choquet representation theorems. See Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015), Cerreia-Vioglio, Maccheroni, and Marinacci (2015), Chateauneuf and Cornet (2018). □

We end this section with a result on the super-hedging price c_+ of the market M under the assumption that all the security payoffs are nonnegative. We note that this assumption can be made without any loss of generality whenever $V^0 \gg 0$, as for example when $V^0 := \mathbf{1}_\Omega$ is the bond.[19] We define $v_{A+} : 2^{\boldsymbol{J}} \to \mathbb{R}$ by:

$$v_{A+}(K) = \min \left\{ \sum_{j\in K} \overline{q}^j \;,\; q^0 - \sum_{j\notin K} (\underline{q}^j)^+ \right\} \;\text{ for all } K \subseteq \boldsymbol{J}.$$

Corollary 2 *Consider a market with independent securities:*

$$M := \big((V^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big) \text{ such that } V^0 = \sum_{j\in \boldsymbol{J}} V^j.$$

(a) Assume $V \geq 0$, that is, $V^j \geq 0$ for all $j \in \boldsymbol{J}$. Then one has:

$$c_+(x) \leq \int_{\boldsymbol{J}} V^{-1}(x) dv_{A+} \quad \text{for all } x \in \langle V \rangle \,.$$

[19] Indeed, if $V^0 \gg 0$, every market $M := \big((V^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$ with independent securities can be transformed in a market $M(t) \equiv M$ with independent securities and with nonnegative security payoffs as follows. Choose $t > 0$ large enough and define
$M(t) := \big((V^0, q^0), (V^1 + tV^0, \underline{q}^1 + tq^0, \overline{q}^1 + tq^0), \ldots, (V^J + tV^0, \underline{q}^J + tq^0, \overline{q}^J + tq^0)\big)$.

(*b*) *Assume* $V \geq 0$ *and* $V^{-1} \geq 0$*, that is,* $V\theta \geq 0$ *implies* $\theta \geq 0$.[20] *Then one has:*

$$c_+(x) = \int_{\boldsymbol{J}} V^{-1}(x) dv_{A+} \quad \textit{for all } x \in \langle V \rangle .$$

The proof is given in the Supplementary Material Appendix B. The next section provides an important class of markets satisfying all the assumptions of Corollary 2, thus for which the super-hedging price is a Choquet integral.

3.5. *Markets with disjoint event securities*

This section is devoted to the study of the market:

$$M_E := \big((\mathbf{1}_\Omega, q^0), (\mathbf{1}_{A_1}, \underline{q}^1, \overline{q}^1), \dots, (\mathbf{1}_{A_J}, \underline{q}^J, \overline{q}^J)\big), \text{ with } (A_j)_{j\in \boldsymbol{J}} \text{ partition of } \Omega.$$

This framework embodies the important case of a market M_A with all Arrow securities, that is, event securities, whose event A_j is a singleton, together with the frictionless bond. The case where A_j are pairwise disjoint but not a partition was discussed in Example 3.4. We define $v_E : \mathcal{A} \to \mathbb{R}$ and $v_{E+} : 2^\Omega \to \mathbb{R}$ by:

$v_E(A) = \min \big\{ \sum_{j \in \boldsymbol{J}(A)} \overline{q}^j \ , \ q^0 - \sum_{j \notin \boldsymbol{J}(A)} \underline{q}^j \big\}$ if $A \in \mathcal{A}$.

$v_{E+}(A) = \min \big\{ \sum_{j \in \boldsymbol{J}(A)} \overline{q}^j \ , \ q^0 - \sum_{j \notin \boldsymbol{J}(A)} (\underline{q}^j)^+ \big\}$ if $A \subseteq \Omega$,

where $\mathcal{A} := \{A_K := \cup_{j \in K} A_j \ : \ K \subseteq \boldsymbol{J}\} \subseteq 2^\Omega$ is the algebra in 2^Ω, generated by the partition $(A_1, \dots, A_J)$, with the convention $A_\emptyset = \emptyset$, and $\boldsymbol{J}(A) := \{j \in \boldsymbol{J} : A_j \cap A \neq \emptyset\}$.

Corollary 3 *Assume the following market is present arbitrage-free:*

$$M_E := \big((\mathbf{1}_\Omega, q^0), (\mathbf{1}_{A_1}, \underline{q}^1, \overline{q}^1), \dots, (\mathbf{1}_{A_J}, \underline{q}^J, \overline{q}^J)\big), \textit{ with } (A_j)_{j\in \boldsymbol{J}} \textit{ partition of } \Omega.$$

- *Then the market* M_E *has hedging complements.*
- *The hedging and super-hedging prices of* M_E *are submodular Choquet integrals:*

  $c_E(x) = \int_\Omega x dv_E$ *if* x *is replicable, i.e.,* $x = \sum_{j=1}^J \theta^j \mathbf{1}_{A_j}$ *for some* $\theta \in \mathbb{R}^J$,

  $c_{E+}(x) = \int_\Omega x dv_{E+}$ *for all* $x \in \mathbb{R}^\Omega$,

  and both v_E *and* v_{E+} *are submodular.*

The proof of Corollary 3 follows directly from Corollaries 1 and 2, and Lemma B.1 in the Supplementary Material Appendix B that summarizes the properties of v_E and v_{E+}. The market M_E has hedging complements by Corollary 1 since

[20] Note that V^{-1} is well defined since $V : \mathbb{R}^J \to \langle V \rangle$ is bijective for a market M with independent securities. Moreover, the assumption $V^{-1} \geq 0$ is stronger than the independence of $V^1, \dots, V^J$. If $V\theta = 0$, then $V\theta \geq 0$ and $V(-\theta) \geq 0$, thus $\theta \geq 0$ and $-\theta \geq 0$, and $\theta = 0$.

M_E has clearly independent securities and $\mathbf{1}_\Omega = \sum_{j \in \boldsymbol{J}} \mathbf{1}_{A_j}$. Moreover, for all replicable payoff x, that is, $x = \sum_{j \in \boldsymbol{J}} \theta^j \mathbf{1}_{A_j}$ for some $\theta \in \mathbb{R}^J$, one has:

$$\begin{aligned} c_E(x) &= \textstyle\int_{\boldsymbol{J}} \theta \, dv_A, \text{ by Corollary 1,} \\ &= \textstyle\int_\Omega x \, dv_E, \text{ by Lemma B.1.d,} \end{aligned}$$

and v_E is submodular by Lemma B.1.*c*. Similarly, for the super-hedging price we have for every replicable payoff x:

$$\begin{aligned} c_{E+}(x) &= \textstyle\int_{\boldsymbol{J}} \theta \, dv_{A+} \text{ by Corollary 2 since } V \geq 0 \text{ and } V^{-1} \geq 0, \\ &= \textstyle\int_\Omega x \, dv_{E+}, \text{ by Lemma B.1.d,} \end{aligned}$$

and v_{E+} is submodular by by Lemma B.1.*c*. Moreover, the super-hedging price is a Choquet integral on the whole payoff space since, for all $x \in \mathbb{R}^\Omega$ one has:

$$\begin{aligned} c_{E+}(x) &= \textstyle\sup_{\mu \in \mathcal{M}_{E+}} x \cdot \mu \text{ by the Fundamental Theorem of Asset Pricing} \\ &= \textstyle\int_\Omega x \, dv_{E+} \text{ by Lemma B.1.d.} \end{aligned}$$

Q.E.D.

We end this section with a remark showing that the set of nonnegative stochastic discount factors $\mathcal{M}_{E+}$ can be characterized as the core of the submodular set function v_{E+}.

REMARK 3.12 ($\mathcal{M}_{E+} = \text{core}\,(v_{E+})$) Consider the present arbitrage-free market:

$$M_E := \big((\mathbf{1}_\Omega, q^0), (\mathbf{1}_{A_1}, \underline{q}^1, \overline{q}^1), \ldots, (\mathbf{1}_{A_J}, \underline{q}^J, \overline{q}^J)\big), \ (A_j)_{j \in \boldsymbol{J}} \text{ partition of } \Omega.$$

Then by Lemma B.1.c in the Supplementary Material Appendix B, the set function v_{E+} is submodular and one has:

$$\mathcal{M}_{E+} = \text{core}\,(v_{E+}) := \{\mu \in \mathbb{R}^\Omega : \forall A \subseteq \Omega,\ \mu(A) \leq v_{E+}(A) \text{ and } \mu(\Omega) = v_{E+}(\Omega)\}.$$ [21]

Typically, we are in a situation of *regular uncertainty*, using the terminology of Jaffray, since after normalizing the price of the bond, i.e., $q^0 = 1$, one has:

$$\text{core}\,(v_{E+}) = \mathcal{M}_{E+} := \{P \in \mathbb{R}^\Omega_+ : \ \forall j \ \underline{q}^j \leq P(A_j) \leq \overline{q}^j \text{ and } P(\Omega) = 1\}.$$

In other words, the risk-neutral *nonadditive* probability v_{E+} is able to summarize the information of the bundle of risk-neutral *standard* probabilities P, that is, in the set $\mathcal{M}_{E+}$ (see Remark 2.2). □

4. CONCLUSION

In this paper, we developed a framework to study financial markets with bid-ask spreads when the market exhibits hedging complements. This class of markets

[21] Note that, as in cost sharing games, the core is naturally defined with the inequality $\mu(A) \leq v(A)$, instead of the converse inequality usually considered in coalition games.

is important conceptually and analytically since it enlarges the well-studied class of frictionless markets and it takes into account a large class of market frictions that can be handled in a tractable way, in a similar way as for frictionless markets, but using and adapting techniques from nonadditive integration (expectation), submodularity and lattice theory.

Our first contribution is to study the different related classes of markets introduced in this paper whose relationship can be summarized as follows, assuming all of them are present arbitrage-free:

Frictionless $\Leftrightarrow$ **Strong Hedging Complements**
$\Rightarrow$ **Independent Securities** $\Rightarrow$ **Hedging Complements**.

The class of markets with hedging complements is defined with a meaningful financial interpretation, by adapting the definition of strategic complements by Bulow, Geanakoplos, and Klemperer (1985) to the case of financial markets. However the definition relies on the decreasing difference property of the hedging price, instead of the primitive features of the financial market. Strengthening this definition by adapting the one by Sharkey (1982), introduced in the study of cost games, it appears that the Strong Hedging Complements Property allows to characterize frictionless markets. Though interesting for itself, this result shows that the strong definition of Hedging Complements is not adapted to study markets with frictions. Finally, the class of markets with Independent Securities, which are simply and directly defined from the primitive features of the market, provide an important subclass of markets with hedging complements, and contains the class of frictionless markets.

Our second contribution is to show that the hedging price of a market with independent securities satisfies an explicit two-part formula, that is, it is the sum of a "generalized" convex (and submodular) Choquet integral and a modular part. The two-part formula directly implies that such markets have hedging complements since their hedging prices are submodular, but it makes also tractable both the computation of the hedging price and the way to find the best strategies to obtain it. It happens that the notion of generalized Choquet integral used in the paper coincides with the notion of Choquet integral on Riesz space introduced by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015). Finally their put-call parity is also proved to be satisfied for markets with independent securities even in the case where the hedging price may not be a generalized Choquet integral.

