Order nearest comparative statics of equilibria

By

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Abstract

Order nearest comparisons of equilibria are intrinsic in many models but general results showing such comparisons have remained elusive in the natural setting of partially ordered sets of actions and monotone selections from best choices. We provide such theorems. We apply these theorems to show rich order nearest comparative statics of equilibria in models where it was not possible do so using earlier results. We don't use any continuity properties and subsume the earlier lattice-based theory seamlessly as a special case. Our framework includes classes of stochastic dynamic environments in decision-making under uncertainty beyond the reach of lattice-based approaches, showing the power of our new analytical tools to deepen the study of equilibrium behavior in broad new settings.

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1 Introduction

Order nearest comparisons of equilibria are motivated intrinsically in many models but theorems showing such comparisons in the natural setting of partially ordered sets (posets) of actions and multiple best choices are lacking. For example, consider the foundational model of stochastic dynamic economies with complementarities (SDEwC,

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Example 2), following Hopenhayn and Prescott (1992) and Stokey and Lucas (1989). Let $(X, \preceq_X)$, $(Z, \preceq_Z)$ be compact metric spaces with closed partial orders, formalizing actions and realizations of uncertainty, respectively, q the structure of serially correlated shocks, $\Gamma : X \times Z \rightrightarrows X$ a feasibility correspondence, $v(x, z) = \sup_{x' \in \Gamma(x,z)} \{F(x', x, z) + \beta \int v(x', z') q(z, dz')\}$ the associated value function, $\gamma(x, z) = \{x' \in \Gamma(x, z) \mid v(x, z) = F(x', x, z) + \beta \int v(x', z') q(z, dz')\}$ the optimal policy correspondence, and $\mathcal{G}$ the set of all measurable selections from γ. Each $g \in \mathcal{G}$ determines an associated stochastic kernel p_g (on the state space $X \times Z$) that governs the stochastic evolution of the economy over time. An *equilibrium* is a pair (g, μ) where $g \in \mathcal{G}$ and μ is a stationary distribution associated with p_g. Equivalently, μ is a fixed point of $\mathcal{T}_{p_g}$ (adjoint of p_g) on $\mathcal{X}$ (the set of probability measures on $X \times Z$). $\mathcal{E}(\mathcal{G})$ is the set of all equilibria. This provides a foundation to study many aspects of dynamic economies, including stochastic growth theory, industry equilibrium, investment theory, and more.

Under their complementarity assumptions, Hopenhayn and Prescott (1992) prove that an equilibrium always exists in SDEwC, by showing that γ has a smallest ($\underline{g}$) and a largest ($\overline{g}$) weakly increasing (or, isotone) measurable selection, the associated adjoint is isotone in the stochastic order (or, first order stochastic dominance), and then using Abian and Brown (1961) or Knaster-Tarski (Dugundji and Granas (1982)). Lattice-based fixed point theorems do not apply, because $\mathcal{X}$ is not a lattice in the stochastic order. It is a chain complete poset.

Other than existence, very little is known about the structure of the equilibrium set or comparative statics of equilibria in such economies, leaving open many deeper questions. For example, suppose policy f identifies a particular economic regime, the economy is at equilibrium $(f, \mu) \in \mathcal{E}(f)$, and economic conditions improve to yield a new policy g with higher actions. A natural prediction for the new equilibrium in the economy is $(g, \nu) \in \mathcal{E}(g)$ where ν is the order nearest higher distribution to μ (in stochastic order) among all equilibria $(g, \nu') \in \mathcal{E}(g)$. Does such an order nearest new equilibrium always exist in these economies? Alternatively, are equilibrium distributions in $\mathcal{E}(f)$ related to those in $\mathcal{E}(g)$ in an order nearest sense? Is there a policy f such that every equilibrium $(g, \mu) \in \mathcal{E}(\mathcal{G})$ can be order approximated (finding an order nearest equilibrium) from below (or from above) with an equilibrium using f? As isotone policies

$\mathcal{G}^{iso} = \{g \in \mathcal{G} \mid g \text{ is isotone}\}$ are frequently of interest, what is the structure of the set of isotone equilibria, $\mathcal{E}(\mathcal{G}^{iso}) = \{(g, \mu) \in \mathcal{E}(\mathcal{G}) \mid g \text{ is isotone}\}$? In the parametric version, the optimal policy correspondence is $(x, z, t) \mapsto \gamma(x, z, t)$, measurable selections from $\gamma(\cdot, \cdot, t)$ define the correspondence $t \mapsto \mathcal{G}_t$, the equilibrium correspondence is $t \mapsto \mathcal{E}(\mathcal{G}_t)$, the isotone equilibrium correspondence is $t \mapsto \mathcal{E}(\mathcal{G}_t^{iso})$, and every selection $t \mapsto g_t$ leads to an associated equilibrium subcorrespondence $t \mapsto \mathcal{E}(g_t)$. With a higher shock (in stochastic order), for a given equilibrium before the shock, monotone arguments predict that the system moves to the nearest higher equilibrium after the shock.[1] Do such equilibria exist for every old equilibrium (or subset of equilibria)? When do we have monotone comparative statics (MCS) of equilibrium (that is, an isotone selection from the equilibrium correspondence)? Do we have MCS of equilibrium subcorrespondences (when t goes up, the entire associated equilibrium set goes up in an order nearest sense)? These are long-standing open questions for SDEwC that we answer as special cases (Corollaries 5 and 9) of our Theorems 3 and 8. Such answers are beyond the reach of lattice-based approaches. As a concrete application, Example 1 demonstrates these results explicitly for a model of stochastic dynamic industry equilibrium with switching costs, and shows limitations of lattice-based approaches in this regard.

We prove poset-based theorems for order nearest comparative statics of equilibria using the following new set comparison relations. For poset X and nonempty $A, B \subseteq X$, *A is $*$-sup-complete in B*, denoted $A \sqsubseteq^{*sc} B$, if for every nonempty $E \subseteq A$, $\sup_X E$ exists $\Rightarrow$ $\sup_B E$ exists. Similarly, *B is $*$-inf-complete in A*, $A \sqsubseteq^{*ic} B$, if for every nonempty $E \subseteq B$, $\inf_X E$ exists $\Rightarrow$ $\inf_A E$ exists. *A is $*$-complete in B*, $A \sqsubseteq^{*c} B$, if $A \sqsubseteq^{*sc} B$ and $A \sqsubseteq^{*ic} B$. *We prove theorems at the level of these definitions.* Our definitions imply the following special cases of interest. If $A \sqsubseteq^{*sc} B$, then for every $a \in A$, there is unique $b \in B$ that is order nearest to a from above. Similarly, $A \sqsubseteq^{*ic} B$ implies that for every $b \in B$, there is unique $a \in A$ that is order nearest to b from below. These provide natural order nearest strengthenings of the upper and lower weak set orders. When X is a complete lattice, $\sup_X E$ and $\inf_X E$ always exist, seamlessly recovering the definitions in Sabarwal (2025).[2] When X is chain complete and E is a

[1] A stability argument for monotone selections is provided by the correspondence principle, see Echenique (2002).

[2] When necessary, we denote the comparable definitions in Sabarwal (2025) by the subscript $s25$, as discussed in Section 6.

chain, $\sup_X E$ and $\inf_X E$ always exist, making our definition stronger than the one in Sabarwal (2023b). Additional comparisons among different set relations are provided in Section 6 and Proposition 21.

A significant technical contribution here is that we develop a theory of order nearest comparative statics of equilibria without using properties of lattices required in the proofs in Sabarwal (2025). This is a critical requirement to move beyond lattices. We achieve this with generalizations of poset-based structure theorems for fixed points which have remained elusive for about 50 years since Markowsky (1976) and by constructing new definitions for comparing equilibria that work with our generalizations. The lattice-based framework remains the foundational framework in the literature. However, the poset-based framework introduced here offers a broader scope of applications extending well beyond the capabilities of lattice-based approaches. The results show that our new techniques are well-suited for comparing equilibria in poset-based models, they subsume the existing theory seamlessly, and they deepen the study of equilibrium behavior in a wider array of problems that were previously out of reach.

Our main results extend to large classes of stochastic dynamic behavior beyond the reach of lattice-based models, such as kernel systems (Example 3, Propositions 11 and 13) and stochastic dynamical systems (Example 4, Propositions 15 and 17), which include stochastic dynamic economies (of which Example 2 is a special case) and Markov decision processes as special cases.

We take individual behavior as given and develop its equilibrium implications. This yields theorems that apply regardless of the manner in which individual choices are made as long as they satisfy our weak conditions, which are satisfied in foundational models. Our results can open the door to new results in additional areas, for example, dynamic supermodular games as in Echenique (2004) and Feng and Sabarwal (2020), or directional MCS as in Barthel and Sabarwal (2018) and Paul and Sabarwal (2023), or matching as in Adachi (2000), Echenique and Oviedo (2004) and Echenique and Oviedo (2006), and more.

We do not make any continuity assumptions for our general results, maintaining a benefit of monotone methods in studying models with discontinuities or non-convexities where the use of standard tools from topology and convex analysis may be limited. A strand of the theory considers monotone sequences that arise from iterating the smallest best response and using continuity, shows that such sequences converge to the smallest equilibrium. Examples can be seen in Milgrom and Roberts (1990), Milgrom and Shannon (1994), Amir (1996), Echenique (2002), Roy and Sabarwal (2012), Sabarwal (2021), Balbus, Dziewulski, Reffett, and Woźny (2022), and others. Balbus, Olszewski, Reffett, and Woźny (2023) study strong set order increasing (respectively, strongly monotone) upper order hemicontinuous correspondences on complete (respectively, σ-complete) lattices. Sabarwal (2023a) studies the general poset-based theory.

The paper is organized as follows. Section 2 presents a concrete motivating application to industry equilibrium with switching costs. Section 3 introduces the general framework and proves the main results. Section 4 extends these results to classes of stochastic dynamic decisions and Section 5 to Bayesian games. Section 6 compares different set relations. Section 7 concludes. The appendix (Section 8) provides additional extensions.

2 Illustrative application

Example 1 (Industry equilibrium with switching costs)**.** Consider an industry with a continuum of symmetric firms with mass 1. The representative firm's demand can be in one of three regimes (r): low (L), medium (M), or high (H), and in each regime, there can be a low or high demand shock. Accordingly, firm demand shocks are modeled as $z \in Z = \{0, 1, 2, 3, 4, 5\}$, where low regime demand shocks are in $Z_L = \{0, 1\}$, medium in $Z_M = \{2, 3\}$, and high in $Z_H = \{4, 5\}$. Low and high shocks in regime r are denoted z_ℓ^r and z_h^r, respectively. In each regime, demand shocks are serially correlated with kernel in Table 1. For convenience, we keep the same kernel for each regime. Quarterly demand

Table 1: Distribution of demand shocks in regime r

		$\mathtt{t}+1$	
$q(z,z')$		z_ℓ^r	z_h^r
$\mathtt{t}$	z_ℓ^r	0.5	0.5
	z_h^r	0.4	0.6

(in millions of units) is

$$f(z) = \begin{cases} z & \text{if } z \in Z_L \\ z+1 & \text{if } z \in Z_M \\ z+2 & \text{if } z \in Z_H. \end{cases}$$

Production technology dictates three plant levels for a firm: shut down (zero output), medium scale (capacity 4 million units per quarter), and large scale (capacity 8 million units per quarter), denoted $x \in X = \{0, 1, 2\}$. Switching costs (in millions of dollars in the switching period) from plant size x to x', $s(x, x')$, are given in Table 2. We assume that incremental switching sizes are moderately expensive ($1 million) and large switching sizes (0 to 2 or vice versa) are prohibitively expensive ($10 million). Market

Table 2: Switching costs ($ millions)

		$\mathtt{t}+1$		
$s(x,x')$		**0**	**1**	**2**
$\mathtt{t}$	**0**	0	1	10
	1	1	0	1
	2	10	1	0

price for output is constant at $2 per unit and marginal cost is constant $1 per unit. Fixed cost of operation for medium and large scale is $k =$ $2 million per quarter. If a firm operates and can't meet demand due to capacity constraint, its payoff is reduced by the profit from unmet demand (opportunity cost of lost sales). Similarly, if a firm is shut down, its payoff is reduced by the profit from unmet demand and increased by k dollars that it doesn't have to pay to generate the profit from unmet demand. In case of a switch in plant scale, payoffs are adjusted for switching costs. Thus, if a firm is operating at scale x and demand shock is z, its next period payoff from operating at scale x' is

$$\pi(x, z, x') = \begin{cases} k - f(z) - s(x, x') & \text{if } x' \in \{0\} \\ f(z) - k - s(x, x') & \text{if } (z, x') \in \{0, 1, 2, 3\} \times \{1\} \\ \pi(x, 3, 1) - (f(z) - f(3)) & \text{if } (z, x') \in \{4, 5\} \times \{1\} \\ f(z) - k - s(x, x') & \text{if } x' \in \{2\}. \end{cases}$$

We assume the per period discount factor is $\beta = 0.95$.

Stochastic dynamic optimization leads to the unique optimal policy $g : X \times Z \to X$ given by

$$g(x,z) = \begin{cases} 0 & \text{if } (x,z) \in \{0,1\} \times Z_L \\ 2 & \text{if } (x,z) \in (\{1,2\} \times Z_H) \cup (\{2\} \times Z_M) \\ 1 & \text{otherwise.} \end{cases}$$

Intuitively, when demand shock is low and the firm is not operating at large scale, the best choice is to shut down, when demand shock is high and the firm is not shut down or when demand shock is medium and the firm is operating at large scale, best choice is to operate at large scale, and best choice is to operate at medium scale otherwise. Policy g is isotone.

The corresponding set $\mathcal{E}$ of equilibrium or stationary distributions is the convex hull of $\{\mu_1, \mu_2, \mu_3, \mu_4\}$, where $\mu_1 = \frac{4}{9}\delta_{(0,0)} + \frac{5}{9}\delta_{(0,1)}$, $\mu_2 = \frac{4}{9}\delta_{(1,2)} + \frac{5}{9}\delta_{(1,3)}$, $\mu_3 = \frac{4}{9}\delta_{(2,2)} + \frac{5}{9}\delta_{(2,3)}$, and $\mu_4 = \frac{4}{9}\delta_{(2,4)} + \frac{5}{9}\delta_{(2,5)}$, where $\delta_{(x,z)}$ is point mass on (x,z). In this industry, for every initial distribution with support in $X \times \{0,1\} \subseteq X \times Z$, the unique equilibrium distribution is μ_1, for every initial distribution with support in $X \times \{4,5\} \subseteq X \times Z$, the unique equilibrium distribution is μ_4, and for every initial distribution with support in $X \times \{2,3\} \subseteq X \times Z$, the equilibrium distribution is a mixture of μ_2 and μ_3. More general initial distributions lead to appropriate mixtures of $\{\mu_1, \mu_2, \mu_3, \mu_4\}$. Thus, $\mathcal{E}$ is a three-dimensional submanifold with boundary in the 17-dimensional simplex $\Delta(X \times Z)$. Lattice-based structure theorems for the equilibrium set do not apply because $\Delta(X \times Z)$ is not a lattice in the stochastic order. It is a chain complete poset and therefore, our Theorem 3 applies and shows that $\mathcal{E}$ is $*$-complete in the stochastic order and has a smallest and largest equilibrium. $\mathcal{E}$ is neither a lattice (and therefore, not a complete lattice, nor $*$-complete in the sense of Sabarwal (2025)) nor a chain. It is chain complete (which follows from the $*$-complete definition here). The smallest equilibrium distribution is μ_1 and the largest is μ_4.

We can conduct monotone comparative statics of the equilibrium set as well. Consider parametric serially correlated demand shocks, where the kernel in each regime is parametrized by $t \in [0,1]$, as listed in Table 3. Notice that $t \mapsto q_t$ is isotone in stochastic order. The case above corresponds to $t = 1/2$. For each $t \in [0,1]$, the set of equilibrium distributions $\mathcal{E}(t)$ is the convex hull of $\{\mu_1(t), \mu_2(t), \mu_3(t), \mu_4(t)\}$, where

Table 3: Parametric distribution of demand shocks in regime r

		$\mathtt{t}+1$	
$q_t(z,z')$		z_ℓ^r	z_h^r
$\mathtt{t}$	z_ℓ^r	$0.6-0.2t$	$0.4+0.2t$
	z_h^r	$0.5-0.2t$	$0.5+0.2t$

$\mu_1(t) = \frac{5-2t}{9}\delta_{(0,0)} + \frac{4+2t}{9}\delta_{(0,1)}$, $\mu_2(t) = \frac{5-2t}{9}\delta_{(1,2)} + \frac{4+2t}{9}\delta_{(1,3)}$, $\mu_3(t) = \frac{5-2t}{9}\delta_{(2,2)} + \frac{4+2t}{9}\delta_{(2,3)}$, and $\mu_4(t) = \frac{5-2t}{9}\delta_{(2,4)} + \frac{4+2t}{9}\delta_{(2,5)}$. The equilibrium correspondence $t \mapsto \mathcal{E}(t)$ maps into three-dimensional submanifolds with boundary in $\Delta(X \times Z)$. Each $\mathcal{E}(t)$ is $*$-complete in the stochastic order but is neither a lattice nor a chain. The smallest equilibrium distribution at t is $\mu_1(t)$ and the largest is $\mu_4(t)$.

Lattice-based monotone comparative statics theorems do not apply, but our Theorem 8 applies and implies that the equilibrium correspondence is isotone in $*$-complete relation. In particular, for arbitrary $\hat{t} \preceq \tilde{t}$ and an arbitrary equilibrium distribution $e(\hat{t}) = \lambda_1\mu_1(\hat{t}) + \lambda_2\mu_2(\hat{t}) + \lambda_3\mu_3(\hat{t}) + \lambda_4\mu_4(\hat{t}) \in \mathcal{E}(\hat{t})$, there is a unique equilibrium distribution in $\mathcal{E}(\tilde{t})$ that is higher than $e(\hat{t})$ and every other equilibrium in $\mathcal{E}(\tilde{t})$ that is higher than $e(\hat{t})$ is also higher than $e(\tilde{t})$. It can be checked that this order nearest higher equilibrium distribution is given by $e(\tilde{t}) = \lambda_1\mu_1(\tilde{t}) + \lambda_2\mu_2(\tilde{t}) + \lambda_3\mu_3(\tilde{t}) + \lambda_4\mu_4(\tilde{t})$. In other words, when demand shocks move toward higher realizations, regardless of the equilibrium distribution before the shock there is always a corresponding nearest higher distribution after the shock and this provides a prediction for the new equilibrium after the shock. The dual statement holds when demand shocks move toward lower realizations. As a special case, the infimum equilibrium selection $t \mapsto \mu_1(t)$ and the supremum equilibrium selection $t \mapsto \mu_4(t)$ are isotone, and for every $\hat{t} \preceq \tilde{t}$, $\mu_1(\tilde{t}) = \sup_{\mathcal{E}(\tilde{t})}\{\mu_1(\hat{t})\}$ and $\mu_4(\tilde{t}) = \sup_{\mathcal{E}(\tilde{t})}\{\mu_4(\hat{t})\}$.

3 Order nearest comparative statics: Theory

A *partially ordered set*, or *poset*, is a pair $(X, \preceq_X)$, where X is a set and $\preceq_X$ is a partial order on X. For $A \subseteq X$, the relative partial order on A is the usual one. For posets $(X, \preceq_X)$ and $(Y, \preceq_Y)$, the Cartesian product $X \times Y$ is a poset under the product partial order, and a function $f : X \to Y$ is *isotone* if for every $\hat{x}, \tilde{x} \in X$,

$\hat{x} \preceq_X \tilde{x} \implies f(\hat{x}) \preceq_Y f(\tilde{x})$. The partial order on the set of all functions from X to poset $(Y, \preceq_Y)$ is the product (or pointwise) partial order: $f \preceq g \Leftrightarrow$ for every x, $f(x) \preceq_Y g(x)$. For notational convenience we may drop the subscript in $\preceq_X$. An *upper bound for* $E \subseteq X$ is an element $x \in X$ such that for every $e \in E$, $e \preceq x$. The *sup of* E *in* X, denoted $\sup_X E$, is an element $\overline{e} \in X$ such that (1) $\overline{e}$ is an upper bound for E and (2) for every $x \in X$ that is an upper bound for E, $\overline{e} \preceq x$. A *lower bound for* E is an element $x \in X$ such that for every $e \in E$, $x \preceq e$. The *inf of* E *in* X, denoted $\inf_X E$, is an element $\underline{e} \in X$ such that (1) $\underline{e}$ is a lower bound for E and (2) for every $x \in X$ that is a lower bound for E, $x \preceq \underline{e}$.

A poset $(X, \preceq)$ is *chain sup-complete (respectively, inf-complete)*, denoted CSC (respectively, CIC) if for every nonempty chain $C \subseteq X$, $\sup_X C$ exists (respectively, $\inf_X C$ exists). A poset X is *chain complete*, denoted CC, if it is CSC and CIC. A *lattice* is a poset $(X, \preceq)$ in which for every $x, y \in X$, $x \wedge y := \inf_X\{x, y\}$ and $x \vee y := \sup_X\{x, y\}$ exist in X. $A \subseteq X$ is a *complete lattice* (in the relative partial order) if for every nonempty $E \subseteq A$, $\inf_A E$ and $\sup_A E$ exist in A. $A \subseteq X$ is *subcomplete* if for every nonempty $E \subseteq A$, $\inf_X E \in A$ and $\sup_X E \in A$. It follows immediately that if X is a nonempty complete lattice then it is CC with smallest and largest points, and it is easy to check that the converse is not true.

A *poset model* is a triple $(X, \preceq, \Phi)$, where $(X, \preceq)$ is a poset and $\Phi : X \rightrightarrows X$ is a correspondence. In a poset model $(X, \preceq, \Phi)$, the *equilibrium set* is the fixed point set of Φ, $\mathcal{E}(\Phi) = \{x \in X \mid x \in \Phi(x)\}$. For arbitrary selection f from Φ, the *f-equilibrium set* is $\mathcal{E}(f) = \{x \in X \mid x = f(x)\} \subseteq \mathcal{E}(\Phi)$.

Recall that for nonempty subsets A and B of lattice X, A *is lower than* B *in strong set order (SSO),* $A \sqsubseteq^s B$, if for every $a \in A$ and for every $b \in B$, (1) $a \wedge b \in A$ and (2) $a \vee b \in B$. Topkis (1978) attributes SSO to Veinott (1989). A weakening of SSO into a lower (respectively, upper) variant is defined if only (1) (respectively, (2)) is assumed and denoted $A \sqsubseteq^\wedge B$ (respectively, $A \sqsubseteq^\vee B$). A still weaker version is defined as $A \sqsubseteq^{wV} B$, if for every $a \in A$ and for every $b \in B$, either $a \wedge b \in A$ or $a \vee b \in B$ (see LiCalzi and Veinott (1992) and Kukushkin (2013)). For nonempty subsets A and B of poset X, A *is lower than* B *in weak set order (WSO)*, denoted $A \sqsubseteq^w B$, if (1) for every $y \in B$, there is $x \in A$, $x \preceq y$ and (2) for every $x \in A$, there is $y \in B$, $x \preceq y$. Its lower (respectively, upper) variant is defined similarly using only (1) (respectively, (2)) and denoted $A \sqsubseteq^{lw} B$ (respectively, $A \sqsubseteq^{uw} B$).

To formalize order nearest comparisons, we follow Sabarwal (2025)'s definition for order nearest points of one set using a different set in a poset X. For arbitrary nonempty $A, E \subseteq X$, $\sup_A E$ is an element $\overline{e} \in A$ such that (1) $\overline{e}$ is an upper bound for E and (2) for every $a \in A$ that is an upper bound for E, $\overline{e} \preceq a$. Similarly, $\inf_A E$, is an element $\underline{e} \in A$ such that (1) $\underline{e}$ is a lower bound for E and (2) for every $a \in A$ that is a lower bound for E, $a \preceq \underline{e}$. Notice that $A = X$ gives the standard definition, and $E \subseteq A$ gives the standard definition in the relative partial order. Sabarwal (2025) generalizes these to arbitrary nonempty subsets of X. Therefore, $\sup_A E$ and $\inf_A E$ might not exist in general even if X is a complete lattice. When they exist, they provide natural and tight order bounds of E from A.

Our set comparison relations are the following. For nonempty $A, B \subseteq X$, A ***is $*$-sup-complete in*** B, denoted $A \sqsubseteq^{*sc} B$, if for every nonempty $E \subseteq A$, $\sup_X E$ exists $\Rightarrow$ $\sup_B E$ exists. Similarly, B ***is $*$-inf-complete in*** A, $A \sqsubseteq^{*ic} B$, if for every nonempty $E \subseteq B$, $\inf_X E$ exists $\Rightarrow$ $\inf_A E$ exists. A ***is $*$-complete in*** B, $A \sqsubseteq^{*c} B$, if $A \sqsubseteq^{*sc} B$ and $A \sqsubseteq^{*ic} B$. We prove theorems at the level of these definitions. Several special cases are of interest. $A \sqsubseteq^{*sc} B$ implies that for every $a \in A$, there is unique $b \in B$ that is order nearest to a from above. Similarly, $A \sqsubseteq^{*ic} B$ implies that for every $b \in B$, there is unique $a \in A$ that is order nearest to b from below. When X is a complete lattice, $\sup_X E$ and $\inf_X E$ always exist, seamlessly recovering the definitions in Sabarwal (2025). More generally, our definitions are weaker. When X is CC, and E is a chain, $\sup_X E$ and $\inf_X E$ always exist, making our definition stronger than the one in Sabarwal (2023b). Section 6 and Proposition 21 provide additional comparisons among different set relations.

Correspondence $\Psi : X \rightrightarrows Y$ is a *subcorrespondence* of $\Phi : X \rightrightarrows Y$, if for every x, $\Psi(x) \subseteq \Phi(x)$. For every relation $\sqsubseteq^r$ on subsets of Y and every correspondence $\Phi : X \rightrightarrows Y$, Φ *is isotone in r-relation*, if $\hat{x} \preceq_X \tilde{x} \Rightarrow \Phi(\hat{x}) \sqsubseteq^r \Phi(\tilde{x})$, and for correspondences $\Psi, \Phi : X \rightrightarrows Y$, $\Psi \sqsubseteq^r \Phi$ is defined pointwise. A correspondence $\Phi : X \rightrightarrows Y$ is *inf-isotone* if Φ has an infimum selection and it is isotone (that is, for every $x \in X, \underline{\Phi}(x) \coloneqq \inf_{\Phi(x)} \Phi(x)$

exists and $x \mapsto \underline{\Phi}(x)$ is isotone).[3] This is equivalent to Φ is isotone in lower weak set order, (that is, $\hat{x} \preceq_X \tilde{x} \Rightarrow \Phi(\hat{x}) \sqsubseteq^{lw} \Phi(\tilde{x})$,) and for every $x, \underline{\Phi}(x)$ exists. Correspondence Φ is *sup-isotone* if it has a supremum selection that is isotone (for every $x \in X, \overline{\Phi}(x) \coloneqq \sup_{\Phi(x)} \Phi(x)$ exists and $x \mapsto \overline{\Phi}(x)$ is isotone). This is equivalent to Φ is isotone in upper weak set order, (that is, $\hat{x} \preceq_X \tilde{x} \Rightarrow \Phi(\hat{x}) \sqsubseteq^{uw} \Phi(\tilde{x})$,) and for every $x, \overline{\Phi}(x)$ exists. A selection f from correspondence $\Phi : X \rightrightarrows Y$ is *upper (respectively, lower) order nearest isotone* if for every $\hat{x} \preceq \tilde{x}$, $f(\tilde{x}) = \sup_{\Phi(\tilde{x})}\{f(\hat{x})\}$ (respectively, $f(\hat{x}) = \inf_{\Phi(\hat{x})}\{f(\tilde{x})\}$). As usual, a singleton valued correpondence is viewed as equivalent to a function, and in this case we use either notation without further mention.

A *universal model with (weak) complementarities* is a poset model in which Φ has an isotone selection. Two special cases are when Φ has an inf-isotone or sup-isotone selection. These weak conditions hold in many applications, as follows.

The lattice-based standard model prevalent in the literature is included here as follows. Following Sabarwal (2025), a *standard model* is a finitely indexed collection $(X_i, \preceq_i, F_i)_{i=1}^I$, where for each i, X_i is a nonempty, complete lattice (and using product order on products of X_i), for each i, $F_i : X_i \times X_{-i} \to \mathbb{R}$ has single crossing property in (x_i, x_{-i}), and for each x_{-i}, $F_i(\cdot, x_{-i})$ is quasisupermodular and upper semicontinuous in Frink (1942)'s order interval topology on X_i. For each x_{-i}, let $\Phi_i(x_{-i}) = \arg\max_{\xi \in X_i} F_i(\xi, x_{-i})$. Let $\Phi : X \rightrightarrows X$ be given by $\Phi(x) = \times_{i=1}^I \Phi_i(x_{-i})$. The *associated poset model* is $(X, \preceq, \Phi)$, where $X = \times_{i=1}^I X_i$, $\preceq$ is product order, and $\Phi = \times_{i=1}^I \Phi_i$ and equilibria are given by fixed points of Φ. As is well-known, for every standard model, in its associated poset model, Φ is inf-isotone and sup-isotone. The standard model includes foundational models of games with strategic complements in Topkis (1978), Topkis (1979), Vives (1990), Milgrom and Roberts (1990), Shannon (1990), and Milgrom and Shannon (1994). New models are included as follows.

Example 2 (Stochastic dynamic economies with complementarities (SDEwC))**.** Fol-

[3] The definitions here use infimum or supremum over $\Phi(x)$ not X. It is easy to check that $\inf_{\Phi(x)} \Phi(x)$ exists $\Rightarrow \inf_{\Phi(x)} \Phi(x) = \inf_X \Phi(x)$, and the same holds if $\inf_X \Phi(x) \in \Phi(x)$, and similarly for supremum. Therefore, in these definitions we can use either version. This is not true more generally: For arbitrary $E \subseteq \Phi(x)$, $\inf_{\Phi(x)} E$ exists $\not\Rightarrow \inf_{\Phi(x)} E = \inf_X E$, and similarly for supremum. The weaker definition using a subset of X rather than X over which infimum or supremum is taken (similar to the distinction between complete and subcomplete lattice) is more relevant for this paper, and therefore, we use $\inf_{\Phi(x)} \Phi(x)$ or $\sup_{\Phi(x)} \Phi(x)$ to keep the notation consistent with other parts of the paper where the distinction is important.

lowing Hopenhayn and Prescott (1992) and Stokey and Lucas (1989), let $(X, \preceq_X)$ and $(Z, \preceq_Z)$ be compact metric spaces with closed partial orders, and Borel sigma-algebras $\mathcal{B}(X), \mathcal{B}(Z)$, q an isotone kernel on $Z \times \mathcal{B}(Z)$,[4] $\Gamma : X \times Z \rightrightarrows X$ a feasibility correspondence, A the graph of Γ, $F : A \to \mathbb{R}$ a bounded one-period return function, $\beta \in (0,1)$ the constant discount rate, and assuming standard conditions for the principle of optimality, let $v : X \times Z \to \mathbb{R}$ be the unique value function associated with this problem, given by $v(x,z) = \sup_{x' \in \Gamma(x,z)} \{F(x', x, z) + \beta \int v(x', z') q(z, dz')\}$, and $\gamma(x,z) = \{x' \in \Gamma(x,z) \mid v(x,z) = F(x', x, z) + \beta \int v(x', z') q(z, dz')\}$ the policy correspondence. The economy satisfies complementarity assumptions in Hopenhayn and Prescott (1992): X and Z are complete lattices, F is supermodular on $X \times X$ for each z and has increasing differences in $(x', x; z)$, Γ has strict complementarity, graph of $\Gamma(\cdot, \cdot, z)$ is a sublattice for each z, and Γ is ascending (isotone in SSO). Hopenhayn and Prescott (1992) show that in this case, γ is isotone in SSO and has an infimum selection $\underline{g}$ and supremum selection $\overline{g}$, and therefore, both selections are isotone. If X is a complete separable metric space with continuous lattice structure and γ is compact valued and upper measurable, then $\underline{g}$ and $\overline{g}$ are measurable. With these assumptions, a *stochastic dynamic economy with complementarities (SDEwC)* is given by $(X, Z, q, \Gamma, F, \beta)$.

Equilibrium is defined as follows. Let $\mathcal{G}$ be the set of measurable selections from γ and $\mathcal{G}^{iso} \subseteq \mathcal{G}$ be isotone selections. For each policy $g \in \mathcal{G}$, let $\hat{g} : X \times Z \times Z \to X \times Z$ be given by $\hat{g}(x, z, z') = (g(x,z), z')$. Let $S = X \times Z$ with product order, sigma-algebra, and topology. For each $g \in \mathcal{G}$, let p_g be a kernel on $S \times \mathcal{B}(S)$ defined by $p_g((x,z), A) = q(z, [\hat{g}^{-1}(A)]_{(x,z)})$, for every $A \in \mathcal{B}(S)$. Let $\mathcal{T}_{p_g}$ be the adjoint of p_g on the set of probability measures on Borel sets of S, denoted $\mathcal{X}$, defined by $\mathcal{T}_{p_g}(\mu)(A) = \int_X p_g(x, A) d\mu(x)$, for every $A \in \mathcal{B}(S)$. The *equilibrium set* is $\mathcal{E}(\mathcal{G}) = \{(g, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{p_g}(\mu)\}$ and the *isotone equilibrium set* is $\mathcal{E}(\mathcal{G}^{iso}) = \{(g, \mu) \in \mathcal{G}^{iso} \times \mathcal{X} \mid \mu = \mathcal{T}_{p_g}(\mu)\}$. Equivalently, $\mathcal{E}(\mathcal{G}^{iso}) = \{(g, \mu) \in \mathcal{E}(\mathcal{G}) \mid g \text{ is isotone}\}$. Hopenhayn and Prescott (1992) prove that every SDEwC has an isotone equilibrium, by showing that $\mathcal{X}$ is CC and $\underline{g}$ and $\overline{g}$ are isotone, whence $\mathcal{T}_{p_{\underline{g}}}$ and $\mathcal{T}_{p_{\overline{g}}}$ are isotone in the stochastic order, and then using

[4] A kernel on $X \times \mathcal{B}(Y)$ is a function $p : X \times \mathcal{B}(Y) \to [0,1]$ that is measurable in x for every $A \in \mathcal{B}(Y)$, and is a probability measure on $\mathcal{B}(Y)$ for every $x \in X$. Kernel p is *isotone*, if for every $\hat{x} \preceq_X \tilde{x}$, $p(\hat{x}, \cdot) \preceq_s p(\tilde{x}, \cdot)$, where $\preceq_s$ is the *stochastic order (or first-order stochastic dominance)* on finite measures on Y, denoted $\mathfrak{M}(Y)$, defined by: $\mu \preceq_s \nu$ if for every increasing set $A \in \mathcal{B}(Y)$, $\mu(A) \leq \nu(A)$, where $A \subseteq Y$ is increasing if for every $a \in A$ and $y \in Y$, $a \preceq_Y y \Rightarrow y \in A$.

Abian and Brown (1961) or Knaster-Tarski (Dugundji and Granas (1982)). Lattice-based fixed point theorems do not apply, because $\mathcal{X}$ is not a lattice in the stochastic order. Using the associated poset model $((\mathcal{X}, \preceq_s, \Phi)$, where $\preceq_s$ is stochastic order and Φ is the adjoint correspondence: $\Phi(\mu) = \{\mathcal{T}_{p_g}(\mu) \mid g \in \mathcal{G}\})$, Lemma 1 strengthens this to show that every SDEwC has a smallest equilibrium distribution μ^* (in stochastic order) such that $(\underline{g}, \mu^*) = \inf_{\mathcal{E}(\mathcal{G})} \mathcal{E}(\mathcal{G}) = \inf_{\mathcal{E}(\mathcal{G}^{iso})} \mathcal{E}(\mathcal{G}^{iso})$ (in product order), and a largest equilibrium distribution ν^* such that $(\overline{g}, \nu^*) = \sup_{\mathcal{E}(\mathcal{G})} \mathcal{E}(\mathcal{G}) = \sup_{\mathcal{E}(\mathcal{G}^{iso})} \mathcal{E}(\mathcal{G}^{iso})$.