Our study appears to be at the crossroad of different fields in economics, game theory, and decision theory. It has benefited from powerful analytical tools developed within these fields, such as Choquet integral (or expectation), first introduced in economics in decision models with ambiguity, and submodularity and lattice theory in economics, extensively studied recently in game theory. We believe that introducing the study of hedging complements in financial markets with frictions opens interesting avenues and perspectives for future research that will benefit from the important existing literature developed in the mentioned related fields.

5. APPENDIX: PROOFS

5.1. *Hedging and strong hedging complements*

This section provides the proofs of Proposition 2.5 and 2.8, which summarize the properties of markets with hedging and strong hedging complements. We first state a claim, the proof of which is standard.

CLAIM 5.1 *If $c : \mathbb{R}^J \to \mathbb{R} \cup \{+\infty\}$ is sublinear and $V^0 \in \mathbb{R}^J$ is frictionless, i.e., $-c(-V^0) = c(V^0)$, then c is constant additive, i.e., $c(x + tV^0) = c(x) + tc(V^0)$ for all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^J$.*

PROOF OF PROPOSITION 2.5: • Since $V^j = V\mathbf{1}_j$ for all j, (1) is equivalent to the fact that $f := c \circ V : \mathbb{R}^J \to \mathbb{R}$ satisfies the following Decreasing Difference Property for all $j \neq k$:

(1′) $f(\theta + \alpha_j \mathbf{1}_j + \alpha_k \mathbf{1}_k) - f(\theta + \alpha_k \mathbf{1}_k) \leq f(\theta + \alpha_j \mathbf{1}_j) - f(\theta)$ for all $\theta \in \mathbb{R}^J, \alpha_j \geq 0, \alpha_k \geq 0$.

⋆ [(1) ⇔ (2)] It follows from Topkis (1998) [Theorem 2.6.1, Corollary 2.6.1].

⋆ [(2) ⇔ (3)] If the vectors $V^1, \ldots, V^J$ are independent, the order $\geq_V$[22] is well defined and:

$(V\theta) \vee_V (V\theta') = V(\theta \vee \theta')$ and $(V\theta) \wedge_V (V\theta') = V(\theta \wedge \theta')$ for all $\theta \in \mathbb{R}^J, \theta' \in \mathbb{R}^J$.

[22] We let $\langle V \rangle$ be the vector space spanned by the vectors $V^j \in \mathbb{R}^\Omega$ ($j = 1, \ldots, J$). The linear mapping $V : \mathbb{R}^J \to \langle V \rangle$ is defined by $V\theta = \sum_{j=1}^J \theta^j V^j$. When the vectors $V^1, \ldots, V^J$ are independent, then V is bijective, its inverse is denoted by V^{-1}, and the **order** $\geq_V$ on $\langle V \rangle$ is defined by $x' \geq_V x$ whenever $V^{-1}(x') \geq V^{-1}(x)$ (denoting $\geq$ the coordinatewise order of $\mathbb{R}^J$).

Consequently, for all x', x in $\langle V \rangle$, $x = V\theta$, $x' = V\theta'$ for some θ, θ' in $\mathbb{R}^J$ (which are uniquely defined since the vectors V^j are independent), and the equivalence $[(2) \Leftrightarrow (3)]$ follows from

$c(x \vee_V x') + c(x \wedge_V x') - c(x) - c(x') = c \circ V(\theta \vee \theta') + c \circ V(\theta \wedge \theta') - c \circ V(\theta) - c \circ V(\theta')$.

$\star$ $[(1) \Leftrightarrow (4)]$ The implication $[(4) \Rightarrow (1)]$ is immediate.

Conversely, for all $x' \in \text{span}\{V^0, V^1, \ldots, V^J\}$, then $x' = x + tV^0$ for some $x \in \langle V \rangle := \text{span}\{V^1, \ldots, V^J\}$ and some $t \in \mathbb{R}$. Since V^0 frictionless and c is sublinear, the function c is constant additive by Claim 5.1. Thus from (1), we get for $\alpha_j \geq 0, \alpha_k \geq 0$, $j \neq k$ in $\{1, \ldots, J\}$

$$\begin{aligned} c(x'+\alpha_j V^j+\alpha_k V^k)+c(x') &= c(x+\alpha_j V^j+\alpha_k V^k)+tc(V^0)+c(x)+tc(V^0) \\ &\leq c(x+\alpha_j V^j)+c(x+\alpha_k V^k)+2tc(V^0) = c(x'+\alpha_j V^j)+c(x'+\alpha_k V^k). \end{aligned}$$

Similarly, for $k = 0$ and $j \in \{1, \ldots, J\}$ one has for $\alpha_0 \geq 0$, $\alpha_j \geq 0$

$c(x' + \alpha_0 V^0 + \alpha_j V^j) + c(x') = c(x' + \alpha_j V^j) + \alpha_0 c(V^0) + c(x') = c(x' + \alpha_j V^j) + c(x' + \alpha_0 V^0)$.

• Assume each V^j $(j = 1, \ldots, J)$ is frictionless. Let $x \in \langle V \rangle := \text{span}\{V^1, \ldots, V^J\}$, $\alpha_j \geq 0, \alpha_k \geq 0$, $j \neq k$. From Claim 5.1, c is constant additive (with respect to V^k). Thus

$c(x + \alpha_j V^j + \alpha_k V^k) + c(x) = c(x + \alpha_j V^j) + \alpha_k c(V^k) + c(x) = c(x + \alpha_j V^j) + c(x + \alpha_k V^k)$.

• Let $M := \big((V^1, q^1), \ldots, (V^J, q^J)\big)$ be a market with frictionless securities, then its hedging price c is sublinear and each V^j is frictionless by the Fundamental Theorem of Asset Pricing (Theorem 3.1). Thus, from above, $(V^1, \ldots, V^J)$ are complement for its hedging price c. Hence the market M has hedging complements.

• Assume $V^1, \ldots, V^J$ are complements for c and $V^1 = V\bar{\theta}$ for some $\bar{\theta} := (0, \theta^2, \ldots, \theta^J) \geq 0$. From Topkis (1998) the complementarity property of $V^1, \ldots, V^J$ is equivalent to:

$c(V\theta + V\alpha + V\beta) + c(V\theta) \leq c(V\theta + V\alpha) + c(V\theta + V\beta)$ for $\theta \in \mathbb{R}^J$, $\alpha \geq 0$, $\beta \geq 0$ $\alpha \wedge \beta = 0$.

Taking $\theta := -\mathbf{1}_1, \alpha := \mathbf{1}_1 \geq 0, \beta := \bar{\theta} \geq 0$, one gets:

$$c(V^1) + c(-V^1) = c(V\bar{\theta}) + c(V(-\mathbf{1}_1)) \leq c(0) + c(0) = 0.$$

From the subadditivity of c we then deduce that $c(V^1) + c(-V^1) \geq c(V^1 - V^1) = 0$. *Q.E.D.*

We now provide a proof of Proposition 2.8.

PROOF OF PROPOSITION 2.8: $[(i) \Rightarrow (ii)]$ Let $M \equiv \bar{M} := \big((V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$ and $V^1, \ldots, V^J$ are strong hedging complements of c. We have $M \equiv \bar{M} \equiv (\bar{M}, \bar{M})$ since the three markets have clearly the same set of stochastic discount factors. Thus, they have also the same hedging price c. We prove now that the security payoffs $(V^1, \ldots, V^J, V^1, \ldots, V^J)$ of $(\bar{M}, \bar{M})$ are strong complements of c. We denote by $[V\ V] : \mathbb{R}^J \times \mathbb{R}^J \to \mathbb{R}^\Omega$ the linear mapping defined by $[V\ V](a, \alpha) = Va + V\alpha$. For all $x \in \operatorname{span}\{V^1, \ldots, V^J, V^1, \ldots, V^J\} = \langle V \rangle$ and all $(a, \alpha), (b, \beta)$ in $\mathbb{R}^J_+ \times \mathbb{R}^J_+$ one has

$$c(x + [V\ V](a, \alpha) + [V\ V](b, \beta)) + c(x) = c(x + V(a + \alpha) + V(b + \beta)) + c(x)$$

$$\leq c(x + V(a + \alpha)) + c(x + V(b + \beta)) \text{ since } (V^1, \ldots, V^J) \text{ are strong}$$

complements

$$= c(x + [V\ V](a, \alpha)) + c(x + [V\ V](b, \beta)).$$

Thus, $(V^1, \ldots, V^J, V^1, \ldots, V^J)$ are strong complements for c (which is the hedging price of both $(\bar{M}, \bar{M})$ and $\bar{M}$), hence they are also complements of c. But every V^j appears twice in the list of complements, thus V^j is frictionless by the last assertion of Proposition 2.5.

$[(ii) \Leftrightarrow (iii) \Leftrightarrow (iv)]$ These are standard equivalent properties of a frictionless market M.

$[(iii) \Rightarrow (i)]$ From (iii) the hedging price c is (finite-valued and) linear on the space of replicable payoffs. It follows easily that M has strong hedging complements from the definition. *Q.E.D.*

5.2. *Proof of the two-part formula (Theorem 2) and its consequences*

5.2.1. *Summary of the properties of v_A, v_{A+} and v_E, v_{E+}*

We prepare the proof of Theorem 3.5 with Lemma 5.2 that summarizes the properties of v_A, v_{A+}, v_E, v_{E+} needed in the proof. Let S be a finite set, let $\mathcal{A} \subseteq 2^S$ such that $\emptyset \in \mathcal{A}, S \in \mathcal{A}$, and let $v : \mathcal{A} \to \mathbb{R}$ such that $v(\emptyset) = 0$, then the core of v is defined by:

$$\operatorname{core}(v, \mathcal{A}) := \{\lambda \in \mathbb{R}^S : \forall K \in \mathcal{A},\ \lambda(K) \leq v(K) \text{ and } \lambda(S) = v(S)\}^{23}$$

[23] Note that, as in cost sharing games, the core is defined with the inequality $\mu(A) \leq v(A)$, instead of the converse inequality considered in coalition games.

and $\operatorname{core}(v, 2^S)$ is simply denoted $\operatorname{core}(v)$ when $v : 2^S \to \mathbb{R}$.

In the following, for $\lambda \in \mathbb{R}^J$, we let

$\lambda(K) := \sum_{j \in K} \lambda^j$ for $\emptyset \neq K \subseteq \boldsymbol{J}$ and $\lambda(\emptyset) = 0$.

LEMMA 5.2 *Let $\boldsymbol{J} := \{1, \ldots, J\}$ be finite, let $q^0 \in \mathbb{R}$, let $\underline{q} := (\underline{q}^1, \ldots, \underline{q}^J) \in \mathbb{R}^J, \overline{q} := (\overline{q}^1, \ldots, \overline{q}^J) \in \mathbb{R}^J$, we define $v_A : 2^{\boldsymbol{J}} \to \mathbb{R}$ and $v_{A+} : 2^{\boldsymbol{J}} \to \mathbb{R}$ by:*

$v_A(K) := \min \left\{ \sum_{j \in K} \overline{q}^j \ , \ q^0 - \sum_{j \notin K} \underline{q}^j \right\}$ *for all* $K \subseteq \boldsymbol{J}$,

$v_{A+}(K) := \min \left\{ \sum_{j \in K} \overline{q}^j \ , \ q^0 - \sum_{j \notin K} (\underline{q}^j)^+ \right\}$ *for all* $K \subseteq \boldsymbol{J}$.