Lemma 1. *For every poset X that is CSC and $\inf_X X$ exists (respectively, CIC and $\sup_X X$ exists), and for every $\Phi : X \rightrightarrows X$ that is inf-isotone (respectively, sup-isotone), $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ exists and $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ (respectively, $\sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ exists and $\sup_{\mathcal{E}(\overline{\Phi})} \mathcal{E}(\overline{\Phi}) = \sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$.*

Proof. $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi})$ exists follows from Markowsky (1976). To see that $\inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$, let $A = \{x \in X \mid x \preceq \underline{\Phi}(x) \text{ and } \forall e \in \mathcal{E}(\Phi), x \preceq e\}$. The set A is nonempty as $\inf_X X \in A$. Let C be a chain in A. If C is empty, then $\sup_X C = \inf_X X \in A$. If C is not empty, let $y = \sup_X C$, which exists because X is chain sup-complete. Notice that for every x, $x \in C$ implies $x \preceq \underline{\Phi}(x)$, and also, $x \preceq y$ implies $\underline{\Phi}(x) \preceq \underline{\Phi}(y)$. Thus, $\underline{\Phi}(y)$ is an upper bound for C, whence $y \preceq \underline{\Phi}(y)$. Moreover, for every $e \in \mathcal{E}(\Phi)$, for every $x \in C, x \preceq e$ implies that for every $e \in \mathcal{E}(\Phi)$, e is an upper bound for C, and therefore, for every $e \in \mathcal{E}(\Phi), y \preceq e$. It follows that $y \in A$. In other words, every chain in A has an upper bound in A. By Zorn's lemma, let e^* be a maximal element in A. Then $e^* \preceq \underline{\Phi}(e^*)$ and for every $e \in \mathcal{E}(\Phi), e^* \preceq e$. Therefore, $\underline{\Phi}(e^*) \preceq \underline{\Phi}(\underline{\Phi}(e^*))$ and for every $e \in \mathcal{E}(\Phi), \underline{\Phi}(e^*) \preceq \underline{\Phi}(e) \preceq e$. Consequently, $\underline{\Phi}(e^*) \in A$. As e^* is maximal in A, it cannot be that $e^* \neq \underline{\Phi}(e^*)$, whence $e^* = \underline{\Phi}(e^*) \in \Phi(e^*)$. In other words, $e^* \in \mathcal{E}(\underline{\Phi}) \subseteq \mathcal{E}(\Phi)$ and for every $e \in \mathcal{E}(\Phi), e^* \preceq e$, that is, $e^* = \inf_{\mathcal{E}(\underline{\Phi})} \mathcal{E}(\underline{\Phi}) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$. ∎

Lemma 1 shows that every poset model in which X is CSC with smallest point and Φ is inf-isotone has a smallest equilibrium, and in which X is CIC with largest point and Φ is sup-isotone has a largest equilibrium. No continuity properties are needed.

We define the following stronger properties useful to prove MCS of equilibrium correspondences in stochastic systems. For partially ordered measurable spaces X and Y, let *mbl*(X, Y) denote the set of measurable functions from X to Y with pointwise partial order. In every SDEwC, for every nonempty $\mathcal{F} \subseteq$ *mbl*$(X \times Z, X)$, $\mathcal{F}$ is *inf-isotone* (respectively, *sup-isotone*) if $\inf_{\mathcal{F}} \mathcal{F}$ (respectively, $\sup_{\mathcal{F}} \mathcal{F}$) exists and is an isotone function. We say that $\mathcal{G}^{iso}$ is *inf-isotone on upper intervals* if $\inf_{\mathcal{G}^{iso}} \mathcal{G}^{iso}$ exists and for every isotone $\hat{g} \in mbl(X \times Z, X)$, $\inf_{\hat{\mathcal{G}}^{iso}} \hat{\mathcal{G}}^{iso}$ exists whenever $\hat{\mathcal{G}}^{iso} = \{g \in \mathcal{G}^{iso} \mid \hat{g} \preceq g\}$ is nonempty. Notice that $\mathcal{G}$ is inf-isotone implies $\inf_{\mathcal{G}^{iso}} \mathcal{G}^{iso}$ exists. Moreover, for every isotone $\hat{g} \in mbl(X \times Z, X)$, $\hat{\mathcal{G}} = \{g \in \mathcal{G} \mid \hat{g} \preceq g\}$ is inf-isotone if it is nonempty implies that $\hat{\mathcal{G}}^{iso}$ is nonempty (because $\inf_{\mathcal{G}} \mathcal{G} \in \hat{\mathcal{G}}^{iso}$) and $\inf_{\hat{\mathcal{G}}^{iso}} \hat{\mathcal{G}}^{iso}$ exists. Similarly, $\mathcal{G}^{iso}$ is *sup-isotone on lower intervals* if $\sup_{\mathcal{G}^{iso}} \mathcal{G}^{iso}$ exists and for every isotone $\hat{g} \in mbl(X \times Z, X)$, $\sup_{\hat{\mathcal{G}}^{iso}} \hat{\mathcal{G}}^{iso}$ exists whenever $\hat{\mathcal{G}}^{iso} = \{g \in \mathcal{G}^{iso} \mid g \preceq \hat{g}\}$ is nonempty. When $\mathcal{G} = \{g\}$ is a singleton, both conditions are equivalent to g is isotone. These properties hold in SDEwC.

Proposition 2. *In every stochastic dynamic economy with complementarities, $\mathcal{G}^{iso}$ is inf-isotone on upper intervals and sup-isotone on lower intervals.*

Proof. Hopenhayn and Prescott (1992) show that $\mathcal{G}$ has a smallest selection $\underline{g}$ and it is isotone and a largest selection $\overline{g}$ and it is isotone, and therefore, $\inf_{\mathcal{G}^{iso}} \mathcal{G}^{iso}$ and $\sup_{\mathcal{G}^{iso}} \mathcal{G}^{iso}$ exist. Let $\hat{g} \in mbl(X \times Z, X)$ be an isotone function such that $\hat{\mathcal{G}}^{iso} = \{g \in \mathcal{G}^{iso} \mid \hat{g} \preceq g\}$ is nonempty. For each $(x, z) \in X \times Z$, consider the restricted problem $\hat{\gamma}(x, z) = \arg\max_{x' \in \hat{\Gamma}(x,z)} F(x', x, z) + \beta \int v(x', z') q(z, dz')$, where $\hat{\Gamma}(x, z) = \Gamma(x, z) \cap [\hat{g}(x, z), \infty)$. The correspondence $\hat{\gamma}$ is nonempty valued because $\overline{g}$ is a selection from $\hat{\gamma}$. Moreover, $\hat{\Gamma}$ remains isotone in SSO and the optimization problem continues to fit the framework of Topkis (1978) and Hopenhayn and Prescott (1992). Let $\underline{\hat{g}}$ be the inf-isotone measurable selection from $\hat{\gamma}$. Then $\underline{\hat{g}} \in \mathcal{G}^{iso}$ and $\hat{g} \preceq \underline{\hat{g}}$, whence $\underline{\hat{g}} \in \hat{\mathcal{G}}^{iso}$. To see that $\underline{\hat{g}} = \inf_{\hat{\mathcal{G}}^{iso}} \hat{\mathcal{G}}^{iso}$, we first show that for every (x, z), $\hat{\gamma}(x, z) = \gamma(x, z) \cap [\hat{g}(x, z), \infty)$, as follows. Suppose $\xi \in \hat{\gamma}(x, z)$. Then $\hat{g}(x, z) \preceq_X \xi$. Moreover, $\hat{g}(x, z) \preceq_X \overline{g}(x, z)$ and $\overline{g}(x, z) \in \gamma(x, z)$ imply $\overline{g}(x, z) \in \hat{\Gamma}(x, z)$, whence $F(\xi, x, z) + \beta \int v(\xi, z') q(z, dz') \geq F(\overline{g}(x, z), x, z) + \beta \int v(\overline{g}(x, z), z') q(z, dz') \geq F(x', x, z) + \beta \int v(x', z') q(z, dz'), \forall x' \in \Gamma(x, y)$, where the second inequality follows from $\overline{g}(x, z) \in \gamma(x, z)$. This shows that $\xi \in \gamma(x, z) \cap [\hat{g}(x, z), \infty)$. In the other direction, suppose $\xi \in \gamma(x, z) \cap [\hat{g}(x, z), \infty)$. Then $\hat{g}(x, z) \preceq \xi$ and $\forall x' \in \Gamma(x, z)$, $F(\xi, x, z) + \beta \int v(\xi, z') q(z, dz') \geq F(x', x, z) + \beta \int v(x', z') q(z, dz')$, whence $\xi \in \hat{\gamma}(x, z)$. Now consider arbitrary $g \in \mathcal{G}$ such that $\hat{g} \preceq g$. Then for every (x, y), $g(x, y) \in \gamma(x, z) \cap [\hat{g}(x, z), \infty) = \hat{\gamma}(x, z)$, and as $\underline{\hat{g}}$ is the smallest such selection, $\underline{\hat{g}} \preceq g$. In particular, $\underline{\hat{g}} = \inf_{\hat{\mathcal{G}}^{iso}} \hat{\mathcal{G}}^{iso}$. This shows that $\mathcal{G}^{iso}$ is inf-isotone on upper intervals.

Similarly, it is sup-isotone on lower intervals. ∎

Similar properties are defined for arbitrary poset models as follows. In a poset model $(X, \preceq_X, \Phi)$, for every $\hat{X} \subseteq X$, Φ *restricted to* $\hat{X}$ is the correspondence given by $\hat{\Phi}(x) = \Phi(x) \cap \hat{X}$. Φ is *inf-isotone on upper intervals*, if it is inf-isotone and for every $\hat{x} \in X$ such that $\Phi(\hat{x}) \cap \hat{X} \neq \emptyset$, (where $\hat{X} = \{x \in X \mid \hat{x} \preceq_X x\}$), Φ restricted to $\hat{X}$ is inf-isotone. Φ is *sup-isotone on lower intervals*, if it is sup-isotone and for every $\hat{x} \in X$ such that $\Phi(\hat{x}) \cap \hat{X} \neq \emptyset$, (where $\hat{X} = \{x \in X \mid x \preceq_X \hat{x}\}$), Φ restricted to $\hat{X}$ is sup-isotone.

Before presenting our first main result, we introduce the following definitions. Let $(X, \preceq_X)$ and $(Y, \preceq_Y)$ be posets and $\Phi : X \rightrightarrows Y$ a correspondence. For a selection f from Φ, define $\Phi^{f-} : X \rightrightarrows Y$ by $\Phi^{f-}(x) = \{y \in \Phi(x) \mid y \preceq_Y f(x)\}$ and $\Phi^{f+} : X \rightrightarrows Y$ by $\Phi^{f+}(x) = \{y \in \Phi(x) \mid f(x) \preceq_Y y\}$. Graph of Φ^{f-} is the graph of Φ that is (weakly) below selection f, graph of Φ^{f+} is the graph of Φ that is (weakly) above selection f, and for every x, $\Phi^{f-}(x) \cap \Phi^{f+}(x) = f(x)$. Moreover, $\mathcal{E}(f) = \mathcal{E}(\Phi^{f+}) \cap \mathcal{E}(\Phi^{f-})$. In the special case that $f = \underline{\Phi}$, for every x, $\Phi^{f-}(x) = f(x)$ and $\Phi^{f+}(x) = \Phi(x)$, and similarly, $f = \overline{\Phi}$ implies that for every x, $\Phi^{f+}(x) = f(x)$ and $\Phi^{f-}(x) = \Phi(x)$. Our first main result follows.

Theorem 3. *For every poset X and every $\Psi, \Phi : X \rightrightarrows X$, the following statements hold whenever the sets being compared are nonempty.*

(1) For every selection f from Ψ and g from Φ with $f \preceq g$,

- *(a) If X is CSC and g is isotone, then $\mathcal{E}(f) \sqsubseteq^{*sc} \mathcal{E}(g)$, $\mathcal{E}(f) \sqsubseteq^{*sc} \mathcal{E}(\Phi^{g+})$, $\mathcal{E}(\Psi^{f-}) \sqsubseteq^{*sc} \mathcal{E}(g)$, and $\mathcal{E}(\Psi^{f-}) \sqsubseteq^{*sc} \mathcal{E}(\Phi^{g+})$.*
- *(b) If X is CSC, $\inf_X X$ exists, and f is isotone, then for every $e \in \mathcal{E}(g)$, $\inf_{\mathcal{E}(f)} \mathcal{E}(f) \preceq e$.*
- *(c) If X is CIC and f is isotone, then $\mathcal{E}(f) \sqsubseteq^{*ic} \mathcal{E}(g)$, $\mathcal{E}(f) \sqsubseteq^{*ic} \mathcal{E}(\Phi^{g+})$, $\mathcal{E}(\Phi^{f-}) \sqsubseteq^{*ic} \mathcal{E}(g)$, and $\mathcal{E}(\Phi^{f-}) \sqsubseteq^{*ic} \mathcal{E}(\Phi^{g+})$.*
- *(d) If X is CIC, $\sup_X X$ exists, and g is isotone, then for every $e \in \mathcal{E}(f)$, $e \preceq \sup_{\mathcal{E}(g)} \mathcal{E}(g)$.*

(2) (a) If X is CSC, $\Psi \sqsubseteq^{uw} \Phi$, Φ is sup-isotone, and Φ is inf-isotone on upper intervals, then $\mathcal{E}(\Psi) \sqsubseteq^{*sc} \mathcal{E}(\Phi)$.

(b) If X is CIC, $\Phi \sqsubseteq^{lw} \Psi$, Φ is inf-isotone, and Φ is sup-isotone on lower intervals, then $\mathcal{E}(\Phi) \sqsubseteq^{*ic} \mathcal{E}(\Psi)$.

Proof. For (1)(a), as $\mathcal{E}(f) \subseteq \mathcal{E}(\Psi^{f-})$, it is sufficient to prove that $\mathcal{E}(\Psi^{f-}) \sqsubseteq^{*sc} \mathcal{E}(g)$ and $\mathcal{E}(\Psi^{f-}) \sqsubseteq^{*sc} \mathcal{E}(\Phi^{g+})$. Consider $E \subseteq \mathcal{E}(\Psi^{f-})$ and suppose $\bar{e} = \sup_X E$ exists. Let $\hat{X} = \{x \in X \mid \bar{e} \preceq x\}$ and $\hat{\Phi} : \hat{X} \to \hat{X}$ be given by $\hat{\Phi}(x) = g(x)$. Then $\forall e \in E$, $e \preceq f(e) \preceq g(e) \preceq g(\bar{e})$, where the second inequality follows from $f \preceq g$ and the third from g is isotone. Therefore, $\bar{e} \preceq g(\bar{e})$. Moreover, g is isotone implies $\forall x \in \hat{X}$, $\bar{e} \preceq g(\bar{e}) \preceq g(x)$. This shows that $\hat{\Phi}$ is well-defined. As $\hat{\Phi}$ is isotone, using Lemma 1 for the poset model $(\hat{X}, \preceq_X, \hat{\Phi})$, let $\tilde{e}$ be the smallest fixed point of $\hat{\Phi}$. Then $\tilde{e} \in \mathcal{E}(g)$ and $\tilde{e}$ is an upper bound for E. If $e \in \mathcal{E}(g)$ is an arbitrary upper bound for E, then $\bar{e} \preceq e$ and $e = g(e) = \hat{\Phi}(e)$ imply $\tilde{e} \preceq e$. Thus, $\sup_{\mathcal{E}(g)} E = \tilde{e}$. To show that $\sup_{\mathcal{E}(\Phi^{g+})} E$ exists, let $\hat{X}$ be as above and let $\hat{\Phi} : \hat{X} \rightrightarrows \hat{X}$ be given by $\hat{\Phi}(x) = \Phi^{g+}(x) \cap \hat{X}$, that is, $\hat{\Phi}(x) = \{y \in \Phi(x) \mid g(x) \preceq y\} \cap \hat{X}$. The argument above shows that $\hat{\Phi}$ is well-defined and has an isotone infimum selection $x \mapsto g(x)$. By Lemma 1, $\tilde{e}$ is the smallest fixed point of $\hat{\Phi}$. If $e \in \mathcal{E}(\Phi^{g+})$ is an arbitrary upper bound for E, then $\bar{e} \preceq e$ and therefore, $e \in \hat{\Phi}(e)$, whence $\tilde{e} \preceq e$, showing that $\tilde{e} = \sup_{\mathcal{E}(\Phi^{g+})} E$. Statement (1)(b) is proved by applying Lemma 1 to $\Lambda : X \rightrightarrows X$ given by $\Lambda(x) = \{f(x), g(x)\}$. Statements (1)(c) and (1)(d) are proved similarly, using the dual argument.

For (2)(a), consider $E \subseteq \mathcal{E}(\Psi)$, suppose $\bar{e} = \sup_X E$ exists, and notice that $\Psi \sqsubseteq^{uw} \Phi$ and $\overline{\Phi}$ is isotone imply $\bar{e} \preceq \overline{\Phi}(\bar{e})$. Let $\hat{X} = \{x \in X \mid \bar{e} \preceq x\}$ and $\hat{\Phi} : \hat{X} \rightrightarrows \hat{X}$ be given by $\hat{\Phi}(x) = \Phi(x) \cap \hat{X}$. As Φ is inf-isotone on upper intervals, $\hat{\Phi}$ is inf-isotone in the poset model $(\hat{X}, \preceq, \hat{\Phi})$. Let $\hat{e}$ be the smallest fixed point of $\hat{\Phi}$. Then $\hat{e} \in \mathcal{E}(\Phi)$. Moreover, for every $e \in \mathcal{E}(\Phi)$ that is an upper bound for E, $\bar{e} \preceq e$, whence $e \in \mathcal{E}(\hat{\Phi})$, and consequently, $\hat{e} \preceq e$. This shows that $\sup_{\mathcal{E}(\Phi)} E = \hat{e}$. Statement (2)(b) is proved similarly. ∎

Theorem 3 provides universal theorems for order nearest comparative statics of equilibria in poset-based models. In statement (1), other than $f \preceq g$, no conditions are placed on f in (1)(a) and (1)(d) or on g in (1)(b) and (1)(c), and in statement (2), other than $\Psi \sqsubseteq^{uw} \Phi$ or $\Phi \sqsubseteq^{lw} \Psi$, no conditions are placed on Ψ. This provides theorems for order nearest comparative statics of equilibria across models with and without complementarities, for example, when comparing an arbitrary stochastic dynamic economy to a SDEwC. The special case of $\Psi = \Phi$ proves order nearest monotone comparative statics of equilibrium sets associated with ranked selections $f \preceq g$ from Φ in statement (1) and proves structural characteristics of the equilibrium set in (2). Similar results apply to subcorrespondences Φ^{f-}, Φ^{f+}, Φ^{g-} and Φ^{g+} as well.

The special case when X is a finite set is important because all finite games fall in this class. As every finite poset is necessarily chain complete, the assumptions on X in these definitions simplify to checking that X has a smallest or largest point. *No additional structure is required for X.* This is considerably weaker than assuming that X is a lattice, which is the corresponding restriction in lattice-based models with finite X. We have the following additional results.

Corollary 4. *For every poset X and $\Phi : X \rightrightarrows X$, if X is CSC and Φ is sup-isotone (respectively, X is CIC and Φ is inf-isotone), then $\mathcal{E}(\Phi) \sqsubseteq^{*sc} \mathcal{E}(\overline{\Phi})$ (respectively, $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*ic} \mathcal{E}(\Phi)$), whenever the sets being compared are nonempty.*

Proof. $\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*ic} \mathcal{E}(\Phi)$ follows from Theorem 3 by setting $\Psi = \Phi$ and $f = g = \underline{\Phi}$ and noting that $\mathcal{E}(\underline{\Phi}) = \mathcal{E}(\Phi^{\underline{\Phi}-}) \sqsubseteq^{*ic} \mathcal{E}(\Phi^{\underline{\Phi}+}) = \mathcal{E}(\Phi)$. Similarly, $\mathcal{E}(\Phi) \sqsubseteq^{*sc} \mathcal{E}(\overline{\Phi})$ follows by setting $f = g = \overline{\Phi}$. ∎

Corollary 4 shows that arbitrary subsets of equilibria can be order approximated from below using equilibria from the infimum selection and from above using the supremum selection. As a special case, in every inf-isotone (respectively, sup-isotone) model, for every $e \in \mathcal{E}(\Phi)$, there is unique $\hat{e} \in \mathcal{E}(\underline{\Phi})$ (respectively, $\hat{e} \in \mathcal{E}(\overline{\Phi})$) that is order-nearest to e from below (respectively, above) among all equilibria in $\mathcal{E}(\underline{\Phi})$ (respectively, $\mathcal{E}(\overline{\Phi})$). This requires very little structure on the model. We have the following applications.

Corollary 5. *In every stochastic dynamic economy with complementarities,*

(1) For every isotone $f, g \in \mathcal{G}$, $f \preceq g \Rightarrow \mathcal{E}(f) \sqsubseteq^{*c} \mathcal{E}(g)$.

(2) $\mathcal{E}(\underline{g}) \sqsubseteq^{*c} \mathcal{E}(\mathcal{G})$ and $\mathcal{E}(\mathcal{G}) \sqsubseteq^{*c} \mathcal{E}(\overline{g})$.

(3) $\mathcal{E}(\mathcal{G}^{iso}) \sqsubseteq^{*c} \mathcal{E}(\mathcal{G}^{iso})$.

Proof. Statements (1) and (2) follow from Theorem 3 in a straightforward manner. For (3), to show that $\mathcal{E}(\mathcal{G}^{iso}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{G}^{iso})$, let $E \subseteq \mathcal{E}(\mathcal{G}^{iso})$ be nonempty and suppose $(\hat{g}, \hat{\mu}) = \sup_{\mathcal{M} \times \mathcal{X}} E$ exists, where $\mathcal{M} = mbl(X \times Z, X)$ and $\mathcal{X}$ is the set of probability measures on Borel sets on S. Then $\hat{g} = \sup_{\mathcal{M}} E_1$, where $E_1 = \{g \in \mathcal{G} \mid \exists \mu, (g, \mu) \in E\}$, and $\hat{\mu} = \sup_{\mathcal{X}} E_2$, where $E_2 = \{\mu \in \mathcal{X} \mid \exists g, (g, \mu) \in E\}$. As every g in E_1 is isotone, it follows that $\hat{g}$ is isotone. Moreover, $\mathcal{G}$ is sup-isotone implies that $\hat{\mathcal{G}}^{iso} = \{g \in \mathcal{G}^{iso} \mid \hat{g} \preceq g\}$ is nonempty. Using Proposition 2, let $\underline{\hat{g}} = \inf_{\hat{\mathcal{G}}^{iso}} \hat{\mathcal{G}}^{iso}$. Let $\hat{\mathcal{X}} = \{\mu \in \mathcal{X} \mid \hat{\mu} \preceq_s \mu\}$ and for $\mu \in \hat{\mathcal{X}}$, let $\hat{\Phi}(\mu) = \{\mathcal{T}_{p_g}(\mu) \mid g \in \hat{\mathcal{G}}^{iso}\}$. Correspondence $\hat{\Phi} : \hat{\mathcal{X}} \rightrightarrows \hat{\mathcal{X}}$ is well defined, because $\hat{\mu} \preceq_s \mathcal{T}_{p_{\underline{\hat{g}}}}(\hat{\mu})$ and moreover, $\hat{\Phi}$ has an isotone infimum selection given by $\mu \mapsto \mathcal{T}_{p_{\underline{\hat{g}}}}(\mu)$. Let $\underline{\hat{\mu}}$ be the smallest fixed point of $\hat{\Phi}$. Then $(\underline{\hat{g}}, \underline{\hat{\mu}}) \in \mathcal{E}(\mathcal{G}^{iso})$. Let $(g', \mu') \in \mathcal{E}(\mathcal{G}^{iso})$ be an upper bound for E. Then $\underline{\hat{g}} \preceq g'$ and $\hat{\mu} \preceq_s \mu'$, and therefore, μ' is a fixed point of $\hat{\Phi}$, whence $\underline{\hat{\mu}} \preceq_s \mu'$. It follows that $(\underline{\hat{g}}, \underline{\hat{\mu}}) = \sup_{\mathcal{E}(\mathcal{G}^{iso})} E$. Similarly, $\mathcal{E}(\mathcal{G}^{iso}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{G}^{iso})$. ∎

Corollary 5 shows new structural features of equilibria in SDEwC and answers several long-standing open questions for SDEwC. Ranked isotone policies have correspondingly ranked equilibrium sets in the $*c$ relation. Moreover, for every equilibrium (g, μ) there is a unique equilibrium distribution μ_- lower than μ in the stochastic order such that $(\underline{g}, \mu_-)$ is the unique equilibrium in $\mathcal{E}(\underline{g})$ that is order nearest to (g, μ) from below and there is a unique equilibrium distribution μ_+ such that $(\overline{g}, \mu_+)$ is the unique equilibrium in $\mathcal{E}(\overline{g})$ that is order nearest to (g, μ) from above. Furthermore, the isotone equilibrium set is $*$-complete.

Corollary 6. *For every poset X and every $\Phi : X \rightrightarrows X$, if X is CSC and $\inf_X X$ (respectively, CIC and $\sup_X X$) exists, Φ is sup-isotone (respectively, inf-isotone), and Φ is inf-isotone on upper intervals (respectively, sup-isotone on lower intervals), then $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ exists and $\mathcal{E}(\Phi) \sqsubseteq^{*sc} \mathcal{E}(\Phi)$ (respectively, $\sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ exists and $\mathcal{E}(\Phi) \sqsubseteq^{*ic} \mathcal{E}(\Phi)$).*

Proof. $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ (respectively, $\sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$) exists follows from Lemma 1, and $\mathcal{E}(\Phi) \sqsubseteq^{*sc} \mathcal{E}(\Phi)$ (respectively, $\mathcal{E}(\Phi) \sqsubseteq^{*ic} \mathcal{E}(\Phi)$) follows from statement (2) in Theorem 3 by setting $\Psi = \Phi$. ∎

Corollary 6 generalizes several existing results in the literature. It generalizes the structure result in Markowsky (1976) to the case of correspondences and strengthens it by showing that $\mathcal{E}(\Phi)$ is $*$-sup-complete (or $*$-inf-complete), which implies (but is stronger than) $\mathcal{E}(\Phi)$ is chain sup-complete (or chain inf-complete). It also generalizes to poset-based models the lattice-based structure results due to Tarski (1955), Zhou (1994), and

Sabarwal (2025). Moreover, it provides a characterization of CSC posets with a smallest point (respectively, CIC posets with a largest point) in terms of correspondences on it.

Corollary 7. *For every poset X, the following are equivalent.*

(1) X is CSC and $\inf_X X$ exists.

(2) For every $\Phi : X \rightrightarrows X$ that is sup-isotone and is inf-isotone on upper intervals, $\mathcal{E}(\Phi) \sqsubseteq^{*sc} \mathcal{E}(\Phi)$ and $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ exists.

(3) For every $\Phi : X \rightrightarrows X$ that is sup-isotone and is inf-isotone on upper intervals, $\mathcal{E}(\Phi)$ is CSC and $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ exists.

(4) For every $\Phi : X \rightrightarrows X$ that is sup-isotone and is inf-isotone on upper intervals, $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ exists.

Proof. (1) $\Rightarrow$ (2) is proved in Corollary 6 and therefore, (1) $\Rightarrow$ (3). Moreover, (2) $\Rightarrow$ (4) and (3) $\Rightarrow$ (4) are obvious. For (4) $\Rightarrow$ (1), notice that (4) implies that for every $f : X \to X$ that is isotone, $\mathcal{E}(f)$ has a least fixed point, because when the correspondence is singleton-valued, the condition on Φ in (4) is equivalent to an isotone function, and using Markowsky (1976), this implies (1). ∎

A dual characterization holds for X is CIC and $\sup_X X$ exists.

Theorem 3 also provides a foundation for MCS of equilibrium subcorrespondences. We include parametric effects using an arbitrary poset T of parameters. A *parametric poset model* is a collection $((X, \preceq_X), (T, \preceq_T), \Phi)$, where $(X, \preceq_X)$ and $(T, \preceq_T)$ are posets and $\Phi : X \times T \rightrightarrows X$ is a correspondence. For each $t \in T$, the *poset model at t* is the triple $(X, \preceq_X, \Phi_t)$ where Φ_t is the t-section of Φ. The *equilibrium set at t* is $\mathcal{E}(\Phi_t) = \{x \in X \mid x \in \Phi(x, t)\}$ and the *equilibrium correspondence* is $\mathcal{E} : T \rightrightarrows X$, $t \mapsto \mathcal{E}(\Phi_t)$. For every selection f from Φ, the *f-equilibrium set at t* is $\mathcal{E}(f_t) = \{x \in X \mid x = f(x, t)\}$ and the *f-equilibrium correspondence* is $t \mapsto \mathcal{E}(f_t)$. Theorem 3 implies the following results for MCS of equilibrium correspondences and subcorrespondences in $*sc$, $*ic$, and $*c$ relations.

Theorem 8. *For every poset X and T such that X is CSC and $\inf_X X$ exists (respectively, X is CIC and $\sup_X X$ exists), for every $\Psi, \Phi : X \times T \rightrightarrows X$,*

(1) *For every isotone selection g from Φ, the equilibrium subcorrespondence $t \mapsto \mathcal{E}(g_t)$ is nonempty valued and isotone in $*sc$ (respectively, $*ic$) relation. Moreover,*

   (a) *The infimum (respectively, supremum) equilibrium selection, $t \mapsto \inf_{\mathcal{E}(g_t)} \mathcal{E}(g_t)$ (respectively, $t \mapsto \sup_{\mathcal{E}(g_t)} \mathcal{E}(g_t)$), exists and is upper (respectively, lower) order nearest isotone, and*

   (b) *If $g = \overline{\Phi}$ (respectively, $g = \underline{\Phi}$), then for every t, $\mathcal{E}(\Phi_t) \sqsubseteq^{*sc} \mathcal{E}(\overline{\Phi}_t)$ (respectively, $\mathcal{E}(\underline{\Phi}_t) \sqsubseteq^{*ic} \mathcal{E}(\Phi_t)$) as well.*

   (c) *For every selection f from Ψ, $f \preceq g$ (respectively, $g \preceq f$) $\Rightarrow$ for every $t, \mathcal{E}(f_t) \sqsubseteq^{*sc} \mathcal{E}(g_t)$ (respectively, $\mathcal{E}(g_t) \sqsubseteq^{*ic} \mathcal{E}(f_t)$), whenever $\mathcal{E}(f_t)$ is nonempty.*

(2) *If Φ is sup-isotone (respectively, inf-isotone) and for every t, Φ_t is inf-isotone on upper intervals (respectively, sup-isotone on lower intervals), then*

   (a) *The full equilibrium correspondence $t \mapsto \mathcal{E}(\Phi_t)$ is nonempty valued and isotone in $*sc$ (respectively, $*ic$) relation, and*

   (b) *For every t, $\Psi_t \sqsubseteq^{uw} \Phi_t$ (respectively, $\Phi_t \sqsubseteq^{lw} \Psi_t$) $\Rightarrow \mathcal{E}(\Psi_t) \sqsubseteq^{*sc} \mathcal{E}(\Phi_t)$ (respectively, $\mathcal{E}(\Phi_t) \sqsubseteq^{*ic} \mathcal{E}(\Psi_t)$), whenever $\mathcal{E}(\Psi_t)$ is nonempty.*

Proof. For the case X is CSC and $\inf_X X$ exists, statement (1) is proved by noting that for every t, $\mathcal{E}(g_t)$ is nonempty, because Lemma 1 implies that $\inf_{\mathcal{E}(g_t)} \mathcal{E}(g_t)$ exists and then applying Theorem 3 by setting $\Psi_t = \Phi_t$ and for $\hat{t} \preceq_T \tilde{t}$, setting $f = g_{\hat{t}}$ and $g = g_{\tilde{t}}$, and this proves (1)(a) as well, because Theorem 3(1)(b) implies that for every $\hat{t} \preceq_T \tilde{t}$, $\inf_{\mathcal{E}(g_{\tilde{t}})} \mathcal{E}(g_{\tilde{t}}) = \sup_{\mathcal{E}(g_{\tilde{t}})}\{\inf_{\mathcal{E}(g_{\hat{t}})} \mathcal{E}(g_{\hat{t}})\}$. Statement (1)(b) follows similarly. Statement (1)(c) is proved by applying the same statement in Theorem 3 to $f_t \preceq g_t$. Statement (2)(a) follows from statement (2) in Theorem 3 by setting $\Psi_t = \Phi_t$ and (2)(b) follows from statement (2) in Theorem 3 as well. The chain inf-complete cases are proved similarly, using the dual arguments. ∎

Theorem 8 provides order nearest isotone equilibrium selections, and order nearest MCS for equilibrium subcorrespondences, the full equilibrium correspondence, and across models with and without complementarities. When Φ is an isotone function, a frequently-desired special case in applications, Theorem 8 implies that the full equilibrium correspondence is isotone in $*sc$ (respectively, $*ic$) relation when X is CSC with smallest point (respectively, X is CIC with largest point). Applications to parametric SDEwC are as follows.

Example 2 continued (Parametric SDEwC) A *parametric stochastic dynamic economy with complementarties* is given by $(X, Z, T, q, \Gamma, F, \beta)$, where X and Z are as in Example 2, $T \subseteq \mathbb{R}^m$ is a compact poset of parameters with standard partial order, $q : T \to ker(Z \times \mathcal{B}(Z))$, $t \mapsto q_t$, is an isotone function such that for every t, q_t is isotone, $\Gamma : X \times Z \times T \rightrightarrows X$ is the feasibility correspondence such that for every t, $\Gamma(\cdot, \cdot, t)$ satisfies conditions for a SDEwC, A the graph of Γ, $F : A \to \mathbb{R}$ the (bounded) one-period return function, $\beta \in (0, 1)$ the constant discount rate, $v : X \times Z \times T \to \mathbb{R}$ the unique value function given by $v(x, z, t) = \sup_{x' \in \Gamma(x,z,t)}\{F(x', x, z, t) + \beta \int v(x', z', t)q_t(z, dz')\}$, and $\gamma(x, z, t) = \{x' \in \Gamma(x, z, t) \mid v(x, z, t) = F(x', x, z, t) + \beta \int v(x', z', t)q_t(z, dz')\}$ the policy correspondence. The *SDEwC at t* is $(X, Z, q(t), \Gamma_t, F_t, \beta)$ defined in terms of the appropriate sections, the policies at t are measurable selections from $\gamma(\cdot, \cdot, t)$, denoted $\mathcal{G}_t$, and isotone measurable policies at t are denoted $\mathcal{G}_t^{iso}$. Let $\mathcal{G}$ denote the correspondence $t \mapsto \mathcal{G}_t$ and $\mathcal{G}^{iso}$ the correspondence $t \mapsto \mathcal{G}_t^{iso}$. The *associated parametric poset model* is $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$, where $\Phi : \mathcal{X} \times T \rightrightarrows \mathcal{X}$ is the adjoint correspondence $(\mu, t) \mapsto \Phi(\mu, t) = \{\mathcal{T}_{p_g}(\mu) \mid g \in \mathcal{G}_t\}$.

Under the natural parametric extension of the complementarity and continuity assumptions in Proposition 2 (page 1395) in Hopenhayn and Prescott (1992) and their compactness assumptions in Corollary 6 (page 1396), the functions $(x, z, t) \mapsto \underline{g}(x, z, t) := \inf \gamma(x, z, t)$ and $(x, z, t) \mapsto \overline{g}(x, z, t) := \sup \gamma(x, z, t)$ exist, are isotone, and are measurable. Therefore, correspondences $\mathcal{G}$ and $\mathcal{G}^{iso}$ are inf-isotone and sup-isotone. Moreover, from Proposition 2, it follows that for every t, $\mathcal{G}_t^{iso}$ is inf-isotone on upper intervals and sup-isotone on lower intervals. In particular, Corollary 7 (page 1396) in Hopenhayn and Prescott (1992) is the special case when $T = \{a, b\}$ with $a \prec b$.

Corollary 9. *In every parametric stochastic dynamic economy with complementarties,*

1. *For every isotone selection g from $\mathcal{G}^{iso}$,*

   (a) *The equilibrium subcorrespondence $t \mapsto \mathcal{E}(g_t)$ is nonempty valued and isotone in $*c$ relation, and*

   (b) *The infimum and supremum equilibrium selections ($t \mapsto \inf_{\mathcal{E}(g_t)} \mathcal{E}(g_t)$ and*

$t \mapsto \sup_{\mathcal{E}(g_t)} \mathcal{E}(g_t))$ *exist and are upper and lower order nearest isotone, respectively.*

2. *The isotone equilibrium correspondence* $t \mapsto \mathcal{E}(\mathcal{G}_t^{iso})$ *is nonempty valued and isotone in* $*c$ *relation.*

Proof. Follows from Proposition 2, Corollary 5, Theorem 8, and definition of parametric SDEwC. ∎

These new results for parametric SDEwC answer additional long-standing open questions and guarantee order-nearest equilibria before or after a parametric environmental change, not only in terms of optimal policies but also for the entire steady state distribution in the economy. Example 1 provides a concrete application to industry equilibrium with switching costs. These results are beyond the grasp of lattice-based theorems. Additional applications to broad classes of stochastic dynamic behavior in decision-making under uncertainty beyond the reach of lattice-based models are presented next.