(a) The following two conditions are equivalent:

- $\mathcal{M}_A := \{\lambda \in \mathbb{R}^J : \underline{q} \leq \lambda \leq \overline{q}$ *and* $\sum_{j \in \boldsymbol{J}} \lambda^j = q^0\} \neq \emptyset$,
- *(WAF)* $\underline{q} \leq \overline{q}$ *and* $\sum_{j \in \boldsymbol{J}} \underline{q}^j \leq q^0 \leq \sum_{j \in \boldsymbol{J}} \overline{q}^j$.

Assume $\mathcal{M}_A \neq \emptyset$, then

- $v_A(\emptyset) = 0$ *and* $v_A(\boldsymbol{J}) = q^0$,
- $\mathcal{M}_A = \operatorname{core}(v_A) \neq \emptyset$,
- v_A *is submodular.*

(b) The following two conditions are equivalent:

- $\mathcal{M}_{A+} := \{\lambda \in \mathbb{R}^J : (\underline{q})^+ \leq \lambda \leq \overline{q}$ *and* $\sum_{j \in \boldsymbol{J}} \lambda^j = q^0\} = \mathcal{M}_A \cap \mathbb{R}^J_+ \neq \emptyset$,
- *(PAF)* $(\underline{q})^+ \leq \overline{q}$ *and* $\sum_{j \in \boldsymbol{J}} (\underline{q}^j)^+ \leq q^0 \leq \sum_{j \in \boldsymbol{J}} \overline{q}^j$.

Assume $\mathcal{M}_{A+} \neq \emptyset$, then:

- $v_{A+}(\emptyset) = 0$ *and* $v_{A+}(\boldsymbol{J}) = q^0$,
- $\operatorname{core}(v_{A+}) = \mathcal{M}_{A+} \neq \emptyset$,
- v_{A+} *is submodular and monotone.*

(c) Let $(A_1, \ldots, A_J)$ be a partition of Ω, we let:

$\mathcal{M}_E := \left\{\mu \in \mathbb{R}^\Omega \ : \ \underline{q}^j \leq \mu(A_j) \leq \overline{q}^j, \ \forall j \in \boldsymbol{J} \text{ and } \mathbf{1}_\Omega \cdot \mu = q^0\right\}$.

Then the following two conditions are equivalent:

- $\mathcal{M}_{E+} := \mathcal{M}_E \cap \mathbb{R}^\Omega_+ \neq \emptyset$,
- *(PAF)* $(\underline{q})^+ \leq \overline{q}$ *and* $\sum_{j \in \boldsymbol{J}} (\underline{q}^j)^+ \leq q^0 \leq \sum_{j \in \boldsymbol{J}} \overline{q}^j$.

Let $\mathcal{A} := \{A_K := \cup_{j \in K} A_j : K \subseteq \boldsymbol{J}\}$ be the algebra generated by the A_j $(j \in \boldsymbol{J})$, we define $v_E : \mathcal{A} \to \mathbb{R}$ and $v_{E+} : 2^\Omega \to \mathbb{R}$ by:

$v_E(A) = v_A(\boldsymbol{J}(A)) := \min \left\{ \sum_{j \in \boldsymbol{J}(A)} \overline{q}^j \ , \ q^0 - \sum_{j \notin \boldsymbol{J}(A)} \underline{q}^j \right\}$ *if* $A \in \mathcal{A}$,

$v_{E+}(A) := v_{A+}(\boldsymbol{J}(A)) := \min \left\{ \sum_{j \in \boldsymbol{J}(A)} \overline{q}^j \ , \ q^0 - \sum_{j \notin \boldsymbol{J}(A)} (\underline{q}^j)^+ \right\}$ *if* $A \subseteq \Omega$,

where $\boldsymbol{J}(A) := \{j \in \boldsymbol{J} : A^j \cap A \neq \emptyset\}$.

Assume $\mathcal{M}_{E+} \neq \emptyset$, then:

- $v_{E+}(\emptyset) = v_E(\emptyset) = 0$ *and* $v_{E+}(\Omega) = v_E(\Omega) = q^0$,

- $\mathcal{M}_E = \mathrm{core}(v_E, \mathcal{A}) \neq \emptyset$ *and* $\mathcal{M}_{E+} = \mathrm{core}(v_{E+}) \neq \emptyset$,
- $v_E : \mathcal{A} \to \mathbb{R}$ *and* $v_{E_+} : 2^\Omega \to \mathbb{R}$ *are submodular and* v_{E_+} *is monotone.*

(*d*) *Assume (PAF), then the following assertion hold:*

- *For all* $\theta \in \mathbb{R}^J$ *and all* $x = \sum_{j=1}^J \theta^j \mathbf{1}_{A_j}$

  $\int_{\boldsymbol{J}} \theta \, dv_A = \int_\Omega x \, dv_E$ *and* $\int_{\boldsymbol{J}} \theta \, dv_{A+} = \int_\Omega x \, dv_{E+}$.
- $\sup_{\lambda \in \mathcal{M}_A} \theta \cdot \lambda = \int_{\boldsymbol{J}} \theta \, dv_A$ *and* $\sup_{\lambda \in \mathcal{M}_{A+}} \theta \cdot \lambda = \int_{\boldsymbol{J}} \theta \, dv_{A+}$ *for all* $\theta \in \mathbb{R}^J$.
- $\sup_{\mu \in \mathcal{M}_{E+}} x \cdot \mu = \int_\Omega x \, dv_{E+}$ *for all* $x \in \mathbb{R}^\Omega$.

Proof: **Part** (a) $[WAF \Rightarrow \mathcal{M}_A \neq \emptyset]$ Define $g : [0,1] \to \mathbb{R}$ by: $g(t) = \sum_{j \in \boldsymbol{J}} [(1-t)\underline{q}^j + t\overline{q}^j]$.

Since g is continuous and $g(0) = \sum_{j \in \boldsymbol{J}} \underline{q}^j \leq q^0 \leq \sum_{j \in \boldsymbol{J}} \overline{q}^j = g(1)$, by the Mean Value Theorem, there exists $t_0 \in [0,1]$ such that $g(t_0) = q^0$. Define $\lambda := (1-t_0)\underline{q} + t_0 \overline{q}$. Then, $\lambda \in \mathcal{M}_A \neq \emptyset$ since $\underline{q} \leq \lambda \leq \overline{q}$ and $\sum_{j \in \boldsymbol{J}} \lambda_j = g(t_0) = q^0$. □

• $[\mathcal{M}_A \neq \emptyset \Rightarrow WAF]$. Let $\lambda \in \mathcal{M}_A \neq \emptyset$, that is, $\underline{q} \leq \lambda \leq \overline{q}$ and $\sum_{j \in \boldsymbol{J}} \lambda^j = q^0$. Consequently, $\underline{q} \leq \overline{q}$ and $\sum_{j \in \boldsymbol{J}} \underline{q}^j \leq \sum_{j \in \boldsymbol{J}} \lambda^j = q^0 \leq \sum_{j \in \boldsymbol{J}} \overline{q}^j$. □

• $[v_A(\boldsymbol{J}) = q^0$ and $v(\emptyset) = 0]$ Indeed, since $\sum_{j \in \boldsymbol{J}} \overline{q}^j \geq q^0$ by WAF one has

$v_A(\boldsymbol{J}) := \min\left\{ \sum_{j \in \boldsymbol{J}} \overline{q}^j \ , \ q^0 - \sum_{j \notin \boldsymbol{J}} \underline{q}^j \right\} = q^0$.

$v_A(\emptyset) := \min\left\{ \sum_{j \in \emptyset} \overline{q}^j \ , \ q^0 - \sum_{j \in \boldsymbol{J}} \underline{q}^j \right\} = 0$ since $\sum_{j \in \boldsymbol{J}} \underline{q}^j \leq q^0$ by WAF.

• $[\mathcal{M}_A \subseteq \mathrm{core}(v_A)]$. Let $\lambda \in \mathcal{M}_A$, that is $\lambda \in \mathbb{R}^J$ such that $\underline{q}^j \leq \lambda^j \leq \overline{q}^j$ for all $j \in \boldsymbol{J}$ and $\lambda(\boldsymbol{J}) = q^0$. Then, for all $K \subseteq \boldsymbol{J}$, $K \neq \emptyset$, one has

$\lambda(K) := \sum_{j \in K} \lambda^j \leq \sum_{j \in K} \overline{q}^j$ and

$\lambda(K) = q^0 - \lambda(\boldsymbol{J}) + \lambda(K) = q^0 - \lambda(K^c) = q^0 - \sum_{j \notin K} \lambda^j \leq q^0 - \sum_{j \notin K} \underline{q}^j$,

Thus, $\lambda(K) \leq v_A(K) := \min\left\{ \sum_{j \in K} \overline{q}^j, q^0 - \sum_{j \notin K} \underline{q}^j \right\}$.

Moreover, $\lambda(\boldsymbol{J}) = v_A(\boldsymbol{J})$ since $\lambda(\boldsymbol{J}) = q^0$ and $q^0 = v_A(\boldsymbol{J})$. Thus, $\lambda \in \mathrm{core}(v_A)$. □

⋆ $[\mathrm{core}(v_A) \subseteq \mathcal{M}_A]$. Let $\lambda \in \mathrm{core}(v_A)$, that is $\lambda(\boldsymbol{J}) = v_A(\boldsymbol{J})(= q^0)$ and $\lambda(K) \leq v_A(K)$ for all $K \subseteq \boldsymbol{J}$. For $K = \{j\}$ and for $K = \{j\}^c$ one has:

$\lambda(\{j\}) \leq v_A(\{j\}) \leq \overline{q}^j$

$q^0 - \lambda(\{j\}) = \lambda(\boldsymbol{J}) - \lambda(\{j\}) = \lambda(\{j\}^c) \leq v_A(\{j\}^c) \leq q^0 - \underline{q}^j$.

Thus, $\underline{q}^j \leq \lambda(\{j\}) \leq \overline{q}^j$ for all $j \in \boldsymbol{J}$ and $\lambda(\boldsymbol{J}) = q^0$, that is $\lambda \in \mathcal{M}_A$. □

⋆ $[\mathrm{core}(v_A) = \mathcal{M}_A \neq \emptyset]$ We proved that $\mathrm{core}(v_A) = \mathcal{M}_A$ and $\mathcal{M}_A \neq \emptyset$, by assumption. □

• $[v_A$ is submodular$]$, that is,

$$v_A(K\cap L)+v_A(K\cup L)\le v_A(K)+v_A(L) \text{ for all } K\subseteq \boldsymbol{J},\ L\subseteq \boldsymbol{J}.$$

The proof will distinguish four cases.