4 Applications to stochastic dynamic decisions

Example 3 (Kernel systems)**.** Kernel systems provide a framework to model state-dependent decisions and serial correlation. A *kernel system* is a collection $(X, \preceq_X, \mathcal{B}(X), \mathcal{P}, \preceq_k)$, where X is a Polish space, $\preceq_X$ is a closed partial order on X, $\mathcal{B}(X)$ are the Borel sets of X, and $\mathcal{P} \subseteq ker(X \times \mathcal{B}(X))$ with kernel order $\preceq_k$.[5] When convenient, $p(x)$ denotes the measure $p(x, \cdot)$. The *associated poset model* is $(\mathcal{X}, \preceq_s, \Phi)$, where $\mathcal{X}$ is the set of probability measures on $\mathcal{B}(X)$, $\preceq_s$ is stochastic order, and Φ is the *adjoint correspondence for* $\mathcal{P}$ defined by $\Phi : \mathcal{X} \rightrightarrows \mathcal{X}$, $\Phi(\mu) = \{\mathcal{T}_p(\mu) \mid p \in \mathcal{P}\}$, where $\mathcal{T}_p$ is the adjoint of p (confer Example 2). For each $p \in \mathcal{P}$, the *p-equilibrium set* is $\mathcal{E}(p) = \{(p, \mu) \mid \mu = \mathcal{T}_p(\mu)\}$ and the *equilibrium set* (or $\mathcal{P}$*-equilibrium set*) is $\mathcal{E}(\mathcal{P}) = \{(p, \mu) \in \mathcal{P} \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$ with the product order. A kernel system may be viewed as a system of Markov processes and the equilibrium set is the collection of all processes along with their stationary distributions. We do not impose uniqueness of

[5] The set of all kernels on $X \times \mathcal{B}(Y)$ is denoted $ker(X, \mathcal{B}(Y))$ and the *kernel order* on it is defined as follows: $p \preceq_k q$ if for every $x \in X, p(x) \preceq_s q(x)$, where $\preceq_s$ is stochastic order on finite measures on Y, denoted $\mathfrak{M}(X)$.

equilibrium for any process and we impose no constraint on the mechanism by which the kernels in $\mathcal{P}$ may arise. Kernel p is *isotone*, if for every $\hat{x}, \tilde{x} \in X, \hat{x} \preceq_X \tilde{x} \Rightarrow p(\hat{x}) \preceq_s p(\tilde{x})$. The isotone kernel set is $\mathcal{P}^{iso} = \{p \in \mathcal{P} \mid p \text{ is isotone}\}$. The *isotone equilibrium set* is $\mathcal{E}(\mathcal{P}^{iso}) = \{(p, \mu) \in \mathcal{P}^{iso} \times \mathcal{X} \mid \mu = \mathcal{T}_p(\mu)\}$. We say that $\mathcal{P}$ is *isotone* if there is a kernel in $\mathcal{P}$ that is isotone, it is *inf-isotone (respectively, sup-isotone)* if $\inf_{\mathcal{P}} \mathcal{P}$ (respectively, $\sup_{\mathcal{P}} \mathcal{P}$) exists and is isotone. Lemma 10 shows connections between kernels and their adjoints.

Lemma 10. *For every $p, q \in \mathcal{P}$ and every $\mu, \nu \in \mathcal{X}$,*
(1) p is isotone and $\mu \preceq_s \nu \Rightarrow \mathcal{T}_p(\mu) \preceq_s \mathcal{T}_p(\nu)$, and (2) $p \preceq_k q \Rightarrow \mathcal{T}_p(\mu) \preceq_s \mathcal{T}_q(\mu)$.

Proof. Statement (1) holds, because for every increasing $A \in \mathcal{B}(X)$, $\mathcal{T}_p(\mu)(A) = \int_X p(x, A) d\mu(x) \leq \int_X p(x, A) d\nu(x) = \mathcal{T}_p(\nu)(A)$, where the inequality follows from p is isotone and $\mu \preceq_s \nu$. Statement (2) holds, because $\mathcal{T}_p(\mu)(A) = \int_X p(x, A) d\mu(x) \leq \int_X q(x, A) d\mu(x) = \mathcal{T}_q(\mu)(A)$, where the inequality follows from $p(x) \preceq_s q(x)$. ∎

$\mathcal{P}$ is *inf-isotone (respectively, strongly inf-isotone) on upper intervals* if $\mathcal{P}$ is inf-isotone and for every isotone kernel (respectively, kernel) $\hat{p} \in ker(X \times \mathcal{B}(X))$, the set $\hat{\mathcal{P}} = \{p \in \mathcal{P}^{iso} \mid \hat{p} \preceq_k p\}$ (respectively, $\hat{\mathcal{P}} = \{p \in \mathcal{P} \mid \hat{p} \preceq_k p\}$) is inf-isotone if it is nonempty. It is *sup-isotone (respectively, strongly sup-isotone) on lower intervals* if it is sup-isotone, and for every isotone kernel (respectively, kernel) $\hat{p} \in ker(X \times \mathcal{B}(X))$, the set $\hat{\mathcal{P}} = \{p \in \mathcal{P}^{iso} \mid p \preceq_k \hat{p}\}$ (respectively, $\hat{\mathcal{P}} = \{p \in \mathcal{P} \mid p \preceq_k \hat{p}\}$) is sup-isotone if it is nonempty. When $\mathcal{P} = \{p\}$ is a singleton, both conditions are equivalent to p is isotone. Theorem 3 translates as follows.

Proposition 11. *In every kernel sytem, for every $\mathcal{P}, \mathcal{Q} \subseteq ker(X \times \mathcal{B}(X))$, the following statements hold whenever the sets being compared are nonempty.*

(1) For every $p \in \mathcal{P}, q \in \mathcal{Q}$ with $p \preceq_k q$,

   (a) q is isotone $\Rightarrow \mathcal{E}(\mathcal{T}_p) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{T}_q)$, $\mathcal{E}(\mathcal{T}_p) \sqsubseteq^{*sc} \mathcal{E}(\Phi^{\mathcal{T}_q +})$, $\mathcal{E}(\Phi^{\mathcal{T}_p -}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{T}_q)$, and $\mathcal{E}(\Phi^{\mathcal{T}_p -}) \sqsubseteq^{*sc} \mathcal{E}(\Phi^{\mathcal{T}_q +})$.

   (b) If $\inf_X X$ exists and p is isotone, then for every $\mu \in \mathcal{E}(\mathcal{T}_q)$, $\inf_{\mathcal{E}(\mathcal{T}_p)} \mathcal{E}(\mathcal{T}_p) \preceq_s \mu$.

   (c) p is isotone $\Rightarrow \mathcal{E}(\mathcal{T}_p) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{T}_q)$, $\mathcal{E}(\mathcal{T}_p) \sqsubseteq^{*ic} \mathcal{E}(\Phi^{\mathcal{T}_q +})$, $\mathcal{E}(\Phi^{\mathcal{T}_p -}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{T}_q)$, and $\mathcal{E}(\Phi^{\mathcal{T}_p -}) \sqsubseteq^{*ic} \mathcal{E}(\Phi^{\mathcal{T}_q +})$.

(d) If $\sup_X X$ *exists and* q *is isotone, then for every* $\mu \in \mathcal{E}(\mathcal{T}_p)$, $\mu \preceq_s \sup_{\mathcal{E}(\mathcal{T}_q)} \mathcal{E}(\mathcal{T}_q)$*.*

(2) (a) If $\mathcal{P} \sqsubseteq^{uw} \mathcal{Q}$, $\mathcal{Q}$ *is sup-isotone and is inf-isotone (respectively, strongly inf-isotone) on upper intervals, then* $\mathcal{E}(\mathcal{P}^{iso}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{Q}^{iso})$ *(respectively,* $\mathcal{E}(\mathcal{P}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{Q})$*).*

(b) If $\mathcal{Q} \sqsubseteq^{lw} \mathcal{P}$, $\mathcal{Q}$ *is inf-isotone and is sup-isotone (respectively, strongly sup-isotone) on lower intervals, then* $\mathcal{E}(\mathcal{Q}^{iso}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{P}^{iso})$ *(respectively,* $\mathcal{E}(\mathcal{Q}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{P})$*).*

Proof. Statement (1) is proved in a manner similar to the corresponding statement in Theorem 3, using Lemma 10 and the fact that $\mathcal{X}$ is chain complete. For (2)(a), consider nonempty $E \subseteq \mathcal{E}(\mathcal{P}^{iso})$ and suppose $(\hat{p}, \hat{\mu}) = \sup_{\mathcal{K}\times\mathcal{X}} E$ exists, where $\mathcal{K} = ker(X \times \mathcal{B}(X))$. Then it can be checked that $\hat{p} = \sup_{\mathcal{K}} E_1$, where $E_1 = \{p \in \mathcal{P} \mid \exists \mu \in \mathcal{X}, (p, \mu) \in E\}$, and $\hat{\mu} = \sup_{\mathcal{X}} E_2$, where $E_2 = \{\mu \in \mathcal{X} \mid \exists p \in \mathcal{P}, (p, \mu) \in E\}$. Moreover, $\hat{p}$ is isotone because for every $p \in E_1$, p is isotone. Using $\mathcal{P} \sqsubseteq^{uw} \mathcal{Q}$ and letting $\overline{q} = \sup_{\mathcal{Q}} \mathcal{Q}$, it follows that $\hat{p} \preceq_k \overline{q}$. Using $\mathcal{Q}$ is inf-isotone on upper intervals, let $\hat{q}$ be the isotone infimum for $\hat{\mathcal{Q}} = \{q \in \mathcal{Q}^{iso} \mid \hat{p} \preceq_k q\}$. Let $\hat{\nu}$ be the smallest fixed point of $\mathcal{T}_{\hat{q}}$ on $\{\mu \in \mathcal{X} \mid \hat{\mu} \preceq_s \mu\}$. Then $(\hat{p}, \hat{\mu}) \preceq (\hat{q}, \hat{\nu})$ and for every $(q, \nu) \in \mathcal{E}(\mathcal{Q}^{iso})$ that is an upper bound of E, $(\hat{p}, \hat{\mu}) \preceq (q, \nu)$ proving that $(\hat{q}, \hat{\nu}) = \inf_{\mathcal{E}(\mathcal{Q}^{iso})} E$. The corresponding statement for the full equilibrium set, $\mathcal{E}(\mathcal{P}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{Q})$, is proved similarly. Statement (2)(b) is proved similarly, using the dual argument. ∎

Corollary 12. *For every* $\mathcal{P} \subseteq ker(X \times \mathcal{B}(X))$*, the following hold whenever the sets being compared are nonempty.*

(1) For every isotone $p, q \in \mathcal{P}$, $p \preceq_k q \Rightarrow \mathcal{E}(p) \sqsubseteq^{*c} \mathcal{E}(q)$*.*

(2) If $\mathcal{P}$ *is inf-isotone (respectively, sup-isotone) then* $\mathcal{E}(\underline{p}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{P})$ *(respectively,* $\mathcal{E}(\mathcal{P}) \sqsubseteq^{*sc} \mathcal{E}(\overline{p})$*).*

(3) If $\mathcal{P}^{iso}$ *is sup-isotone and is inf-isotone on upper intervals (respectively, inf-isotone and is sup-isotone on lower intervals), then* $\mathcal{E}(\mathcal{P}^{iso}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{P}^{iso})$ *(respectively,* $\mathcal{E}(\mathcal{P}^{iso}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{P}^{iso})$*).*

(4) If $\mathcal{P}$ is sup-isotone and is strongly inf-isotone on upper intervals (respectively, inf-isotone and is strongly sup-isotone on lower intervals), then $\mathcal{E}(\mathcal{P}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{P})$ (respectively, $\mathcal{E}(\mathcal{P}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{P})$).

Proof. Follows from Proposition 11. ∎

A *parametric kernel system* is a collection $((X, \preceq_X, \mathcal{B}(X)), (T, \preceq_T), \mathcal{P})$, where $\mathcal{P}$ is a correspondence $\mathcal{P} : T \rightrightarrows ker(X \times \mathcal{B}(X))$, $t \mapsto \mathcal{P}_t$, and for every $t \in T$, $((X, \preceq_X, \mathcal{B}(X)), \mathcal{P}_t)$ is a kernel system. The *associated parametric poset model* is $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$, where $\Phi : \mathcal{X} \times T \rightrightarrows \mathcal{X}$ is the adjoint correspondence $(\mu, t) \mapsto \Phi(\mu, t) = \{\mathcal{T}_p(\mu) \mid p \in \mathcal{P}_t\}$. $\mathcal{P}$ is *sup-isotone* (respectively, *inf-isotone*), if it has an isotone selection $t \mapsto p_t$ such that for every t, $p_t = \sup_{\mathcal{P}_t} \mathcal{P}_t$ (respectively, $p_t = \inf_{\mathcal{P}_t} \mathcal{P}_t$) and p_t is isotone. MCS of equilibrium subcorrespondences follows.

Proposition 13. *For every poset X and T in which X is a Polish space and $\inf_X X$ (respectively, $\sup_X X$) exists, for every $\mathcal{P}, \mathcal{Q} : T \rightrightarrows \mathcal{K}$,*

1. *For every isotone selection p from $\mathcal{P}^{iso}$,*
   - *(a) The equilibrium subcorrespondence $t \mapsto \mathcal{E}(p_t)$ is nonempty valued and isotone in $*c$ relation,*
   - *(b) The infimum (respectively, supremum) equilibrium selection, $t \mapsto \inf_{\mathcal{E}(p_t)} \mathcal{E}(p_t)$ (respectively, $t \mapsto \sup_{\mathcal{E}(p_t)} \mathcal{E}(p_t)$), exists and is upper (respectively, lower) order nearest isotone,*
   - *(c) For every selection q from $\mathcal{Q}$, $q \preceq p$ (respectively, $p \preceq q$) $\Rightarrow$ for every t, $\mathcal{E}(q_t) \sqsubseteq^{*sc} \mathcal{E}(p_t)$ (respectively, $\mathcal{E}(p_t) \sqsubseteq^{*ic} \mathcal{E}(q_t)$), whenever $\mathcal{E}(q_t)$ is nonempty.*
2. *If $\mathcal{P}^{iso}$ is sup-isotone (respectively, inf-isotone) and for every t, $\mathcal{P}^{iso}_t$ is inf-isotone on upper intervals (respectively, sup-isotone on lower intervals),*
   - *(a) The isotone equilibrium correspondence, $t \mapsto \mathcal{E}(\mathcal{P}^{iso}_t)$ is nonempty valued and isotone in $*sc$ (respectively, $*ic$) relation,*
   - *(b) For every t, $\mathcal{Q}^{iso}_t \sqsubseteq^{uw} \mathcal{P}^{iso}_t$ (respectively, $\mathcal{P}^{iso}_t \sqsubseteq^{lw} \mathcal{Q}^{iso}_t$) $\Rightarrow \mathcal{E}(\mathcal{Q}^{iso}_t) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{P}^{iso}_t)$ (respectively, $\mathcal{E}(\mathcal{P}^{iso}_t) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{Q}^{iso}_t)$), whenever $\mathcal{E}(\mathcal{Q}^{iso}_t)$ is nonempty.*

3. *If $\mathcal{P}$ is sup-isotone (respectively, inf-isotone) and for every t, $\mathcal{P}_t$ is strongly inf-isotone on upper intervals (respectively, strongly sup-isotone on lower intervals),*

   (a) *The full equilibrium correspondence, $t \mapsto \mathcal{E}(\mathcal{P}_t)$ is nonempty valued and isotone in $*sc$ (respectively, $*ic$) relation,*

   (b) *For every t, $\mathcal{Q}_t \sqsubseteq^{uw} \mathcal{P}_t$ (respectively, $\mathcal{P}_t \sqsubseteq^{lw} \mathcal{Q}_t$) $\Rightarrow \mathcal{E}(\mathcal{Q}_t) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{P}_t)$ (respectively, $\mathcal{E}(\mathcal{P}_t) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{Q}_t)$), whenever $\mathcal{E}(\mathcal{Q}_t)$ is nonempty.*

Proof. To prove (1) for the case $\inf_X X$ exists, notice that for every t, $\mathcal{E}(p_t)$ is nonempty because Lemma 10 and Lemma 1 imply that $\inf_{\mathcal{E}(p_t)} \mathcal{E}(p_t)$ exists. Moreover, $t \mapsto \mathcal{E}(p_t)$ is isotone in $*c$ relation, $t \mapsto \inf_{\mathcal{E}(p_t)} \mathcal{E}(p_t)$ is upper order nearest isotone, and $\mathcal{E}(q_t) \sqsubseteq^{*sc} \mathcal{E}(p_t)$ follow from statement (1) in Proposition 11. To prove (2)(a), $\mathcal{P}^{iso}$ is sup-isotone implies that for every $\hat{t} \preceq_T \tilde{t}$, $\mathcal{P}^{iso}_{\hat{t}} \sqsubseteq^{uw} \mathcal{P}^{iso}_{\tilde{t}}$ and moreover, $\mathcal{P}^{iso}_{\tilde{t}}$ is inf-isotone on upper intervals, and therefore, Proposition 11 implies that $\mathcal{E}(\mathcal{P}^{iso}_{\hat{t}}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{P}^{iso}_{\tilde{t}})$. Statement $\mathcal{E}(\mathcal{P}^{iso}_{\hat{t}}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{P}^{iso}_{\tilde{t}})$ is proved similarly. Statements (2)(b) and (3) are proved similarly. ∎

Example 4 (Stochastic dynamical systems)**.** In addition to economic systems, stochastic dynamics model many stochastically evolving phenomena including physical, human, animal, societal, and socioeconomic. A *stochastic dynamical system* is a collection $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$, where S is a Polish space, $\preceq_S$ is a closed partial order on S, $\mathcal{B}(S)$ are Borel sets of S, Z is a Polish space, $\preceq_Z$ is a closed partial order on Z, $\mathcal{B}(Z)$ are Borel sets of Z, q is an isotone kernel on $S \times \mathcal{B}(Z)$, and $\mathcal{G} \subseteq mbl(S \times Z, S)$ with pointwise partial order. We view S as the state space, Z as the set of possible realizations of exogenous shocks to the system (or exogenous uncertainty), q as the state dependent distribution of exogenous shocks, and $\mathcal{G}$ as a set of policies or update rules. The dynamical system evolves as follows. For a given policy $g \in \mathcal{G}$, in period t, if system state is s_t and system shock is z_t, the state next period is $s_{t+1} = g(s_t, z_t)$, where shocks are state dependent and governed by $q(s_t, \cdot)$. Each $g \in \mathcal{G}$ defines a kernel $p_g(s, A) = q(s, [g^{-1}(A)]_s)$, where $[g^{-1}(A)]_s$ is the s-section of $g^{-1}(A)$ and a corresponding adjoint $\mathcal{T}_{p_g}$. An equilibrium is a pair (g, μ), where $g \in \mathcal{G}$ and μ is a steady state distribution for kernel p_g, given by $\mu = \mathcal{T}_{p_g}(\mu)$. For each $g \in \mathcal{G}$, the *g-equilibrium set* is $\mathcal{E}(g) = \{(g, \mu) \mid \mu = \mathcal{T}_{p_g}(\mu)\}$ and the *equilibrium set* (or *$\mathcal{G}$-equilibrium set*) is $\mathcal{E}(\mathcal{G}) = \{(g, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{p_g}(\mu)\}$

with the product order. Equilibrium (g, μ) is *isotone* if g is isotone, and the *isotone equilibrium set* is $\mathcal{E}(\mathcal{G}^{iso}) = \{(g, \mu) \in \mathcal{E}(\mathcal{G}) \mid g \text{ is isotone}\}$. We impose no constraint on the mechanism by which the policies in $\mathcal{G}$ may arise. The underlying behavior can be the solution to an optimization problem, but we do not require that in the general case. Anything that is an accurate description of the situation being studied is permissible as long as it satisfies our weak conditions. $\mathcal{G}$ is *isotone* if there is $g \in \mathcal{G}$ that is isotone, it is *inf-isotone (respectively, sup-isotone)* if $\inf_{\mathcal{G}} \mathcal{G}$ (respectively, $\sup_{\mathcal{G}} \mathcal{G}$) exists and is isotone. Lemma 14 shows connections between policies and the kernels they generate.

Stochastic dynamic economies and Markov decision processes are two large special cases that can be analyzed with our techniques. A *stochastic dynamic economy* is a collection $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$, where $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$ are as above and q is an isotone kernel on $Z \times \mathcal{B}(Z)$. In dynamic macro models (e.g. Stokey and Lucas (1989)), the state space is $S = X \times Z$, and given state $s_t = (x_t, z_t)$ in period t and serially correlated shock z_{t+1} governed by $q(z_t, \cdot)$, the state next period is $s_{t+1} = (g(x_t, z_t), z_{t+1})$. The *associated stochastic dynamical system* is $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \tilde{\mathcal{G}})$, where $S = X \times Z$, and $\tilde{\mathcal{G}}$ are extensions of policies $g \in \mathcal{G}$ given by $\tilde{g} : X \times Z \times Z \to X \times Z$, $\tilde{g}(x, z, z') = (g(x, z), z')$. The equilibrium set is $\mathcal{E}(\mathcal{G}) = \{(g, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{p_{\tilde{g}}(\mu)}\}$. SDEwC (Example 2) are a special case. A *Markov decision process* is a collection $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \mathcal{G})$, where $((X, \preceq_X, \mathcal{B}(X)), (Z, \preceq_Z, \mathcal{B}(Z)), \mathcal{G})$ are as above, the state space is $S = X \times Z$ and q is an isotone kernel on $S \times \mathcal{B}(Z)$. Shocks are simultaneously state dependent and serially correlated. The *associated stochastic dynamical system* is $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q, \tilde{\mathcal{G}})$, where $S = X \times Z$, and $\tilde{\mathcal{G}}$ are extensions of policies in $g \in \mathcal{G}$ given by $\tilde{g} : X \times Z \times Z \to X \times Z$, $\tilde{g}(x, z, z') = (g(x, z), z')$. The equilibrium set is $\mathcal{E}(\mathcal{G}) = \{(g, \mu) \in \mathcal{G} \times \mathcal{X} \mid \mu = \mathcal{T}_{p_{\tilde{g}}(\mu)}\}$.

Lemma 14. *In every stochastic dynamical system, for every $g, h \in \mathcal{G}$,*
(1) g is isotone $\Rightarrow p_g$ is isotone, and (2) $g \preceq h \Rightarrow p_g \preceq_k p_h$.

Proof. Statement (1) holds, because for every $\hat{s} \preceq_S \tilde{s}$ and for every increasing $A \in \mathcal{B}(S)$, $p_g(\hat{s}, A) = q(\hat{s}, [g^{-1}(A)]_{\hat{s}}) \leq q(\hat{s}, [g^{-1}(A)]_{\tilde{s}}) \leq q(\tilde{s}, [g^{-1}(A)]_{\tilde{s}}) = p_g(\tilde{s}, A)$, where the first inequality follows from g is isotone and A is increasing, and the second inequality follows from q is isotone and A is increasing. Statement (2) holds, because for every $s \in S$ and for every increasing $A \in \mathcal{B}(S)$, $p_g(s, A) = q(s, [g^{-1}(A)]_s) \leq q(s, [h^{-1}(A)]_s) = p_h(s, A)$, where the inequality follows from $g \preceq h$ and A is increasing. ∎

$\mathcal{G}$ is *inf-isotone (respectively, strongly inf-isotone) on upper intervals* if $\mathcal{G}$ is inf-isotone and for every isotone function (respectively, function) $\hat{g} \in mbl(X \times Z, X)$, the set $\hat{\mathcal{G}} = \{g \in \mathcal{G}^{iso} \mid \hat{g} \preceq g\}$ (respectively, $\hat{\mathcal{G}} = \{g \in \mathcal{G} \mid \hat{g} \preceq g\}$) is inf-isotone if it is nonempty. It is *sup-isotone (respectively, strongly sup-isotone) on lower intervals* if it is sup-isotone, and for every isotone function (respectively, function) $\hat{g} \in mbl(X \times Z, X)$, the set $\hat{\mathcal{G}} = \{g \in \mathcal{G}^{iso} \mid g \preceq \hat{g}\}$ (respectively, $\hat{\mathcal{G}} = \{g \in \mathcal{G} \mid g \preceq \hat{g}\}$) is sup-isotone if it is nonempty. When $\mathcal{G} = \{g\}$ is a singleton, all four conditions are equivalent to g is isotone. Theorem 3 translates as follows.

Proposition 15. *In every stochastic dynamical system, for every $\mathcal{F}, \mathcal{G} \subseteq mbl(S \times Z, S)$, the following statements hold whenever the sets being compared are nonempty.*

(1) For every $f \in \mathcal{F}, g \in \mathcal{G}$ with $f \preceq g$,

- *(a) g is isotone $\Rightarrow$ $\mathcal{E}(\mathcal{T}_{p_f}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{T}_{p_g})$, $\mathcal{E}(\mathcal{T}_{p_f}) \sqsubseteq^{*sc} \mathcal{E}(\Phi^{\mathcal{T}_{p_g}+})$, $\mathcal{E}(\Phi^{\mathcal{T}_{p_f}-}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{T}_{p_g})$, and $\mathcal{E}(\Phi^{\mathcal{T}_{p_f}-}) \sqsubseteq^{*sc} \mathcal{E}(\Phi^{\mathcal{T}_{p_g}+})$.*
- *(b) If $\inf_X X$ exists and f is isotone, then for every $\mu \in \mathcal{E}(\mathcal{T}_{p_g})$, $\inf_{\mathcal{E}(\mathcal{T}_{p_f})} \mathcal{E}(\mathcal{T}_{p_f}) \preceq_s \mu$.*
- *(c) f is isotone $\Rightarrow$ $\mathcal{E}(\mathcal{T}_{p_f}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{T}_{p_g})$, $\mathcal{E}(\mathcal{T}_{p_f}) \sqsubseteq^{*ic} \mathcal{E}(\Phi^{\mathcal{T}_{p_g}+})$, $\mathcal{E}(\Phi^{\mathcal{T}_{p_f}-}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{T}_{p_g})$, and $\mathcal{E}(\Phi^{\mathcal{T}_{p_f}-}) \sqsubseteq^{*ic} \mathcal{E}(\Phi^{\mathcal{T}_{p_g}+})$.*
- *(d) If $\sup_X X$ exists and g is isotone, then for every $\mu \in \mathcal{E}(\mathcal{T}_{p_f})$, $\mu \preceq_s \sup_{\mathcal{E}(\mathcal{T}_{p_g})} \mathcal{E}(\mathcal{T}_{p_g})$.*

(2)

- *(a) If $\mathcal{F} \sqsubseteq^{uw} \mathcal{G}$, $\mathcal{G}$ is sup-isotone and is inf-isotone (respectively, strongly inf-isotone) on upper intervals, then $\mathcal{E}(\mathcal{F}^{iso}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{G}^{iso})$ (respectively, $\mathcal{E}(\mathcal{F}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{G})$).*
- *(b) If $\mathcal{G} \sqsubseteq^{lw} \mathcal{F}$, $\mathcal{G}$ is inf-isotone and is sup-isotone (respectively, strongly sup-isotone) on lower intervals, then $\mathcal{E}(\mathcal{G}^{iso}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{F}^{iso})$ (respectively, $\mathcal{E}(\mathcal{G}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{F})$).*

Proof. Similar to proof of Proposition 11, using Lemma 14. ∎

Corollary 16. *For every $\mathcal{G} \subseteq mbl(S \times Z, S)$, the following hold whenever the sets being compared are nonempty.*

(1) *For every isotone* $f, g \in \mathcal{G}$, $f \preceq g \Rightarrow \mathcal{E}(f) \sqsubseteq^{*c} \mathcal{E}(g)$.

(2) *If* $\mathcal{G}$ *is inf-isotone (respectively, sup-isotone) then* $\mathcal{E}(\underline{g}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{G})$ *(respectively,* $\mathcal{E}(\mathcal{G}) \sqsubseteq^{*sc} \mathcal{E}(\overline{g})$*).*

(3) *If* $\mathcal{G}^{iso}$ *is sup-isotone and is inf-isotone on upper intervals (respectively, inf-isotone and is sup-isotone on lower intervals), then* $\mathcal{E}(\mathcal{G}^{iso}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{G}^{iso})$ *(respectively,* $\mathcal{E}(\mathcal{G}^{iso}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{G}^{iso})$*).*

(4) *If* $\mathcal{G}$ *is sup-isotone and is strongly inf-isotone on upper intervals (respectively, inf-isotone and is strongly sup-isotone on lower intervals), then* $\mathcal{E}(\mathcal{G}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{G})$ *(respectively,* $\mathcal{E}(\mathcal{G}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{G})$*).*

Proof. Follows from Proposition 15. ∎

A *parametric stochastic dynamical system* is a collection $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), (T, \preceq_T), q, \mathcal{G})$, where $q : T \to ker(S \times \mathcal{B}(Z))$, $t \mapsto q_t$, and $\mathcal{G} : T \rightrightarrows mbl(S \times Z, S)$, $t \mapsto \mathcal{G}_t$, and for every $t \in T$, $((S, \preceq_S, \mathcal{B}(S)), (Z, \preceq_Z, \mathcal{B}(Z)), q_t, \mathcal{G}_t)$ is a stochastic dynamical system. The *associated parametric poset model* is $((\mathcal{X}, \preceq_s), (T, \preceq_T), \Phi)$, where $\Phi : \mathcal{X} \times T \rightrightarrows \mathcal{X}$ is the adjoint correspondence $(\mu, t) \mapsto \Phi(\mu, t) = \{\mathcal{T}_g(\mu) \mid g \in \mathcal{G}_t\}$. $\mathcal{G}$ is *sup-isotone* (respectively, *inf-isotone*), if it has an isotone selection $t \mapsto g_t$ such that for every t, $g_t = \sup_{\mathcal{G}_t} \mathcal{G}_t$ (respectively, $g_t = \inf_{\mathcal{G}_t} \mathcal{G}_t$) and g_t is isotone. MCS of equilibrium subcorrespondences follows.

Proposition 17. *In every parametric stochastic dynamical system, in which* $\inf_S S$ *(respectively,* $\sup_S S$*) exists, for every* $\mathcal{F}, \mathcal{G} : T \rightrightarrows mbl(S \times Z, S)$*,*

(1) *For every isotone selection* g *from* $\mathcal{G}^{iso}$*,*

   (a) *The equilibrium subcorrespondence* $t \mapsto \mathcal{E}(g_t)$ *is nonempty valued and isotone in* $*c$ *relation,*

   (b) *The infimum (respectively, supremum) equilibrium selection,* $t \mapsto \inf_{\mathcal{E}(g_t)} \mathcal{E}(g_t)$ *(respectively,* $t \mapsto \sup_{\mathcal{E}(g_t)} \mathcal{E}(g_t)$*), exists and is upper (respectively, lower) order nearest isotone*

   (c) *For every selection* f *from* $\mathcal{F}$, $f \preceq g$ *(respectively,* $g \preceq f$*)* $\Rightarrow$ *for every* t, $\mathcal{E}(f_t) \sqsubseteq^{*sc} \mathcal{E}(g_t)$ *(respectively,* $\mathcal{E}(g_t) \sqsubseteq^{*ic} \mathcal{E}(f_t)$*), whenever* $\mathcal{E}(f_t)$ *is nonempty.*

(2) *If $\mathcal{G}^{iso}$ is sup-isotone (respectively, inf-isotone) and for every t, $\mathcal{G}_t^{iso}$ is inf-isotone on upper intervals (respectively, sup-isotone on lower intervals),*

(a) *The isotone equilibrium correspondence, $t \mapsto \mathcal{E}(\mathcal{G}_t^{iso})$ is nonempty valued and isotone in $*sc$ (respectively, $*ic$) relation,*

(b) *For every t, $\mathcal{F}_t^{iso} \sqsubseteq^{uw} \mathcal{G}_t^{iso}$ (respectively, $\mathcal{G}_t^{iso} \sqsubseteq^{lw} \mathcal{F}_t^{iso}$) $\Rightarrow \mathcal{E}(\mathcal{F}_t^{iso}) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{G}_t^{iso})$ (respectively, $\mathcal{E}(\mathcal{G}_t^{iso}) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{F}_t^{iso})$), whenever $\mathcal{E}(\mathcal{F}_t^{iso})$ is nonempty.*

(3) *If $\mathcal{G}$ is sup-isotone (respectively, inf-isotone) and for every t, $\mathcal{G}_t$ is strongly inf-isotone on upper intervals (respectively, strongly sup-isotone on lower intervals),*

(a) *The full equilibrium correspondence, $t \mapsto \mathcal{E}(\mathcal{G}_t)$ is nonempty valued and isotone in $*sc$ (respectively, $*ic$) relation,*

(b) *For every t, $\mathcal{F}_t \sqsubseteq^{uw} \mathcal{G}_t$ (respectively, $\mathcal{G}_t \sqsubseteq^{lw} \mathcal{F}_t$) $\Rightarrow \mathcal{E}(\mathcal{F}_t) \sqsubseteq^{*sc} \mathcal{E}(\mathcal{G}_t)$ (respectively, $\mathcal{E}(\mathcal{G}_t) \sqsubseteq^{*ic} \mathcal{E}(\mathcal{F}_t)$), whenever $\mathcal{E}(\mathcal{F}_t)$ is nonempty.*

Proof. Similar to proof of Proposition 13. ∎

5 Applications to Bayesian games

Applications to Bayesian games over lattices are presented in this section.

Example 5 (Bayesian games with complementarities (BGwC))**.** Bayesian games are a flexible model to study a variety of situations with incomplete information. Following Vives (1990) and Van Zandt and Vives (2007), we define a *Bayesian game with complementarities (BGwC)* as a finitely indexed collection $((A_i, \preceq_{X_i}, \Theta_i, \preceq_{\Theta_i}, \mathfrak{T}_i, u_i)_{i=1}^I, \mu)$, where for each i, $(A_i, \preceq_{A_i}) \subseteq \mathbb{R}^{n_i}$ is a nonempty, compact sublattice of actions for i with Borel sigma-algebra, $(\Theta_i, \preceq_{\Theta_i}, \mathfrak{T}_i)$ is a partially ordered type space for i with sigma-algebra $\mathfrak{T}_i$, $A = \times_{i=1}^I A_i$ is the joint action space, $\Theta = \times_{i=0}^I \Theta_i$ is the joint type space where Θ_0 captures residual uncertainty, $u_i : A \times \Theta \to \mathbb{R}$ is the payoff function, and μ is a common prior on Θ. For every i, $\theta_i \mapsto \mu_{-i}(\cdot \mid \theta_i)$ is the measurable conditional belief mapping of player i on types of others Θ_{-i} and μ_i is the marginal on Θ_i. Product spaces are endowed with product order, topology, and sigma-algebra. Strategies for i are measurable functions $\sigma_i : \Theta_i \to A_i$, with $\mathcal{S}_i = \{[\sigma_i] \mid \sigma_i \text{ is measurable}\}$

the set of all strategies, where $[\sigma_i]$ is the equivalence class of σ_i mod μ_i, and similarly, $\mathcal{S}_i^{iso} = \{[\sigma_i] \in \mathcal{S}_i \mid \sigma_i \text{ is isotone}\}$ the set of all isotone strategies mod μ_i, each with pointwise partial order mod μ_i. When convenient, we write $\sigma_i \in \mathcal{S}_i$ instead of $[\sigma_i] \in \mathcal{S}_i$. Let $\mathcal{S} = \times_{i=1}^{I} \mathcal{S}_i$ be the joint strategy space with product partial order $\preceq$. As shown in Vives (1990) and Van Zandt and Vives (2007), $\mathcal{S}_i$ and $\mathcal{S}$ are complete lattices. Let $\mathcal{U}_i(a_i, \theta_i, \sigma_{-i}) = \int_{\Theta_{-i}} u_i(a_i, \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i}) d\mu_{-i}(\theta_{-i} \mid \theta_i)$ and notice that it does not depend on the version of σ_{-i} used, let $\phi_i(\theta_i, \sigma_{-i}) = \arg\max_{A_i} \mathcal{U}_i(a_i, \theta_i, \sigma_{-i})$, let $\Phi_i : \mathcal{S}_{-i} \rightrightarrows \mathcal{S}_i$ be i's best response correspondence, where $\Phi_i(\sigma_{-i})$ is the set of all measurable selections from $\phi_i(\cdot, \sigma_{-i})$ mod μ_i, and let $\Phi : \mathcal{S} \rightrightarrows \mathcal{S}$ be the joint best response correspondence, $\Phi(\sigma) = \times_{i=1}^{I} \Phi_i(\sigma_{-i})$. The associated poset model is $(\mathcal{S}, \preceq, \Phi)$. An *equilibrium* or *Bayesian Nash equilibrium* is a fixed point of Φ and the *equilibrium set* is $\mathcal{E}(\Phi)$. Under their assumptions about continuity, compactness, and measurability, and their complementarity assumptions (each u_i is supermodular in a_i, has increasing differences in (a_i, a_{-i}), has increasing differences in (a_i, θ), and each belief mapping μ_{-i} is isotone in stochastic order), Van Zandt and Vives (2007) show that for every i and every (θ_i, σ_{-i}), $\inf_{A_i} \phi_i(\theta_i, \sigma_{-i})$ exists, is isotone in σ_{-i} and is measurable in θ_i, whence Φ_i is inf-isotone with $\underline{\Phi}_i(\sigma_{-i})(\theta_i) = \inf_{A_i} \phi_i(\theta_i, \sigma_{-i})$ mod μ_i, and therefore, Φ is inf-isotone. Similarly, Φ_i and Φ are sup-isotone.