(1) $v_A(K)=\sum_{j\in K}\overline{q}^j$ and $v_A(L)=\sum_{j\in L}\overline{q}^j$. Then

$$v_A(K)+v_A(L)=\sum_{j\in K\cap L}\overline{q}^j+\sum_{j\in K\cup L}\overline{q}^j\ge v_A(K\cap L)+v_A(K\cup L).$$

(2) $v_A(K)=q^0-\sum_{j\in K^c}\underline{q}^j$ and $v_A(L)=q^0-\sum_{j\in L^c}\underline{q}^j$. Then,

$$\begin{aligned}v_A(K)+v_A(L)&=q^0-\textstyle\sum_{j\in K^c\cup L^c}\underline{q}^j+q^0-\sum_{j\in K^c\cap L^c}\underline{q}^j\\&=q^0-\textstyle\sum_{j\in(K\cap L)^c}\underline{q}^j+q^0-\sum_{j\in(K\cup L)^c}\underline{q}^j\\&\ge v_A(K\cap L)+v_A(K\cup L).\end{aligned}$$

(3) $v_A(K)=\sum_{j\in K}\overline{q}^j$ and $v_A(L)=q^0-\sum_{j\in L^c}\underline{q}^j$. Then

$$\begin{aligned}v_A(K)+v_A(L)&=\textstyle\sum_{j\in K\cap L}\overline{q}^j+\sum_{j\in K\setminus L}\overline{q}^j+q^0-\sum_{j\in(K\cup L)^c}\underline{q}^j-\sum_{j\in K\setminus L}\underline{q}^j\\&=\textstyle\sum_{j\in K\cap L}\overline{q}^j+[q^0-\sum_{j\in(K\cup L)^c}\underline{q}^j]+\sum_{j\in K\setminus L}(\overline{q}^j-\underline{q}^j).\\&\ge v_A(K\cap L)+v_A(K\cup L) \text{ since } \underline{q}\le\overline{q} \text{ by WAF.}\end{aligned}$$

(4) $v_A(K)=q^0-\sum_{j\in K^c}\underline{q}^j$ and $v_A(L)=\sum_{j\in L}\overline{q}^j$. The proof is similar to (3). □

Part (b) We notice that v_{A+} has the same form as v_A but with $(q^0,\underline{q},\overline{q})$ replaced by $(q^0,(\underline{q})^+,\overline{q})$. Thus, from Part (a) we directly deduce the following properties: $[PAF\Leftrightarrow\mathcal{M}_{A+}\neq\emptyset]$, $v_{A+}(\boldsymbol{J})=q^0=v_A(\boldsymbol{J})$, $v_{A+}(\emptyset)=0$, v_{A+} is submodular, together with

$\operatorname{core}(v_{A+})=\mathcal{M}_{A+}:=\{\lambda\in\mathbb{R}^J:(\underline{q})^+\le\lambda\le\overline{q} \text{ and } \sum_{j\in\boldsymbol{J}}\lambda^j=q^0\}=\mathcal{M}_A\cap\mathbb{R}^J_+\neq\emptyset.$

• $[v_{A+}$ is monotone$]$. That is, $v_{A+}(K)\le v_{A+}(L)$ for $K\subseteq L$. This follows from the two following inequalities:

$$\textstyle\sum_{j\in K}\overline{q}^j\le\sum_{j\in L}\overline{q}^j \text{ since } K\subseteq L \text{ and } \overline{q}^j\ge0 \text{ for all } j, [\text{for } \overline{q}^j\ge(\underline{q}^j)^+\ge0],$$
$$q^0-\textstyle\sum_{j\notin K}(\underline{q}^j)^+\le q^0-\sum_{j\notin L}(\underline{q}^j)^+ \text{ since } L^c\subseteq K^c \text{ and } (\underline{q}^j)^+\ge0 \text{ for all } j.$$

□

Part (c) • $[\mathcal{M}_{E+}\neq\emptyset\Leftrightarrow(\text{PAF})]$ One has

$$\begin{aligned}&\mathcal{M}_{E+}:=\{\mu\in\mathbb{R}^\Omega_+:\mu(\Omega)=q^0,\ \text{and } \underline{q}^j\le\mu(A_j)\le\overline{q}^j,\ \forall j\in\boldsymbol{J}\}\neq\emptyset\\&\Leftrightarrow\{\lambda\in\mathbb{R}^J_+:\lambda(\boldsymbol{J})=q^0,\ \text{and } \underline{q}^j\le\lambda^j\le\overline{q}^j,\ \forall j\in\boldsymbol{J}\}\neq\emptyset\\&\Leftrightarrow\{\lambda\in\mathbb{R}^J:\lambda(\boldsymbol{J})=q^0,\ \text{and } (\underline{q}^j)^+\le\lambda^j\le\overline{q}^j,\ \forall j\in\boldsymbol{J}\}\neq\emptyset\\&\Leftrightarrow(\text{PAF})\ (\underline{q})^+\le\overline{q} \text{ and } \textstyle\sum_{i\in\boldsymbol{J}}(\underline{q}^j)^+\le q^0\le\sum_{i\in\boldsymbol{J}}\overline{q}^j \ [\text{by Part } (b)].\end{aligned}$$

□

In the following, we will use extensively the following properties

$$A\in\mathcal{A}\Leftrightarrow A=A_K:=\cup_{j\in K}A_j \text{ for some } K\subseteq\boldsymbol{J},$$
$$v_E(A_K)=v_A(K):=\min\{\textstyle\sum_{j\in K}\overline{q}^j\ ,\ q^0-\sum_{j\notin K}\underline{q}^j\},$$

$$v_{E+}(A_K) = v_{A+}(K) := \min\left\{ \sum_{j\in K} \overline{q}^j \ , \ q^0 - \sum_{j\notin K} (\underline{q}^j)^+ \right\},$$

and the properties of v_E and v_{E+} are deduced from those of v_A and v_{A+}.

• $[v_{E+}(\Omega) = v_E(\Omega) = q^0,\ v_{E+}(\emptyset) = v_E(\emptyset) = 0]$

We have $v_E(\Omega) = v_A(\boldsymbol{J}(\Omega)) = v_A(\boldsymbol{J}) = q^0$ by Part (a). Moreover, $v_E(\emptyset) = v_A(\boldsymbol{J}(\emptyset)) = v_A(\emptyset) = 0$ by Part (a). Similarly, we have $v_{E+}(\Omega) = q^0$, $v_{E+}(\emptyset) = 0$ from Part (b).

• $[\mathcal{M}_E = \mathrm{core}(v_E, \mathcal{A})]$ Indeed one has

$$\begin{aligned} &\mu \in \mathcal{M}_E \Leftrightarrow \lambda := (\mu(A_1), \ldots, \mu(A_J)) \in \mathcal{M}_A \\ \Leftrightarrow\ &\lambda := (\mu(A_1), \ldots, \mu(A_J)) \in \mathrm{core}(v_A) \text{ (since } \mathcal{M}_A = \mathrm{core}(v_A)) \\ \Leftrightarrow\ &\mu \in \{\mu \in \mathbb{R}^\Omega : \mu(\Omega) = v_E(\Omega) \text{ and } \mu(A_K) \le v_E(A_K) = v_A(K)\ \forall K \subseteq \boldsymbol{J}\} \\ \Leftrightarrow\ &\mu \in \mathrm{core}(v_E) \text{ (since } v_E(A) = +\infty \text{ whenever } A \notin \mathcal{A}). \qquad \square \end{aligned}$$

• $[v_E : \mathcal{A} \to \mathbb{R}$ is submodular$]$ Let $A \in \mathcal{A}$ and $B \in \mathcal{A}$, that is $A = A_K$ and $B = A_L$ for some $K \subseteq \boldsymbol{J}$, $L \subseteq \boldsymbol{J}$. Then one has

$$\begin{aligned} v_E(A_K \cap A_L) + v_E(A_K \cup A_L) &= v_E(A_{K\cap L}) + v_E(A_{K\cup L}) \\ &= v_A(K \cap L) + v_A(K \cup L) \\ &\le v_A(K) + v_A(L) \text{ since } v_A \text{ is submodular from Part } (a) \\ &= v_E(A_K) + v_E(A_L). \qquad \square \end{aligned}$$

• $[\mathcal{M}_{E+} = \mathrm{core}(v_{E+})]$ Let $\mu \in \mathcal{M}_{E+} := \mathcal{M}_E \cap \mathbb{R}^\Omega_+$, then $\mu \ge 0$, $\underline{q}^j \le \mu(A_j) \le \overline{q}^j$ for all $j \in \boldsymbol{J}$ (thus $(\underline{q}^j)^+ \le \mu(A_j)$) and $\mu(\Omega) = q^0$. Then $\mu \in \mathrm{core}(v_{E+})$ since $\mu(\Omega) = q^0 = v_{E+}(\Omega)$, and for all $A \subseteq \Omega$ one has $\mu(A) \le v_{E+}(A)$, which is a consequence of the two following inequalities:

$$\begin{aligned} &\mu(A) \le \mu(\cup_{j\in \boldsymbol{J}(A)} A_j) = \sum_{j\in \boldsymbol{J}(A)} \mu(A_j) \le \sum_{j\in \boldsymbol{J}(A)} \overline{q}^j, \\ &q^0 - \mu(A) = \mu(\Omega) - \mu(A) = \mu(A^c) \ge \mu(\cup_{j\notin \boldsymbol{J}(A)} A_j) \\ &\qquad\qquad = \sum_{j\notin \boldsymbol{J}(A)} \mu(A_j) \ge \sum_{j\notin \boldsymbol{J}(A)} (\underline{q}^j)^+. \end{aligned}$$

Conversely, let $\mu \in \mathrm{core}(v_{E+})$, that is, $\mu(\Omega) = v_{E+}(\Omega) = v_E(\Omega)$ and for all $A \subseteq \Omega$, $\mu(A) \le v_{E+}(A) \le v_E(A)$. Thus $\mu \in \mathrm{core}(v_E)$ and $\mathrm{core}(v_E) = \mathcal{M}_E$ from Part (a). It only remains to prove that $\mu \ge 0$, that is $\mu(\omega) \ge 0$ for all $\omega \in \Omega$, which is a consequence of

$$\mu(\Omega) - \mu(\{\omega\}) = \mu(\{\omega\}^c) \le v_{E+}(\{\omega\}^c) \le v_{E+}(\Omega) = \mu(\Omega). \qquad \square$$

• $[v_{E+}$ is submodular$]$ For all $A \subseteq \Omega$, $B \subseteq \Omega$ we first notice that

$(*)$ $\boldsymbol{J}(A \cap B) \subseteq \boldsymbol{J}(A) \cap \boldsymbol{J}(B)$ and $\boldsymbol{J}(A \cup B) \subseteq \boldsymbol{J}(A) \cup \boldsymbol{J}(B)$.

Consequently

$$v_{E+}(A \cap B) + v_{E+}(A \cup B) = v_{A+}(\boldsymbol{J}(A \cap B)) + v_{A+}(\boldsymbol{J}(A \cup B))$$

$\leq v_{A+}(\boldsymbol{J}(A) \cap \boldsymbol{J}(B)) + v_{A+}(\boldsymbol{J}(A) \cup \boldsymbol{J}(B))$ from (*) and the monotonicity of v_{A+}

$\leq v_{A+}(\boldsymbol{J}(A)) + v_{A+}(\boldsymbol{J}(B))$ since v_{A+} is submodular from Part (b)

$= v_{E+}(A) + v_{E+}(B)$.

• [v_{E+} is monotone] Let $A \subseteq B \subseteq \Omega$, then $\boldsymbol{J}(A) \subseteq \boldsymbol{J}(B)$. Hence $v_{E+}(A) := v_{A+}(\boldsymbol{J}(A)) \leq v_{A+}(\boldsymbol{J}(B)) =: v_{E+}(B)$ since v_{A+} is monotone by Part (b).