For each player i, we define i's isotone best response correspondence as $\Phi_i^{iso} : \mathcal{S}_{-i}^{iso} \rightrightarrows \mathcal{S}_i^{iso}$ by $\Phi_i^{iso}(\sigma_{-i}) = \Phi_i(\sigma_{-i}) \cap \mathcal{S}_i^{iso}$, and the isotone joint best response correspondence $\Phi^{iso} : \mathcal{S}^{iso} \rightrightarrows \mathcal{S}^{iso}$ by $\Phi^{iso}(\sigma) = \times_{i=1}^{I} \Phi_i^{iso}(\sigma_{-i})$. It follows that $\mathcal{S}_i^{iso}$ and $\mathcal{S}^{iso}$ are complete lattices. The associated poset model is $(\mathcal{S}^{iso}, \preceq, \Phi^{iso})$. The *isotone equilibrium set* is $\mathcal{E}(\Phi^{iso})$. Equivalently, $\mathcal{E}(\Phi^{iso}) = \{\sigma \in \mathcal{E}(\Phi) \mid \forall i, \sigma_i \text{ is isotone}\}$. It can be checked that for every isotone σ_{-i}, $\mathcal{U}_i(a_i, \theta_i, \sigma_{-i})$ has increasing differences in (a_i, θ_i), and therefore, $\theta_i \mapsto \underline{\Phi}_i(\sigma_{-i})(\theta_i)$ is isotone in θ_i, that is, $\underline{\Phi}_i(\sigma_{-i}) \in \mathcal{S}_i^{iso}$, and similarly, for every isotone σ_{-i}, $\overline{\Phi}_i(\sigma_{-i}) \in \mathcal{S}_i^{iso}$. Therefore, Φ_i^{iso} and Φ^{iso} are inf-isotone and sup-isotone.

Proposition 18. *In every Bayesian game with complementarities,* Φ *and* Φ^{iso} *are inf-isotone on upper intervals and sup-isotone on lower intervals.*

Proof. To show that Φ is inf-isotone on upper intervals, we already know that Φ is inf-isotone. Let $\hat{\sigma} = (\hat{\sigma}_i)$ be a profile of strategies such that $\Phi(\hat{\sigma}) \cap \hat{\mathcal{S}} \neq \emptyset$, where $\hat{\mathcal{S}} = \{\sigma \in \mathcal{S} \mid \hat{\sigma} \preceq \sigma\}$. For every i, let $\hat{\mathcal{S}}_i = \{\sigma_i \in \mathcal{S}_i \mid \hat{\sigma}_i \preceq \sigma_i\}$. As $\Phi(\hat{\sigma}) \cap \hat{\mathcal{S}} \neq \emptyset$ and Φ is sup-isotone, it

follows that for every i, $\hat{\sigma}_i \preceq \overline{\Phi}_i(\hat{\sigma}_{-i})$ and therefore, for every $\sigma_{-i} \in \hat{\mathcal{S}}_{-i}$, $\Phi_i(\sigma_{-i}) \cap \hat{\mathcal{S}}_i \neq \emptyset$. For every $\sigma_{-i} \in \hat{\mathcal{S}}_{-i}$ and for every θ_i, let $\psi_i(\theta_i, \sigma_{-i}) = \arg\max_{a_i \in [\hat{\sigma}_i(\theta_i), \overline{a}_i]} \mathcal{U}_i(a_i, \theta_i, \sigma_{-i})$. As $\mathcal{U}_i$ has increasing differences in (a_i, σ_{-i}), it follows that $\sigma_{-i} \mapsto \psi_i(\theta_i, \sigma_{-i})$ is inf-isotone and sup-isotone, and $\theta_i \mapsto \inf \psi_i(\theta_i, \sigma_{-i})$ and $\theta_i \mapsto \sup \psi_i(\theta_i, \sigma_{-i})$ are measurable. Moreover, for every θ_i and every $\sigma_{-i} \in \hat{\mathcal{S}}_{-i}$, $\phi_i(\theta_i, \sigma_{-i}) \cap [\hat{\sigma}_i(\theta_i), \overline{a}_i] = \psi_i(\theta_i, \sigma_{-i})$, as follows. It is clear that for every $a_i \in \phi_i(\theta_i, \sigma_{-i}) \cap [\hat{\sigma}_i(\theta_i), \overline{a}_i]$, $a_i \in \psi_i(\theta_i, \sigma_{-i})$. Let $a_i \in \psi_i(\theta_i, \sigma_{-i})$. Then $a_i \in [\hat{\sigma}_i(\theta_i), \overline{a}_i]$. Moreover, Φ is sup-isotone implies that $\hat{\sigma}_i(\theta_i) \preceq \overline{\Phi}_i(\sigma_{-i})(\theta_i)$, and therefore, $\overline{\Phi}_i(\sigma_{-i})(\theta_i) \in [\hat{\sigma}_i(\theta_i), \overline{a}_i]$, whence $a_i \in \phi_i(\theta_i, \sigma_{-i})$, and consequently, $a_i \in \phi_i(\theta_i, \sigma_{-i}) \cap [\hat{\sigma}_i(\theta_i), \overline{a}_i]$. Let $\Psi_i(\sigma_{-i})$ be measurable selections from $\psi_i(\cdot, \sigma_{-i})$ mod μ_i and $\Psi_i : \hat{\mathcal{S}}_{-i} \rightrightarrows \hat{\mathcal{S}}_i$ be given by $\sigma_{-i} \mapsto \Psi_i(\sigma_{-i})$. Then Ψ_i is inf-isotone and $\Psi_i(\sigma_{-i}) = \Phi_i(\sigma_{-i}) \cap \hat{\mathcal{S}}_i$. It follows that Φ is inf-isotone on upper intervals. Similarly, Φ is sup-isotone on lower intervals. Φ^{iso} is inf-isotone on upper intervals and sup-isotone on lower intervals is shown similarly, using the additional facts that $\hat{\sigma}_i$ is isotone implies that $\theta_i \mapsto [\hat{\sigma}_i(\theta_i), \overline{a}_i]$ is isotone in SSO, and therefore, $\theta_i \mapsto \inf \psi_i(\theta_i, \sigma_{-i})$ and $\theta_i \mapsto \sup \psi_i(\theta_i, \sigma_{-i})$ are isotone.
▮

Corollary 19. *In every Bayesian game with complementarities,*

(1) *For every isotone selection f, g from Φ, $f \preceq g \Rightarrow \mathcal{E}(f) \sqsubseteq^{*c} \mathcal{E}(g)$.*

(2) *$\mathcal{E}(\underline{\Phi}) \sqsubseteq^{*c} \mathcal{E}(\Phi)$ and $\mathcal{E}(\Phi) \sqsubseteq^{*c} \mathcal{E}(\overline{\Phi})$.*

(3) *$\mathcal{E}(\Phi) \sqsubseteq^{*c} \mathcal{E}(\Phi)$ and $\mathcal{E}(\Phi^{iso}) \sqsubseteq^{*c} \mathcal{E}(\Phi^{iso})$.*

Proof. Statements (1) and (2) follow from Theorem 3. In (3), $\mathcal{E}(\Phi) \sqsubseteq^{*c} \mathcal{E}(\Phi)$ follows from Proposition 18 and Theorem 3 by setting $X = \mathcal{S}$ and $\Psi = \Phi$, and $\mathcal{E}(\Phi^{iso}) \sqsubseteq^{*c} \mathcal{E}(\Phi^{iso})$ follows by setting $X = \mathcal{S}^{iso}$ and setting Ψ and Φ equal to Φ^{iso}. ▮

Corollary 19 shows new structural features of equilibria in BGwC. Ranked isotone selections from the joint best response correspondence have correspondingly ranked equilibrium sets in the $*c$ relation. Every equilibrium profile can be uniquely order approximated from below by an equilibrium profile using smallest best responses and from above by an equilibrium profile using largest best responses. Both the full equilibrium set and the isotone equilibrium set are $*$-complete.

Example 5 continued (Parametric BGwC). A *parametric Bayesian game with complementarities* is a collection $((A_i, \preceq_{A_i}, \Theta_i, \preceq_{\Theta_i}, \mathfrak{T}_i, u_i)_{i=1}^{I}, \mu, (T, \preceq_T))$, where $(A_i, \preceq_{X_i}$

$, \Theta_i, \preceq_{\Theta_i}, \mathfrak{T}_i)_{i=1}^I$ are as in Example 5, $T = T_1 \times T_2$ with product partial order, where $(T_1, \preceq_{T_1})$ and $(T_2, \preceq_{T_2})$ are nonempty posets,[6] $u_i : A \times \Theta \times T_1 \to \mathbb{R}$ is i's payoff function, for each t_2, $\mu(t_2)$ is a common prior on Θ, such that for each $t = (t_1, t_2)$, $((A_i, \preceq_{X_i}, \Theta_i, \preceq_{\Theta_i}, \mathfrak{T}_i, u_i(\cdot, \cdot, t_1)_{i=1}^I, \mu(t_2))$ is a BGwC. We assume that each u_i has increasing differences in (a_i, t_1), and $t_2 \mapsto \mu(t_2)$ is isotone in stochastic order, and the measures $\{\mu(t_2) \mid t_2 \in T_2\}$ are mutually absolutely continuous (share the same null sets). Denote the conditional of $\mu(t_2)$ on Θ_{-i} by $\mu_{-i}(\cdot \mid \theta_i, t_2)$ and the marginal on Θ_i by $\mu_i(\cdot \mid t_2)$ or by $\mu_i(t_2)$. Let $\mathcal{U}_i(a_i, \theta_i, \sigma_{-i}, t) = \int_{\Theta_{-i}} u_i(a_i, \sigma_{-i}(\theta_{-i}), \theta_i, \theta_{-i}, t_1) d\mu_{-i}(\theta_{-i} \mid \theta_i, t_2)$, let $\phi_i(\theta_i, \sigma_{-i}, t) = \arg\max_{A_i} \mathcal{U}_i(a_i, \theta_i, \sigma_{-i}, t)$, and let $\Phi_i(\sigma_{-i}, t)$ be the set of all measurable selections from $\phi_i(\cdot, \sigma_{-i}, t)$ mod $\mu_i(t_2)$. The associated parametric poset model is $((\mathcal{S}, \preceq), (T, \preceq_T), \Phi)$, where $\Phi(\sigma, t) = \times_{i=1}^I \Phi_i(\sigma_{-i}, t)$. The parametric isotone best response correspondence $\Phi_i^{iso}(\sigma_{-i}, t)$ is defined analogously, $\Phi_i^{iso} : \mathcal{S}_{-i}^{iso} \times T \rightrightarrows \mathcal{S}_i^{iso}$, $\Phi_i^{iso}(\sigma_{-i}, t) = \Phi_i(\sigma_{-i}, t) \cap \mathcal{S}_i^{iso}$. Let $\Phi^{iso}(\sigma, t) = \times_{i=1}^I \Phi_i^{iso}(\sigma_{-i}, t)$, and the associated parametric poset model is $((\mathcal{S}^{iso}, \preceq), (T, \preceq_T), \Phi^{iso})$.

Corollary 20. *In every parametric Bayesian game with complementarities,*

1. *For every isotone selection f from Φ^{iso},*
   - *(a) The equilibrium subcorrespondence $t \mapsto \mathcal{E}(f_t)$ is nonempty valued and isotone in $*c$ relation, and*
   - *(b) The infimum and supremum equilibrium selections ($t \mapsto \inf_{\mathcal{E}(f_t)} \mathcal{E}(f_t)$ and $t \mapsto \sup_{\mathcal{E}(f_t)} \mathcal{E}(f_t)$) exist and are isotone.*
2. *The isotone equilibrium correspondence ($t \mapsto \mathcal{E}(\Phi_t^{iso})$) is nonempty valued and isotone in $*c$ relation.*
3. *When T_2 is a singleton,*
   - *(a) For every isotone selection f from Φ, the equilibrium subcorrespondence $t \mapsto \mathcal{E}(f_t)$ is nonempty valued and isotone in $*c$ relation, and the infimum and supremum equilibrium selections exist and are isotone.*

[6] We model parametric impact on payoffs using T_1 and on distributions using T_2, because of their different impact on expected utility for profiles of arbitrary strategies or of isotone strategies, as seen in proof of Corollary 20.

(b) *The full equilibrium correspondence* $(t \mapsto \mathcal{E}(\Phi_t))$ *is nonempty valued and isotone in* $*c$ *relation.*

Proof. Applying the results in Example 5, for every $(\theta_i, \sigma_{-i}, t_1, t_2)$, $\phi_i(\theta_i, \sigma_{-i}, t_1, t_2)$ is a complete sublattice and therefore, $\underline{\phi}_i(\theta_i, \sigma_{-i}, t_1, t_2)$ and $\overline{\phi}_i(\theta_i, \sigma_{-i}, t_1, t_2)$ exist. Moreover, $\theta_i \mapsto \underline{\phi}_i(\theta_i, \sigma_{-i}, t_1, t_2)$ and $\overline{\phi}_i(\theta_i, \sigma_{-i}, t_1, t_2)$ are measurable follows from Van Zandt and Vives (2007). Furthermore, for every (θ_i, t_2), $(\sigma_{-i}, t_1) \mapsto \phi_i(\theta_i, \sigma_{-i}, t_1, t_2)$ is isotone in SSO. Therefore, for a fixed t_2, Φ_i and Φ are inf-isotone and sup-isotone. Moreover, Proposition 18 implies that for a fixed t_2, and for every t_1, Φ_{i,t_1} and Φ_{t_1} are inf-isotone on upper intervals and sup-isotone on lower intervals. Applying Theorem 8 proves statement (3). Moreover, for every σ_{-i} isotone, $(\theta_i, t_1, t_2) \mapsto \phi_i(\theta_i, \sigma_{-i}, t_1, t_2)$ is isotone in SSO, and therefore, Φ_i^{iso} and Φ^{iso} are inf-isotone and sup-isotone and Proposition 18 implies that for every $t = (t_1, t_2)$, $\Phi_{i,t}^{iso}$ and Φ_t^{iso} are inf-isotone on upper intervals and sup-isotone on lower intervals. Applying Theorem 8 proves statements (1) and (2). ∎

These are new results for parametric BGwC and guarantee a variety of order-nearest comparative statics of equilibria in decentralized models with incomplete information and complementarities.

6 Comparing set relations

Our $*ic$, $*sc$, and $*c$ relations can be viewed as natural strengthenings of the upper weak, lower weak, and weak set orders (see Proposition 21 below). Those relations provide a larger higher and/or smaller lower point in the comparison whereas our relations strengthen these to smallest higher and/or largest lower point. Our $*ic$, $*sc$, and $*c$ relations are strictly weaker than those in Sabarwal (2025). To facilitate comparisons, we denote those relations by the subscript *s25*: $A \sqsubseteq_{s25}^{*ic} B$ if for every nonempty $E \subseteq B$, $\inf_A E$ exists, $A \sqsubseteq_{s25}^{*sc} B$ if for every nonempty $E \subseteq A$, $\sup_B E$ exists, and $A \sqsubseteq_{s25}^{*c} B$ if $A \sqsubseteq_{s25}^{*ic} B$ and $A \sqsubseteq_{s25}^{*sc} B$. When X is a complete lattice, these definitions coincide with ours. More generally, our definitions are weaker. Let $X = \{(0,0), (1,0), (0,1), (2,0), (0,2)\} \cup \{(t,t) \mid 2 < t \leq 3\}$ with the relative partial order from $\mathbb{R}^2$. Then X is CSC and $\inf_X X$ exists. Let $A = \{(0,0), (1,0), (0,1)\}$ and $B = \{(0,0), (2,0), (0,2)\} \cup \{(t,t) \mid 2 < t \leq 3\}$. Let $g : X \to X$ be the isotone function given by $g(x) = x$ if $x \in B$, $g(1,0) = (2,0)$, and $g(0,1) = (0,2)$. Let $f : X \to X$ be given by $f(x) = x$ if $x \in A$, $f(3,3) = (0,0)$, and $f(x) = (3,3)$ if $x \in X \backslash (A \cup \{(3,3)\})$. Theorem 3(1)(a) shows that $A = \mathcal{E}(f) \sqsubseteq^{*sc} \mathcal{E}(g) = B$. It is easy to check that $A \not\sqsubseteq^{*sc}_{s25} B$. Similarly, it can be shown that $A \sqsubseteq^{*ic} B$ does not necessarily imply $A \sqsubseteq^{*ic}_{s25} B$. Proposition 21 provides additional comparisons.

We may similarly generalize the relations $*\wedge$, $*\vee$, and $*\ell$ relations in Sabarwal (2025). For nonempty $A, B \subseteq X$, $A \sqsubseteq^{*\wedge} B$, if for every $x \in A$ and $y \in B$, $\inf_X\{x, y\}$ exists $\Rightarrow \inf_A\{x, y\}$ exists, $A \sqsubseteq^{*\vee} B$, if for every $x \in A$ and $y \in B$, $\sup_X\{x, y\}$ exists $\Rightarrow \sup_B\{x, y\}$ exists, and $A \sqsubseteq^{*\ell} B$, if $A \sqsubseteq^{*\wedge} B$ and $A \sqsubseteq^{*\vee} B$. The relations in Sabarwal (2025) are defined by deleting the hypothesis $\inf_X\{x, y\}$ exists or $\sup_X\{x, y\}$ exists in the corresponding definitions. We denote those relations as follows: $A \sqsubseteq^{*\wedge}_{s25} B$, $A \sqsubseteq^{*\vee}_{s25} B$, and $A \sqsubseteq^{*\ell}_{s25} B$. When X is meet-complete (respectively, join-complete), that is, for every $x, y \in X$, $\inf_X\{x, y\} \in X$, (respectively, $\sup_X\{x, y\} \in X$,) our definitions coincide. More generally, our definitions are weaker. The example above shows that $A \sqsubseteq^{*\vee} B$ and $A \not\sqsubseteq^{*\vee}_{s25} B$.