Part (d) • [$\int_{\boldsymbol{J}} \theta \, dv_A = \int_\Omega x \, dv_E$ for $\theta \in \mathbb{R}^J$ and $x = \sum_{j=1}^J \theta^j \mathbf{1}_{A_j}$] Indeed, without any loss of generality we assume that $\theta^1 \geq \theta^2 \geq \ldots, \geq \theta^J$. By definition of the Choquet integral one has:

$$\int_\Omega x \, dv_E = (\theta^1 - \theta^2) v_E(A_1) + \cdots + (\theta^j - \theta^{j+1}) v_E(A_1 \cup \cdots \cup A_j) + \cdots + \theta^J v_E(\Omega).$$

But $v_E(A_1 \cup \cdots \cup A_j) = v_A(\{1, \ldots, j\})$ for all j. Consequently

$$\begin{aligned}\int_\Omega x \, dv_E &= (\theta^1 - \theta^2) v_A(\{1\}) + \cdots + (\theta^j - \theta^{j+1}) v_A(\{1, \ldots, j\}) + \cdots + \theta^J v_A(\Omega) \\ &= \int_{\boldsymbol{J}} \theta \, dv_A.\end{aligned}$$

• [$\int_{\boldsymbol{J}} \theta \, dv_{A+} = \int_\Omega x \, dv_{E+}$ for $\theta \in \mathbb{R}^J$, $x = \sum_{j=1}^J \theta^j \mathbf{1}_{A_j}$] The proof is similar to the previous one. □

• [$\sup_{\lambda \in \mathcal{M}_{A+}} \theta \cdot \lambda = \int_{\boldsymbol{J}} \theta \, dv_{A+}$ for all $\theta \in \mathbb{R}^J$] From Part (a) we have $\mathcal{M}_A = \mathrm{core}\,(v_A) \neq \emptyset$ and v_A is submodular. Consequently, from the extension of Schmeidler's theorem (Schmeidler (1986)) to the non-monotone case (e.g., Marinacci and Montrucchio (2004)), one has:

$$\sup_{\mu \in \mathrm{core}\,(v_A)} \lambda \cdot \theta = \int_{\boldsymbol{J}} \theta \, dv_A \text{ for all } \theta \in \mathbb{R}^J.$$

• [$\sup_{\lambda \in \mathcal{M}_{A+}} \theta \cdot \lambda = \int_{\boldsymbol{J}} \theta \, dv_{A+}$ for all $\theta \in \mathbb{R}^J$] The proof is similar to the previous one. Under (PAF), from Part (b), $\mathcal{M}_{A+} = \mathrm{core}\,(v_{A+}) \neq \emptyset$ and v_{A+} is submodular. Thus, from Schmeidler (1986), one deduces:

$$\sup_{\lambda \in \mathcal{M}_{A+}} \theta \cdot \lambda = \sup_{\lambda \in \mathrm{core}\,(v_{A+})} \theta \cdot \lambda = \int_{\boldsymbol{J}} \theta \, dv_{A+} \text{ for all } \theta \in \mathbb{R}^J.$$

• [$\sup_{\mu \in \mathcal{M}_{E+}} x \cdot \mu = \int_\Omega x \, dv_{E+}$ for all $x \in \mathbb{R}^\Omega$] Under (PAF), from Part (c), we have $\mathcal{M}_{E+} = \mathrm{core}\,(v_{E+})$ and v_{E+} is submodular. Thus, from Schmeidler (1986), one deduces:

$$\sup_{\mu \in \mathcal{M}_{E+}} x \cdot \mu = \sup_{\mu \in \mathrm{core}\,(v_{E+})} x \cdot \mu = \int_\Omega x \, dv_{E+} \text{ for all } x \in \mathbb{R}^\Omega. \quad \textit{Q.E.D.}$$

5.2.2. *Proof of the two-part formula (Theorem 3.5)*

STEP 1: Proof under the additional assumption: $V^0 = \sum_{j \in \boldsymbol{J}} V^j$, that is, $\bar{\boldsymbol{J}} = \boldsymbol{J}$. Let $\theta \in \mathbb{R}^J$, then one has:

$$c(V\theta) := \inf \left\{ \sum_{j=0}^J \overline{q}^j \alpha^j - \underline{q}^j \beta^j : \sum_{j=0}^J (\alpha^j - \beta^j) V^j = V\theta, (\alpha, \beta) \in \mathbb{R}_+^{1+J} \times \mathbb{R}_+^{1+J} \right\}.$$

Since $V^0 = \sum_{j=1}^{J} V^j = V\mathbf{1}_{\boldsymbol{J}}$ and $V^j = V\mathbf{1}_j$ and V is one-to-one:

$$V\theta = \sum_{j=0}^{J}(\alpha^j - \beta^j)V^j = V\big((\alpha^0 - \beta^0)\mathbf{1}_{\boldsymbol{J}} + \sum_{j=1}^{J}(\alpha^j - \beta^j)\mathbf{1}_j\big),$$

holds if and only if $\theta = (\alpha^0 - \beta^0)\mathbf{1}_{\boldsymbol{J}} + \sum_{j=1}^{J}(\alpha^j - \beta^j)\mathbf{1}_j$. Thus

$$c(V\theta) = \inf\Big\{ \sum_{j=0}^{J} \overline{q}^j\alpha^j - \underline{q}^j\beta^j : (\alpha^0 - \beta^0)\mathbf{1}_{\boldsymbol{J}} + \sum_{j=1}^{J}(\alpha^j - \beta^j)\mathbf{1}_j = \theta,$$
$$(\alpha,\beta) \in \mathbb{R}_+^{1+J} \times \mathbb{R}_+^{1+J}\Big\} =: c_A(\theta).$$

We notice that the above right-hand side $c_A(\theta)$ is the hedging price of the following market:

$$M_A := \big((\mathbf{1}_{\boldsymbol{J}}, q^0), (\mathbf{1}_1, \underline{q}^1, \overline{q}^1), \ldots, (\mathbf{1}_J, \underline{q}^J, \overline{q}^J)\big).$$

whose set of stochastic discount factors is:

$$\mathcal{M}_A := \{\lambda \in \mathbb{R}^J : \underline{q} \leq \lambda \leq \overline{q} \text{ and } \sum_{j\in \boldsymbol{J}} \lambda^j = q^0\}.$$

Moreover, the market M is present arbitrage-free, hence from the Fundamental Theorem of Asset Pricing $\mathcal{M} \neq \emptyset$. This clearly implies that $\mathcal{M}_A \neq \emptyset$. Thus, for all $\theta \in \mathbb{R}^J$, one has:

$$c_A(\theta) = \sup_{\lambda\in\mathcal{M}_A} \lambda\cdot\theta \text{ from the Fundamental Theorem of Asset Pricing,}$$
$$= \int_{\boldsymbol{J}} \theta\, dv_A \text{ by Lemma 5.2.d.}$$

Recalling that since $\bar{\boldsymbol{J}} = \boldsymbol{J}$, the second term of the two-part formula of Theorem 3.5 is null, and from above we deduce that:

$$c(V\theta) = c_A(\theta) = \int_{\boldsymbol{J}} \theta\, dv_A \text{ for all } \theta \in \mathbb{R}^J. \qquad \textit{Q.E.D.}$$

STEP 2: General case: $V^0 = \sum_{j\in\bar{\boldsymbol{J}}} V^j$ for some $\bar{\boldsymbol{J}} \subseteq \boldsymbol{J}$.

We first prove a lemma.

LEMMA 5.3 *Consider the market* $M := \big((V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$, *and assume that the vectors* $V^1, \ldots, V^J$ *are independent and* $\underline{q} \leq \overline{q}$. *Then, one has:*

- $c(V\theta) = \overline{q}\cdot\theta^+ - \underline{q}\cdot\theta^-$ *for all* $\theta \in \mathbb{R}^J$.
- $f := c \circ V : \mathbb{R}^J \to \mathbb{R}$ *is modular (for the coordinate-wise order* $\geq$ *on* $\mathbb{R}^J$*), that is,*

$$f(\theta \vee \theta') + f(\theta \wedge \theta') = f(\theta) + f(\theta') \text{ for all } \theta, \theta' \text{ in } \mathbb{R}^J.$$

- $c : \langle V\rangle \to \mathbb{R}$ *is modular (for the order* $\geq_V$ *on* $\langle V\rangle$ *).*

PROOF OF LEMMA 5.3: First, since the V^j ($j \in \boldsymbol{J}$) are independent, one has for all $\theta \in \mathbb{R}^J$

$$f(\theta) := c(V\theta) := \inf\{\overline{q}\cdot\alpha - \underline{q}\cdot\beta : \alpha \geq 0, \beta \geq 0, V(\alpha - \beta) = V\theta\}$$
$$= \inf\{\overline{q}\cdot\alpha - \underline{q}\cdot\beta : \alpha \geq 0, \beta \geq 0, \alpha - \beta = \theta\} \leq \overline{q}\cdot\theta^+ - \underline{q}\cdot\theta^-.$$

Conversely, if $\theta = \alpha - \beta$ with $\alpha \geq 0, \beta \geq 0$, then $\beta \geq \theta^- = \max\{-\theta, 0\}$ and $\overline{q} - \underline{q} \geq 0$ imply:

$$\overline{q}\cdot\theta^+ - \underline{q}\cdot\theta^- = \overline{q}\cdot(\theta^+ - \theta^-) + (\overline{q} - \underline{q})\cdot\theta^- \leq \overline{q}\cdot(\alpha - \beta) + (\overline{q} - \underline{q})\cdot\beta = \overline{q}\cdot\alpha - \underline{q}\cdot\beta.$$
$$\overline{q}\cdot\theta^+ - \underline{q}\cdot\theta^- \leq \inf\{\overline{q}\cdot\alpha - \underline{q}\cdot\beta : \alpha \geq 0, \beta \geq 0, \alpha - \beta = \theta\} = f(\theta). \quad Q.E.D.$$

We now come back to the proof of Theorem 3.5.

Proof of Theorem 3.5: We have $V^0 = \sum_{j\in\bar{\boldsymbol{J}}} V^j$ for some $\bar{\boldsymbol{J}} \subseteq \boldsymbol{J}$. and the proof will distinguish the following three cases.

- Assume first that $\bar{\boldsymbol{J}} = \boldsymbol{J}$. The result follows from Step 1.
- Assume now that $\bar{\boldsymbol{J}} = \emptyset$, that is, $V^0 = 0$. Then $q^0 = 0$ since M is present arbitrage-free. Let $\tilde{M} := \big((V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big)$, then

$$M := \big((0,0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big) \equiv \tilde{M}.$$

Hence the markets M and $\tilde{M}$ have the same hedging price c and from Lemma 5.3 one deduces that

$$c(V\theta) = \overline{q}\cdot\theta^+ - \underline{q}\cdot\theta^- \text{ for all } \theta \in \mathbb{R}^J. \qquad \square$$

- Finally, assume that $\emptyset \neq \bar{\boldsymbol{J}} \subsetneq \boldsymbol{J}$, say $\bar{\boldsymbol{J}} := \{1, \ldots, \bar{J}\}$ with $1 \leq \bar{J} < J$. We claim that:

$$c(V\theta)\ [:= c(\textstyle\sum_{j=1}^{J}\theta^j V^j)] = c(\sum_{j=1}^{\bar{J}}\theta^j V^j) + c(\sum_{j=\bar{J}}^{J}\theta^j V^j).$$

Indeed, since the vectors $V^1, \ldots, V^J$ are independent and $V^0 = \sum_{j=1}^{\bar{J}} V^j$ since M is written in normal form (see Definition 3.1), one gets:

$$\begin{aligned} c(V\theta) &= \inf\big\{\textstyle\sum_{j=0}^{J}\overline{q}^j\alpha^j - \underline{q}^j\beta^j : \sum_{j=0}^{J}(\alpha^j - \beta^j)V^j = V\theta, (\alpha,\beta) \geq 0\},\\ &= \inf\big\{\textstyle\sum_{j=0}^{\bar{J}}\overline{q}^j\alpha^j - \underline{q}^j\beta^j : \sum_{j=0}^{\bar{J}}(\alpha^j - \beta^j)V^j = \sum_{j=1}^{\bar{J}}\theta^j V^j, (\alpha,\beta) \geq 0\}\\ &\quad + \inf\big\{\textstyle\sum_{j=\bar{J}}^{J}\overline{q}^j\alpha^j - \underline{q}^j\beta^j : \sum_{j=\bar{J}}^{J}(\alpha^j - \beta^j)V^j = \sum_{j=\bar{J}}^{J}\theta^j V^j, (\alpha,\beta) \geq 0\},\\ &= c(\textstyle\sum_{j=1}^{\bar{J}}\theta^j V^j) + c(\sum_{j=\bar{J}}^{J}\theta^j V^j). \end{aligned}$$

Then one sees that $f : \mathbb{R}^{\bar{J}} \to \mathbb{R}$ and $g : \mathbb{R}^{J\setminus\bar{J}} \to \mathbb{R}$ defined by

$$f(\theta_1, \ldots, \theta_{\bar{J}}) = c(\textstyle\sum_{j=1}^{\bar{J}}\theta^j V^j) \text{ and } g(\theta_{\bar{J}}, \ldots, \theta_J) = c(\sum_{j=\bar{J}}^{J}\theta^j V^j)$$

are the hedging prices, respectively, of the markets

$$\big((V^0, q^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^{\bar{J}}, \underline{q}^{\bar{J}}, \overline{q}^{\bar{J}})\big) \text{ and } \big((V^{\bar{J}}, \underline{q}^{\bar{J}}, \overline{q}^{\bar{J}}), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big).$$

The first market has independent securities (Definition 3.1) and $V^0 = \sum_{j=1}^{\bar{J}} V^j$, and for the second market, the payoff vectors $V^{\bar{J}}, \ldots, V^J$ are independent. Thus from Step 1 of the proof and Lemma 5.3, one gets:

$$c(V\theta) = \textstyle\int_{\bar{\boldsymbol{J}}}(\theta_1, \ldots, \theta_{\bar{J}})\, dv_A + \sum_{j\in\boldsymbol{J}\setminus\bar{\boldsymbol{J}}}\overline{q}^j(\theta^j)^+ - \underline{q}^j(\theta^j)^- \ \forall\theta = (\theta_1, \ldots, \theta_J) \in \mathbb{R}^J$$

where $v_A : 2^{\bar{\boldsymbol{J}}} \to \mathbb{R}$ is submodular and defined by:

$v_A(K) = \min\left\{ \sum_{j\in K} \overline{q}^j \ , \ q^0 - \sum_{j\notin K} \underline{q}^j \right\}$ if $K \subseteq \bar{\boldsymbol{J}}$. *Q.E.D.*

5.2.3. *Proofs of Corollary 1 and Corollary 2*

We first provide a proof of Corollary 1.

Proof of Corollary 1: • If $c \circ V$ is a Choquet integral, then $c(V\mathbf{1}_{\boldsymbol{J}}) = -c(-V\mathbf{1}_{\boldsymbol{J}})$. But

$$c(V\mathbf{1}_{\boldsymbol{J}}) = v_A(\bar{\boldsymbol{J}}) + \sum_{j\in \boldsymbol{J}\setminus\bar{\boldsymbol{J}}} \overline{q}^j \text{ and } -c(-V\mathbf{1}_{\boldsymbol{J}}) = v_A(\bar{\boldsymbol{J}}) + \sum_{j\in \boldsymbol{J}\setminus\bar{\boldsymbol{J}}} \underline{q}^j.$$

Thus $\sum_{j\in \boldsymbol{J}\setminus\bar{\boldsymbol{J}}} \overline{q}^j = \sum_{j\in \boldsymbol{J}\setminus\bar{\boldsymbol{J}}} \underline{q}^j$. But $\underline{q} \leq \overline{q}$ from PAF, hence $\overline{q}^j = \underline{q}^j$ for all $j \in \boldsymbol{J} \setminus \bar{\boldsymbol{J}}$.

Conversely, assume that $\overline{q}^j = \underline{q}^j$ for all $j \in \boldsymbol{J} \setminus \bar{\boldsymbol{J}}$. We first notice that

$$M \equiv \tilde{M} := \big((\tilde{V}^0, \tilde{q}^0), (V^1, \underline{q}^1, \overline{q}^1), \ldots, (V^J, \underline{q}^J, \overline{q}^J)\big),$$

with $\tilde{V}^0 := V^0 + \sum_{j\in \boldsymbol{J}\setminus\bar{\boldsymbol{J}}} V^j$ and $\tilde{q}^0 := q^0 + \sum_{j\in \boldsymbol{J}\setminus\bar{\boldsymbol{J}}} q^j$.

We note that $\tilde{V}^0 := V^0 + \sum_{j\in \boldsymbol{J}} V^j$ and $c \circ V$ is a Choquet integral from Theorem 3.5.

• The proofs of the two remaining assertions of Corollary 1 are immediate. *Q.E.D.*

We now provide a proof of Corollary 2.

Proof of Corollary 2: • *Part (a)* $[V \geq 0]$ Assume that $V^j \geq 0$ for all $j \in \boldsymbol{J}$. Let $x \in \langle V \rangle$, then $x = V\theta$ for some $\theta \in \mathbb{R}^J$ and

$$c_+(V\theta) := \inf\Big\{ \sum_{j=0}^J \overline{q}^j \alpha^j - \underline{q}^j \beta^j : (\alpha, \beta) \in A \Big\}$$

where $A := \Big\{(\alpha, \beta) \in \mathbb{R}_+^{1+J} \times \mathbb{R}_+^{1+J} \ : \ \sum_{j=0}^J (\alpha^j - \beta^j) V^j \geq V\theta \Big\}$.

Since $V^j = V\mathbf{1}_j$, $V^0 = \sum_{j=1}^J V^j = V\mathbf{1}_{\boldsymbol{J}}$, and $V \geq 0$, one has:

$$A \supseteq B := \Big\{(\alpha, \beta) \in \mathbb{R}_+^{1+J} \times \mathbb{R}_+^{1+J} \ : \ (\alpha^0 - \beta^0))\mathbf{1}_{\boldsymbol{J}} + \sum_{j=1}^J (\alpha^j - \beta^j)\mathbf{1}_j \geq \theta \Big\}.$$

Consequently, from the definition of c_+, one gets:

$$c_+(V\theta) \leq \inf\Big\{ \sum_{j=0}^J \overline{q}^j \alpha^j - \underline{q}^j \beta^j : (\alpha, \beta) \in B \Big\} =: (c_A)_+(\theta).$$

We notice that $(c_A)_+$ is the super-hedging price of the following market:

$$M_A := \big((\mathbf{1}_J, q^0, q^0), (\mathbf{1}_1, \underline{q}^1, \overline{q}^1), \ldots, (\mathbf{1}_J, \underline{q}^J, \overline{q}^J)\big).$$

We denote by $\mathcal{M}$ and $\mathcal{M}_A$ the set of stochastic discount factors of the markets M and M_A. Since the market M is present arbitrage-free, from the Fundamental Theorem of Asset Pricing, there exists $\mu \in \mathbb{R}^\Omega$, such that

$$\mu \in \mathcal{M}_+ := \{\mu \in \mathbb{R}_+^\Omega : \forall j \ \underline{q}^j \leq V^j \cdot \mu \leq \overline{q}^j \text{ and } V^0 \cdot \mu = q^0\}.$$

Thus, $\lambda := (V^1 \cdot \mu, \dots, V^J \cdot \mu) \in \mathcal{M}_A := \{\lambda \in \mathbb{R}^J : \underline{q} \leq \lambda \leq \overline{q}$ and $\sum_{j\in \boldsymbol{J}} \lambda^j = q^0\}$

and $\lambda \geq 0$ since $V \geq 0$. Hence, $\mathcal{M}_{A+} := \mathcal{M}_A \cap \mathbb{R}^J_+ \neq \emptyset$, and by Lemma 5.2.b, one has:

$\mathcal{M}_{A+} = \text{core}(v_{A+})$, where $v_{A+} : 2^{\boldsymbol{J}} \to \mathbb{R}$ is submodular and defined by:

$v_{A+}(K) := \min\left\{ \sum_{j\in K} \overline{q}^j \ ,\ q^0 - \sum_{j\notin K} (\underline{q}^j)^+ \right\}$ for all $K \subseteq \boldsymbol{J}$.

Consequently, since $\mathcal{M}_{A+} \neq \emptyset$, for all $\theta \in \mathbb{R}^J$, one has:

$c_+(V\theta) \leq (c_A)_+(\theta) = \sup_{\lambda\in\mathcal{M}_{A+}} \lambda\cdot\theta$ by the Fundamental Theorem of Asset Pricing

$= \int_{\boldsymbol{J}} \theta \, dv_{A+}$ by Lemma 5.2.d. *Q.E.D.*

• *Part (b). $V \geq 0$ and $V^{-1} \geq 0$* Let $x \in \langle V \rangle$, then $x = \sum_{j=1}^J V^j \theta^j = V\theta$ for some $\theta \in \mathbb{R}^J$. Since $V\theta \geq 0$ implies $\theta \geq 0$, by a similar argument as the one made in Part (a), one gets:

$c_+(V\theta) := \inf\left\{ \sum_{j=0}^J \overline{q}^j \alpha^j - \underline{q}^j \beta^j : \sum_{j=0}^J (\alpha^j - \beta^j) V^j \geq V\theta, (\alpha, \beta) \in \mathbb{R}^{1+J}_+ \times \mathbb{R}^{1+J}_+ \right\}$

$= \inf\left\{ \sum_{j=0}^J \overline{q}^j \alpha^j - \underline{q}^j \beta^j : (\alpha^0 - \beta^0)\mathbf{1}_{\boldsymbol{J}} + \sum_{j=1}^J (\alpha^j - \beta^j)\mathbf{1}_j \geq \theta, (\alpha, \beta) \geq 0 \right\}$

$=: (c_A)_+(\theta)$.

The end of the proof is the same as in the first part. Since $\mathcal{M}_{A+} \neq \emptyset$ (from Part (a)):

$c_+(V\theta) \geq (c_A)_+(\theta) = \sup_{\lambda\in\mathcal{M}_{A+}} \lambda \cdot \theta$ by the Fundamental Theorem of Asset Pricing

$= \int_{\boldsymbol{J}} \theta \, dv_{A+}$ by Lemma 5.2.d.

Consequently, from above and Part (a) we get $c_+(V\theta) = \int_{\boldsymbol{J}} \theta \, dv_{A+}$ for all θ. *Q.E.D.*

5.3. *Generalized Choquet integral*

The goal of this section is to show that the notion of generalized Choquet integral (used in Theorem 3.5) on the finite dimensional space X of replicable payoffs is equivalent to the notion of generalized Choquet integral on Riesz spaces, introduced by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015) (and studied in infinite dimensional Archimedean spaces).

Note that, hereafter the function $f : X \to \mathbb{R}$ is assumed to be finite-valued but f is neither assumed to be monotone, nor assumed to be defined on the whole space $\mathbb{R}^\Omega$. This level of generality is important to allow f to be the hedging price of a market (whereas the super-hedging price f_+ is always monotone, and is defined and finite on the whole space $\mathbb{R}^\Omega$ under mild assumptions).

Some general definitions are recalled hereafter; see Aliprantis and Border (1999) for a general reference on the subject. A *partially ordered set (poset)* $(X, \geq_X)$ is a set endowed with a partial order $\geq_X$, that is, $\geq_X$ is a transitive, reflexive, antisymmetric relation (but not necessarily total). We define $y >_X x$ by $y \geq_X x$ and $y \neq x$. The notation $x \leq_X y$ means $y \geq_X x$ and $x <_X y$ means $y >_X x$. A partial ordered set X is said to be a *lattice* if, for each pair x, y of elements in X, the set $\{x, y\}$ has a supremum (least upper bound) in X and an infimum (greatest lower bound) in X. A real vector space X, endowed with a partial order $\geq$, is said to be a Riesz space (also called a vector lattice) if it satisfies the following two properties:

$x \geq_X y \Rightarrow \alpha x + z \geq_X \alpha y + z \ \forall \alpha \in \mathbb{R}, \forall z \in X;$ [Ordered Vector Space]

X is a lattice with respect to $\geq_X$. [Riesz Property]

The Riesz space X is said to be Archimedean if

$0 \leq_X nx \leq_X y$ for all $n \in \mathbb{N}$ and some $y \in X$ implies $x = 0$. [Archimedean Property]

An element e is said to be an **order unit** of X if

$e >_X 0$ and $X = \cup_{n \in \mathbb{N}} \{x \in X : |x| \leq ne\}$, where $|x| := x \wedge -x$. [Order Unit Property]

Examples of Archimedean Riesz spaces with order unit used in our paper are:

- $(\mathbb{R}^J, \geq, \mathbf{1}_{\boldsymbol{J}})$, where $\geq$ is the coordinate-wise order,
- $(\langle V \rangle, \geq_V, e := V\mathbf{1}_{\boldsymbol{J}})$ where $\langle V \rangle := \operatorname{span}\{V^1, \ldots, V^J\}$ for independent vectors $V^1, \ldots, V^J$

in $\mathbb{R}^\Omega$ (Ω finite), and the order $x \geq_V x'$ is defined by $\theta \geq \theta'$ whenever $x = V\theta, x' = V\theta'$.

THEOREM 5.4 *Let Ω be a finite set, let $X \subseteq \mathbb{R}^\Omega$ be a vector subspace, let $e \in X$, and let $f : X \to \mathbb{R}$. Then the following assertions are equivalent:*

(1) f is a generalized Choquet functional on X, i.e., there exist an integer J, a set function $v : 2^{\boldsymbol{J}} \to \mathbb{R}$ such that $v(\emptyset) = 0$ and a linear isomorphism $V : \mathbb{R}^J \to X$ such that $e = V\mathbf{1}_{\boldsymbol{J}}$ and

$$f(V\theta) = \int_{\boldsymbol{J}} \theta \, dv \text{ for all } \theta \in \mathbb{R}^J \quad [\Leftrightarrow f(x) = \int_{\boldsymbol{J}} V^{-1}(x) \, dv \text{ for all } x \in X].$$

(2) $(X, \geq_X, e)$ is an Archimedean Riesz space with order unit and f satisfies

[Unit Additivity] $f(x+te) = f(x) + f(te)$ *for all* $x \in X$ *and all* $t \geq 0$*;*
[Unit Modularity] $f(x \vee_X te) + f(x \wedge_X te) = f(x) + f(te)$ *for all* $x \in X$ *and all* $t \in \mathbb{R}$*;*
[Bounded Variation] For all $x \in X_+$*,*
$Var_f(0,x) := \sup\left\{ \sum_{k=1}^{K} |f(x_k) - f(x_{k-1})| : 0 = x_0 \leq_X \cdots \leq_X x_K = x \text{ and } K \in \mathbb{N}\right\} < +\infty.$

(3) $(X, \geq_X, e)$ *is an Archimedean Riesz space with order unit and* $f : X \to \mathbb{R}$ *is a Choquet functional in the sense of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015).*

The equivalence $[(2) \Leftrightarrow (3)]$ is the fundamental result proved by Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015) that allows them to characterize their notion of Choquet functional on the Archimedean Riesz space X (see their paper for the formal definition) by the properties of Unit Additivity, Unit Modularity, and Bounded Variation that have some intuitive interpretation when applied to the hedging and super-hedging prices (see Cerreia-Vioglio, Maccheroni, and Marinacci (2015) and Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2015)). The proof below will thus only prove the equivalence $[(1) \Leftrightarrow (2)]$.

Proof: $[(1) \Rightarrow (2)]$ • $[(X, \geq_X, e)$ is an Archimedean Riesz space with unit$]$ It is standard routine argument to show that the linear isomorphism, $V : \mathbb{R}^J \to X$ allows to transport the Archimedean Riesz structure with order unit of $(\mathbb{R}^J, \geq, \mathbf{1_J})$ as an Archimedean Riesz space with order unit $(X, \geq_X, e)$, with $\geq$ denoting the coordinate-wise order of $\mathbb{R}^J$, $e := V(\mathbf{1_J})$, and the order $x \geq_X x'$ on the space X is defined by $V^{-1}(x) \geq V^{-1}(x')$.

• $[f$ is unit additive$]$ We have for every $x \in X$ and every $t \geq 0$

$f(x+te) := \int_{\boldsymbol{J}} V^{-1}(x+te)\,dv = \int_{\boldsymbol{J}} [V^{-1}(x) + t\mathbf{1_J}]\,dv$ [since $e := V(\mathbf{1_J})$]

$= \int_{\boldsymbol{J}} V^{-1}(x)\,dv + \int_{\boldsymbol{J}} t\mathbf{1_J}\,dv$ [since the Choquet integral $\int_{\boldsymbol{J}}$ is unit additive for $\mathbf{1_J}$]

$= f(x) + f(te)$ [since $f(te) := \int_{\boldsymbol{J}} V^{-1}(te)\,dv = \int_{\boldsymbol{J}} t\mathbf{1_J}\,dv$].

• $[f$ is unit modular$]$ We have for every $x \in X$ and every $t \in \mathbb{R}$

$f(x \vee_X te) + f(x \wedge_X te) := \int_{\boldsymbol{J}} V^{-1}(x \vee_X te)\,dv + \int_{\boldsymbol{J}} V^{-1}(x \wedge_X te)\,dv$

$= \int_{\boldsymbol{J}} [V^{-1}(x) \vee V^{-1}(te)]\,dv + \int_{\boldsymbol{J}} [V^{-1}(x) \wedge V^{-1}(te)]\,dv$ [from the definition of $\geq_X$]

$= \int_{\boldsymbol{J}}[V^{-1}(x) \vee t\mathbf{1}_{\boldsymbol{J}}]\, dv + \int_{\boldsymbol{J}}[V^{-1}(x) \wedge t\mathbf{1}_{\boldsymbol{J}}]\, dv$ [since $e := V(\mathbf{1}_{\boldsymbol{J}})$]

$= \int_{\boldsymbol{J}} V^{-1}(x)\ \ dv + \int_{\boldsymbol{J}} t\mathbf{1}_{\boldsymbol{J}}\, dv$ [since the Choquet integral $\int_{\boldsymbol{J}}$ is unit modular for $\mathbf{1}_{\boldsymbol{J}}$]

$= f(x) + f(te)$ [since $f(te) := \int_{\boldsymbol{J}} t\mathbf{1}_{\boldsymbol{J}}\, dv$].

• [f is of bounded variation] The proof follows Marinacci and Montrucchio (2004). From Aumann-Shapley, we can write v (which is of bounded variation since Ω is finite) as the difference of two capacities, that is, $v = v^+ - v^-$ where v^+ and v^- are set functions defined on $2^{\boldsymbol{J}}$ such that $v^+(\emptyset) = v^-(\emptyset) = 0$ and both v^+ and v^- are monotone. Then, for all $x \in X$, $K \in \mathbb{N}$, and $(x_0, \ldots, x_K) \in X^{1+K}$ such that $0 = x_0 \leq_X x_1 \leq_X \cdots \leq_X x_K = x$ one has

$$f(x_k) - f(x_{k-1}) := \int_{\boldsymbol{J}} V^{-1}(x_k)\, dv - \int_{\boldsymbol{J}} V^{-1}(x_{k-1})\, dv = a^+ - a^-$$

where $a^+ := \int_{\boldsymbol{J}} V^{-1}(x_k)\, dv^+ - \int_{\boldsymbol{J}} V^{-1}(x_{k-1})\, dv^+$,

$$a^- := \int_{\boldsymbol{J}} V^{-1}(x_k)\, dv^- - \int_{\boldsymbol{J}} V^{-1}(x_{k-1})\, dv^-.$$

But $x_k \geq_V x_{k-1}$ is equivalent to $V^{-1}(x_k) \geq V^{-1}(x_{k-1})$, and since v^+ and v^- are capacities we deduce that $a^+ \geq 0$ and $a^- \geq 0$. Thus

$$\begin{aligned} |f(x_k) - f(x_{k-1})| \leq a^+ + a^- = &\big(\textstyle\int_{\boldsymbol{J}} V^{-1}(x_k)\, dv^+ - \int_{\boldsymbol{J}} V^{-1}(x_{k-1})\, dv^+\big) \\ &+\big(\textstyle\int_{\boldsymbol{J}} V^{-1}(x_k)\, dv^- - \int_{\boldsymbol{J}} V^{-1}(x_{k-1})\, dv^-\big). \end{aligned}$$

Summing up these inequalities we get

$$\begin{aligned} &\textstyle\sum_{k=1}^K |f(x_k) - f(x_{k-1})| \\ &\quad\leq \big(\textstyle\int_{\boldsymbol{J}} V^{-1}(x_K)\, dv^+ - \int_{\boldsymbol{J}} V^{-1}(x_0)\, dv^+\big) + \big(\int_{\boldsymbol{J}} V^{-1}(x_K)\, dv^- - \int_{\boldsymbol{J}} V^{-1}(x_0)\, dv^-\big) \\ &\quad= \textstyle\int_{\boldsymbol{J}} V^{-1}(x)\, dv^+ + \int_{\boldsymbol{J}} V^{-1}(x)\, dv^- \text{ since } [x_0 = 0 \text{ and } x_K = x]. \end{aligned}$$

Thus, $\mathrm{Var}_f(0, x) \leq \int_{\boldsymbol{J}} V^{-1}(x)\, dv^+ + \int_{\boldsymbol{J}} V^{-1}(x)\, dv^- < +\infty$.

• • [(2) ⇒ (1)] We first recall that if $(X, \geq_X)$ is an Archimedean Riesz space of finite dimension $J \geq 1$, then there exists a lattice isomorphism V from $(\mathbb{R}^J, \geq)$ onto $(X, \geq_X)$, that is, $\geq$ is the coordinatewise order of $\mathbb{R}^J$, and $V : \mathbb{R}^J \to X$ is a linear isomorphism which is lattice preserving, i.e., $V(\theta \vee \theta') = V(\theta) \vee_X V(\theta')$ and $V(\theta \wedge \theta') = V(\theta) \wedge_X V(\theta')$ for all θ, θ' in $\mathbb{R}^J$ (Luxemburg and Zaanen, 1971, Theorem 26.11).

• We now prove that we can assume that $e = V(\mathbf{1}_{\boldsymbol{J}})$ by eventually modifying the lattice isomorphism V. Since V is a lattice isomorphism from $(\mathbb{R}^J, \geq)$ onto $(X, \geq_X)$, a standard routine argument shows that $V^{-1}(e)$ is an order unit of $(\mathbb{R}^J, \geq)$ and moreover $\tilde{\theta} := V^{-1}(e) \gg 0$. We now define the mapping $\tilde{V} : \mathbb{R}^J \to \mathbb{R}^\Omega$ by $\tilde{V}(\theta) := \sum_{j=1}^J \tilde{\theta}^j \theta^j V(\mathbf{1}_j)$ and one checks that $\tilde{V} : \mathbb{R}^J \to \mathbb{R}^\Omega$ is a lattice isomorphism satisfying $e = \tilde{V}(\mathbf{1_J})$. In the following, we will thus consider the lattice isomorphism $\tilde{V}$, *simply denoted by* V *for the sake of simpler notations.*

We now prove that the three properties of Unit Additivity, Unit Modularity, and of Bounded Variation, satisfied by f, are also satisfied by $f \circ V : \mathbb{R}^J \to \mathbb{R}$ for the unit $\mathbf{1_J}$.

• [$f \circ V$ is unit additive for $\mathbf{1_J}$] We have for all $\theta \in \mathbb{R}^J$ and all $t \geq 0$

$f \circ V(\theta + t\mathbf{1_J}) = f\big(V(\theta)\big) + f\big(te\big)$ [since since f is unit-additive for $e := V(\mathbf{1_J})$]

$= f \circ V(\theta) + f \circ V(t\mathbf{1_J})$ [since $e := V(\mathbf{1_J})$].

• [$f \circ V$ is unit modular for $\mathbf{1_J}$] For every $\theta \in \mathbb{R}^J$ and every $t \in \mathbb{R}$ we have

$f \circ V(\theta \vee t\mathbf{1_J}) + f \circ V(\theta \wedge t\mathbf{1_J}) = f\big(V(\theta) \vee_X tV(\mathbf{1_J})\big) + f\big(V(\theta) \wedge_X tV(\mathbf{1_J})\big)$

$= f\big(V(\theta) \vee_X te\big) + f\big(V(\theta) \wedge_X te\big)$ [since $e := V(\mathbf{1_J})$]

$= f\big(V(\theta)\big) + f\big(te\big)$ [since the f is unit modular on X]

$= f \circ V(\theta) + f \circ V(t\mathbf{1_J})$ [since $e := V(\mathbf{1_J})$].

• [$f \circ V$ is of bounded variation] For all $\theta \in \mathbb{R}^J$, all $K \in \mathbb{N}$, all $(\theta_0, \ldots, \theta_K) \in (\mathbb{R}^J)^{1+K}$ such that $0 = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_K = \theta$ one has $0 = V(\theta_0) \leq_X V(\theta_1) \leq \cdots \leq_X V(\theta_K) = V(\theta)$ and:

$\sum_{k=1}^K |f \circ V(\theta_k) - f \circ V(\theta_{k-1})|$

$\leq \sup \big\{ \sum_{k=1}^K |f(x_k) - f(x_{k-1})| : 0 = x_0 \leq_X \cdots \leq_X x_K = V(\theta) \text{ and } K \in \mathbb{N}\big\}$

$=: \mathrm{Var}_f(0, V(\theta)) < +\infty$ [since f is of Bounded Variation].

Hence $f \circ V$ is of Bounded Variation.

• We can now apply the Choquet Representation Theorem (Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2015, Theorem 22) (taking $L := \mathbb{R}^J$) that claims that if the function $g : \mathbb{R}^J \to \mathbb{R}$ satisfies Unit Additivity, Unit Modularity (both for the unit $\mathbf{1_J}$), and is of Bounded Variation, then it is a standard Choquet integral on the whole space $\mathbb{R}^J$. Applying it to $g := f \circ V$ we deduce the existence of a set function $v : 2^{\mathbf{J}} \to \mathbb{R}$ such that $v(\emptyset) = 0$ and

$g(\theta) := f \circ V(\theta) = \int_{\mathbf{J}} \theta \, dv$ for all $\theta \in \mathbb{R}^J$. *Q.E.D.*

REFERENCES

Aliprantis, C. D., and Border, K. C. (1999): *Infinite dimensional analysis.* Springer-Verlag, New York.

Amihud, Y., and Mendelson, H. (1986): Asset pricing and the Bid-Ask Spread. *Journal of Financial Economics* **17** 223–249.

Araujo, A., Chateauneuf A., and Faro J.H. (2012): Pricing rules and Arrow-Debreu ambiguous valuation. *Economic Theory* **49** 1–35.

——— (2018): Financial market structures revealed by pricing rules: Efficient complete markets are prevalent. *Journal of Economic Theory* **173** 257–288.

Bulow, J. I., Geanakoplos, J. D., and Klemperer, P. D. (1985): Multimarket Oligopoly: Strategic Substitutes and Complements. *Journal of Political Economy* **93** 488–511.

Cerreia-Vioglio, S., Maccheroni, F., and Marinacci, M. (2015): Put-Call parity and market frictions. *Journal of Economic Theory* **157** 730–762.

Cerreia-Vioglio, S., Maccheroni, F., Marinacci, M., and Montrucchio, L. (2015): Choquet integration on Riesz spaces and dual comonotonicity. *Transactions of the American Mathematical Society* **367** 8521–8542.

Chateauneuf, A., Kast, R., and Lapied, A.(1996): Choquet pricing for financial markets with frictions. *Mathematical Finance* **6**, 323–330.

Chateauneuf, A., and Cornet, B. (2018): Choquet representability of submodular functions. *Mathematical Programming* **168** 615–629.

Choquet, G. (1954): Theory of capacities. *Annales de l'Institut Fourier* **5**, 131–295.

Cox, J. C., and Ross, S.A. (1976): A survey of some new results in financial option pricing theory. *Journal of Finance* **31** 383–402.

——— (2006): *The Mathematics of Arbitrage*. Springer, Berlin.

Denneberg, D. (1994): *nonadditive measure and integral*. Kluwer, Dordrecht.

Duffie, D., and Huang, C. (1986): Multiperiod security markets with differential information. *Journal of Mathematical Economics* **15** 283–303.

Dybvig, P. H., and Ross, S.A. (1986): Tax clienteles and asset pricing. *Journal of Finance* **41** 751–762.

Follmer, H., and Schied, A. (2004): *Stochastic Finance. An Introduction in Discrete Time.* 2nd Edition. de Gruyter Studies in Mathematics 27.

Gilboa, I.(1987): Expected utility with purely subjective nonadditive probabilities. *Journal of Mathematical Economics* **16** 65–88.

——— (2009): *Theory of Decision under Uncertainty.* Econometric Society Monographs, 45, Cambridge University Press, Cambridge.

Gilboa, I., and Schmeidler, D. (1989): Maxmin expected utility with a non-unique prior. *Journal of Mathematical Economics* **18**, 141–153.

——— (1994): Additive representations of nonadditive measures and the Choquet integral. *Annals of Operations Research* **52** 43–65.

Gould, J.P., and Galai, D. (1974): Transactions costs and the relationship between put and call prices *Journal of Financial Economics* **1** 105–129.

Harrison, M.J., and Kreps, D.M. (1979): Martingales and arbitrage in multiperiod securities markets. *Journal of Economic Theory* **20** 381–408.

Jouini, E. (2000): Price Functionals with bid-ask spreads: An axiomatic approach. *Journal of Mathematical Economics* **34** 547–558.

Jouini, E., and Kallal, H. (1995): Martingales and arbitrage in securities markets with transactions costs. *Journal of Economic Theory* **66** 178–197.

——— (2001): Efficient trading strategies in the presence of market frictions. *The Review of Financial Studies* **14** 343–369.

Kamara, A., and Miller, T.W. (1995): Daily and intradaily tests of European Put-Call Parity *J.Finan. Quant. Anal.* **30** 519–539.

Klemkosky, R.C., and Miller, B.G. (1980): An ex ante analysis of Put-Call Parity *J.Finan. Econ* **8** 363–378.

Luttmer, E.G. (1996): Asset pricing in economies with frictions. *Econometrica* **64** 1439–1467.

Luxemburg, W. A. J., and Zaanen, A. C. (1971): *Riesz Spaces, Volume 1.* North Holland, Amsterdam.

Marinacci, M., and Montrucchio, L. (2004): Introduction to the Mathematics of Ambiguity. 46–107, in I. Gilboa, Ed. *Uncertainty in Economic Theory, Essays in honor of David Schmeidler's 65th birthday.* Routledge, New York.

——— (2005): Ultramodular functions. *Mathematics of Operations Research* **30** 311–332.

Milgrom, P., and Shannon, C. (1994): Monotone comparative statics. *Econometrica*, **62** 157–180.

Milgrom, P., and Roberts, J. (1990): Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica* **58** 1255–78.

Moulin, H. (1992): An application of the Shapley value to fair division with money. *Econometrica* **60** 1331–1349.

Müller A., and Scarsini, M. (2012): Fear of loss, inframodularity, and transfers. *Journal of Economic Theory* **147** 41–48.

Prisman, E. (1986): Valuation of risky assets in arbitrage-free economies with frictions. *Journal of Finance* **56** 545–557.

Rockafellar, R. T. (1970): *Convex Analysis.* Princeton University Press, Princeton.

Ross, S.A. (1976) Options and efficiency. *The Quarterly Journal of Economics* **90** 75–89.

——— (1987): Arbitrage and Martingales with Taxation. *Journal of Political Economy* **95** 371–393.

——— (2005): *Neoclassical Finance.* Princeton University Press, Princeton.

Samuelson, P. (1974): Complementarity: an essay on the 40th anniversary of the Hicks-Allen revolution in demand theory. *Journal of Economic Literature* **12** 1255–1289.

Schmeidler, D. (1986): Integral Representation withoutAdditivity. *Proceedings of the American Mathematical Society* **97** 255–261.

Schmeidler, D. ——— (1989): Subjective Probability and Expected Utility without Additivity. *Econometrica* **57** 571–587.

Shapley, S.C. (1971): Cores of convex games. *International Journal of GameTheory* **1** 12–26.

Sharkey, W. W. *The theory of natural monopoly.* Cambridge, Cambridge University Press.

Stoll, H.R. (1969): The relationship between put and call option prices *Journal of Finance* 801–824.

Sternberg, J. (1984): A re-examination of put-call parity on index futures. *Journal of futures markets* **14** 79–101.

Topkis, D. (1998): *Supermodularity and Complementarity.* Princeton, New Jersey.

Vives, X. (1990): Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics* **19** 305–321.