As our $*c$ and $*\ell$ relations are weaker than $\sqsubseteq^{*c}_{s25}$ and $\sqsubseteq^{*\ell}_{s25}$ relations, our relations are reflexive on the larger classes of $*c$ complete and $*\ell$ complete subsets of a poset, respectively. Moreover, restricting comparisons to $\sqsubseteq^{*c}_{s25}$ and $\sqsubseteq^{*\ell}_{s25}$ relations shows that our relations remain not antisymmetric and not transitive (WSO is not antisymmetric as well but is transitive). Therefore, transitivity cannot be presumed and must be established explicitly. We prove this in all the general results in the paper. Additional comparisons among the relations are as follows.

Proposition 21. *For every poset X and nonempty $A, B \subseteq X$,*

(1) $A \sqsubseteq^{*ic}_{s25} B \Rightarrow A \sqsubseteq^{*ic} B \Rightarrow A \sqsubseteq^{lw} B$ *and* $A \sqsubseteq^{*sc}_{s25} B \Rightarrow A \sqsubseteq^{*sc} B \Rightarrow A \sqsubseteq^{uw} B$

(2) $A \sqsubseteq^{*\wedge}_{s25} B \Rightarrow A \sqsubseteq^{*\wedge} B \Rightarrow A \sqsubseteq^{lw} B$ *and* $A \sqsubseteq^{*\vee}_{s25} B \Rightarrow A \sqsubseteq^{*\vee} B \Rightarrow A \sqsubseteq^{uw} B$

(3) $A \sqsubseteq^{*ic} B$ *and A is meet complete* $\Rightarrow A \sqsubseteq^{*\wedge}_{s25} B$, *and* $A \sqsubseteq^{*sc} B$ *and B is join complete* $\Rightarrow A \sqsubseteq^{*\vee}_{s25} B$

(4) If A and B are subcomplete in X, then $A \sqsubseteq^{*ic}_{s25} B \Leftrightarrow A \sqsubseteq^{*ic} B \Leftrightarrow A \sqsubseteq^{*\wedge}_{s25} B \Leftrightarrow A \sqsubseteq^{*\wedge} B \Leftrightarrow A \sqsubseteq^{lw} B$, *and* $A \sqsubseteq^{*sc}_{s25} B \Leftrightarrow A \sqsubseteq^{*sc} B \Leftrightarrow A \sqsubseteq^{*\vee}_{s25} B \Leftrightarrow A \sqsubseteq^{*\vee} B \Leftrightarrow A \sqsubseteq^{uw} B$

Proof. Statements (1) and (2) follow immediately from the definitions. For (3), suppose $A \sqsubseteq^{*ic} B$ and A is meet complete. Let $x \in A$ and $y \in B$. Let $\hat{y} = \inf_A\{y\} \in A$, which exists because $A \sqsubseteq^{*ic} B$. As A is meet complete, let $\hat{a} = \inf_A\{x, \hat{y}\} \in A$. Then $\hat{y} \preceq y$ implies that $\hat{a}$ is a lower bound for $\{x, y\}$. Suppose $z \in A$ is a lower bound for $\{x, y\}$. Then $z \in A$ and $z \preceq y$ implies $z \preceq \hat{y}$ and therefore, z is a lower bound for $\{x, \hat{y}\}$, whence $z \preceq \hat{a}$. This shows that $\inf_A\{x, y\} = \hat{a} \in A$. The other statement, $A \sqsubseteq^{*sc} B$ and B is join complete implies $A \sqsubseteq^{*\vee}_{s25} B$, is proved similarly. For (4), statements (1)-(3) imply $A \sqsubseteq^{*ic}_{s25} B \Rightarrow A \sqsubseteq^{*ic} B \Rightarrow A \sqsubseteq^{*\wedge}_{s25} B \Rightarrow A \sqsubseteq^{*\wedge} B \Rightarrow A \sqsubseteq^{lw} B$, and $A \sqsubseteq^{lw} B \Rightarrow A \sqsubseteq^{*ic}_{s25} B$ follows from Theorem 14 in Sabarwal (2025). The second part of statement (4) is proved similarly. ∎

Notably, $A \sqsubseteq^{lw} B \Rightarrow A \sqsubseteq^{*ic} B$ is not necessarily true when subcompleteness of A and B is weakened to complete lattices. Consider $X = [0, 3], A = [0, 1) \cup \{2\}$ and $B = \{1\} \cup (2, 3]$. Then A and B are complete lattices and $A \sqsubseteq^{w} B$, but $A \not\sqsubseteq^{*ic} B$, because for $E = \{1\} \subseteq B$, $\inf_X\{1\}$ exists but $\inf_A\{1\}$ does not exist in A. Similarly, $A \not\sqsubseteq^{*sc} B$. Subcompleteness properties are easily violated for sets of equilibria. We show that our completeness-type properties do hold for order nearest comparative statics under natural conditions, making our approach more suitable for such comparisons.

7 Conclusion

We introduce a powerful and general framework for order nearest comparative statics of equilibria in partially ordered settings, significantly advancing the theory beyond the limitations of lattice-based approaches. By developing new poset-based definitions to compare equilibria and structure theorems for fixed points that do not rely on continuity or lattice structure, our framework provides a unified and robust method to analyze equilibrium behavior in a wide range of models, including stochastic dynamic economies and more general stochastic dynamic environments in decision-making under uncertainty such as Markov decision processes. An illustrative application to dynamic industry equilibrium with switching costs highlights the results concretely and underscores the practical relevance of the theory. The contributions are both deep and broadly applicable. Overall, this work represents a substantial theoretical advance with the potential to reshape how economists approach comparative statics and equilibrium analysis in partially ordered settings.

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8 Appendix: Additional extensions

When $\Phi(x)$ does not have a smallest or largest point but has minimal or maximal points, we may define a generalization of inf-isotone or sup-isotone selection using minimal or maximal subcorrespondences. Following Kukushkin (2013), consider the minimal subcorrespondence, denoted $\Phi_- : X \rightrightarrows Y$, where $\Phi_-(x)$ are the minimal elements of $\Phi(x)$, and the maximal subcorrespondence, $\Phi^+ : X \rightrightarrows Y$, where $\Phi^+(x)$ are the maximal elements of $\Phi(x)$. As noted in Kukushkin (2013), when images $\Phi(x)$ are chain lower complete and Φ is isotone in $\wedge$ relation ($\hat{x} \prec \tilde{x} \Rightarrow \Phi(\hat{x}) \sqsubseteq^\wedge \Phi(\tilde{x})$), then $\Phi_- \sqsubseteq^{lw} \Phi$ and Φ_- is uniformly isotone ($\hat{x} \prec \tilde{x} \Rightarrow \Phi_-(\hat{x}) \sqsubseteq^u \Phi_-(\tilde{x})$),[7] and the dual statement holds when images $\Phi(x)$ are chain upper complete and Φ is isotone in $\vee$ relation. It is easy to check that the same statements hold if we replace $\wedge$ relation with $*\wedge$ relation and if we replace $\vee$ relation with $*\vee$ relation due to Sabarwal (2025).

These necessary implications are used to define the following. A nonempty valued correspondence $\Phi : X \rightrightarrows Y$ is *lower-isotone* (respectively, *upper-isotone*), if it has a subcorrespondence Ψ such that $\Psi \sqsubseteq^{lw} \Phi$ (respectively, $\Phi \sqsubseteq^{uw} \Psi$) and Ψ is uniformly isotone. It follows that if Φ is inf-isotone (respectively, sup-isotone) then it is lower-isotone (respectively, upper-isotone). Notably, Ψ is uniformly isotone is a strong condition. We can weaken it as follows. It can be checked that for every subcorrespondence $\Psi : X \rightrightarrows Y$ of Φ, $\Psi \sqsubseteq^{lw} \Phi$ implies that Φ_- is a subcorrespondence of Ψ, and therefore, Ψ is uniformly isotone implies that Φ_- is uniformly isotone as well. In fact, if $\Phi_- \sqsubseteq^{lw} \Phi$, then for every

[7] $A \sqsubseteq^u B$ if for every $a \in A$, for every $b \in B$, $a \preceq b$.

x, $\Phi_-(x) = \bigcap\{\Psi(x) \mid \Psi$ is a subcorrespondence of Φ and $\Psi \sqsubseteq^{lw} \Phi\}$ showing that Φ_- is the smallest subcorrespondence under which Φ is lower-isotone, and similarly, Φ^+ is the smallest subcorrespondence under which Φ is upper-isotone. For this reason, in what follows, when Φ is lower-isotone (respectively, upper-isotone) we'll use the notation Φ_- (respectively, Φ^+) for Ψ.

A caveat is that weak conditions in standard models imply that $\Phi_- = \underline{\Phi}$ and $\Phi^+ = \overline{\Phi}$. For example, under standard conditions such as those in Topkis (1978), LiCalzi and Veinott (1992), and Milgrom and Shannon (1994), the best response correspondence is isotone in $\wedge$ relation as follows: For every $\hat{x} \preceq \tilde{x}$, $\Phi(\hat{x}) \sqsubseteq^{\wedge} \Phi(\tilde{x})$ (as compared to for every $\hat{x} \prec \tilde{x}$, $\Phi(\hat{x}) \sqsubseteq^{\wedge} \Phi(\tilde{x})$). This implies $\Phi(x) \sqsubseteq^{\wedge} \Phi(x)$ and combined with $\Phi_-(x) \sqsubseteq^{lw} \Phi(x)$ it follows that $\Phi_-(x) = \{\inf_{\Phi(x)} \Phi(x)\}$, and therefore, $\Phi_- = \underline{\Phi}$. The same conclusion holds if we replace $\wedge$ relation with the bound isotone relation in Calciano (2007) that arises from their bound-modular functions or the $*\wedge$ relation in Sabarwal (2025). The dual statements hold for $\vee$ and $*\vee$ relations. In other words, the benefit of the generalization to minimal or maximal subcorrespondences depends on classes of applications for which Φ satisfies isotone property for $\hat{x} \prec \tilde{x}$ but not at $\hat{x} = \tilde{x}$. This remains an open question. With this caveat, Lemma 1 generalizes as follows.[8]

Lemma 22. *For every X that is CSC and $\inf_X X$ exists (respectively, X is CIC and $\sup_X X$ exists), for every $\Phi : X \rightrightarrows X$, if Φ is lower-isotone (respectively, upper-isotone), then $\inf_{\mathcal{E}(\Phi_-)} \mathcal{E}(\Phi_-) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ (respectively, $\sup_{\mathcal{E}(\Phi^+)} \mathcal{E}(\Phi^+) = \sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$).*

Proof. Notice that Φ is lower-isotone implies Φ_- is nonempty valued and every selection from Φ_- is isotone, and using Abian and Brown (1961), it follows that $\mathcal{E}(\Phi_-) \neq \emptyset$. To see that $\inf_{\mathcal{E}(\Phi_-)} \mathcal{E}(\Phi_-) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$, let $A = \{x \in X \mid \{x\} \sqsubseteq^u \underline{\Phi}(x)$ and $\forall e \in \mathcal{E}(\Phi), x \preceq e\}$. Let C be a chain in A. If C is empty, then $\sup_X C = \inf_X X \in A$. If C is not empty, let $y = \sup_X C$, which exists because X is CSC. If $y \in A$, then C has an upper bound in A. Otherwise, for every $x \in C$, $x \prec y$, implying $\Phi_-(x) \sqsubseteq^u \Phi_-(y)$, and therefore, $\{x\} \sqsubseteq^u \Phi_-(y)$, whence $\{y\} \sqsubseteq^u \Phi_-(y)$. Moreover, for every $x \in C$ and every $e \in \mathcal{E}(\Phi)$, $x \preceq e$ implies that for every $e \in \mathcal{E}(\Phi)$, $y \preceq e$. It follows that $y \in A$. By Zorn's lemma, let e^* be a maximal element in A. As one case, if there is $e \in \mathcal{E}(\Phi)$ such that $e^* = e$, then $e^* = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$. Moreover, $\Phi_- \sqsubseteq^{lw} \Phi$ implies there is $x \in \Phi_-(e^*)$ with $x \preceq e^*$, and $\{e^*\} \sqsubseteq^u \Phi_-(e^*)$ implies $e^* \preceq x$, whence $e^* \in \Phi_-(e^*)$. It follows that $e^* = \inf_{\mathcal{E}(\Phi_-)} \mathcal{E}(\Phi_-) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi) \in \mathcal{E}(\Phi)$. As the other case, if for every $e \in \mathcal{E}(\Phi)$, $e^* \prec e$, then it must be that $\{e^*\} = \Phi_-(e^*)$, as follows. We already know that $\{e^*\} \sqsubseteq^u \Phi_-(e^*)$. If there is $x \in \Phi_-(e^*)$, $e^* \prec x$, then $\Phi_-(e^*) \sqsubseteq^u \Phi_-(x)$, whence $\{x\} \sqsubseteq^u \Phi_-(x)$. Moreover, $e^* \prec e$ implies $\Phi_-(e^*) \sqsubseteq^u \Phi_-(e)$, whence $\{x\} \sqsubseteq^u \Phi_-(e)$. As e is arbitrary and $e \in \Phi_-(e)$, it follows that for every $e \in \mathcal{E}(\Phi)$, $x \preceq e$. Thus, $x \in A$, and therefore, $e^* \not\prec x$, a contradiction. Therefore, $\{e^*\} = \Phi_-(e^*)$, and it follows that $e^* = \inf_{\mathcal{E}(\Phi_-)} \mathcal{E}(\Phi_-) = \inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi) \in \mathcal{E}(\Phi)$. The statement for upper-isotone Φ is proved similarly, using the dual argument. ∎

[8]Yu (2024) proves a related result. We do not make their assumption that $\sup_X\{x \in X \mid F(x) \cap [\hat{x}, \infty) \neq \emptyset\}$ exists. Our proof is based on Lemma 1, proved earlier in Sabarwal (2023b).

We say that Φ is *lower-isotone on upper intervals*, if it is lower-isotone and for every $\hat{x} \in X$ such that $\Phi(\hat{x}) \cap \hat{X} \neq \emptyset$, (where $\hat{X} = \{x \in X \mid \hat{x} \preceq_X x\}$,) Φ restricted to $\hat{X}$ is lower-isotone. Φ is *upper-isotone on lower intervals*, if it is upper-isotone and for every $\hat{x} \in X$ such that $\Phi(\hat{x}) \cap \hat{X} \neq \emptyset$, (where $\hat{X} = \{x \in X \mid x \preceq_X \hat{x}\}$,) Φ restricted to $\hat{X}$ is upper-isotone. Theorem 3(2), Corollary 4, and related results generalize as follows.

Proposition 23. *For every poset X and every $\Psi, \Phi : X \rightrightarrows X$, the following statements hold whenever the sets being compared are nonempty.*

(1) (a) If X is CIC (respectively, X is CSC and $\inf_X X$ exists) and Φ is lower-isotone, then $\mathcal{E}(\Phi_-) \sqsubseteq^{*ic} \mathcal{E}(\Phi)$ (respectively, $\mathcal{E}(\Phi_-) \sqsubseteq^{*sc} \mathcal{E}(\Phi)$).

(b) If X is CSC (respectively, X is CIC and $\sup_X X$ exists) and Φ is upper-isotone, then $\mathcal{E}(\Phi) \sqsubseteq^{*sc} \mathcal{E}(\Phi^+)$ (respectively, $\mathcal{E}(\Phi) \sqsubseteq^{*ic} \mathcal{E}(\Phi^+)$).

(2) (a) If X is CSC, $\Psi \sqsubseteq^{uw} \Phi$, Φ is upper-isotone, and Φ is lower-isotone on upper intervals, then $\mathcal{E}(\Psi) \sqsubseteq^{*sc} \mathcal{E}(\Phi)$.

(b) If X is CIC, $\Phi \sqsubseteq^{lw} \Psi$, Φ is lower-isotone, and Φ is upper-isotone on lower intervals, then $\mathcal{E}(\Phi) \sqsubseteq^{*ic} \mathcal{E}(\Psi)$.

Proof. For (1)(a), to show that $\mathcal{E}(\Phi_-) \sqsubseteq^{*ic} \mathcal{E}(\Phi)$, consider nonempty $E \subseteq \mathcal{E}(\Phi)$, suppose $\underline{e} = \inf_X E$ exists, let $\hat{X} = \{x \in X \mid x \preceq \underline{e}\}$, and let $\hat{\Phi} : \hat{X} \rightrightarrows \hat{X}$ be given by $\hat{\Phi}(x) = \Phi_-(x)$. As Φ_- is uniformly isotone, it is upper-isotone on lower intervals, and using Lemma 22, let $\hat{e}$ be the largest fixed point of $\hat{\Phi}$. Then $\hat{e} \in \mathcal{E}(\Phi_-)$, $\hat{e}$ is a lower bound for E and for every $e \in \mathcal{E}(\Phi_-)$, e is a lower bound for E implies $e \in \hat{X}$ and therefore, $e \in \mathcal{E}(\hat{\Phi})$, whence $e \preceq \hat{e}$. To show that $\mathcal{E}(\Phi_-) \sqsubseteq^{*sc} \mathcal{E}(\Phi)$, consider nonempty $E \subseteq \mathcal{E}(\Phi_-)$, suppose $\overline{e} = \sup_X E$ exists, let $\hat{X} = \{x \in X \mid \overline{e} \preceq x\}$, and let $\Psi : \hat{X} \rightrightarrows \hat{X}$ be given $\Psi(x) = \Phi(x) \cap \hat{X}$. If there is $e \in E$ such that $e = \overline{e}$, then it is easy to check that $\overline{e} = \sup_{\mathcal{E}(\Phi)} E$. Otherwise, for every $e \in E$, $e \prec \overline{e}$, and therefore, $e \in \Phi_-(e) \sqsubseteq^{u} \Phi_-(\overline{e})$, whence $\{\overline{e}\} \sqsubseteq^{u} \Phi_-(\overline{e})$. It follows that for every $x \in \hat{X}$, $\hat{\Phi}(x) = \Phi(x)$, proving that $\hat{\Phi}$ is lower-isotone. Using Lemma 22, let $\hat{e}$ be the smallest fixed point of $\hat{\Phi}$. Then $\hat{e} = \sup_{\mathcal{E}(\Phi)} E$. Statement (1)(b) is proved similarly.

For (2)(a), consider nonempty $E \subseteq \mathcal{E}(\Psi)$, suppose $\overline{e} = \sup_X E$ exists, let $\hat{X} = \{x \in X \mid \overline{e} \preceq x\}$, and $\hat{\Phi} : \hat{X} \rightrightarrows \hat{X}$ be given by $\hat{\Phi}(x) = \Phi(x) \cap \hat{X}$. As case one, suppose there is $e \in E$, $e = \overline{e}$. Then $\overline{e} \in \Phi(\overline{e})$ implies $\Phi(\overline{e}) \cap \hat{X} \neq \emptyset$. As case two, suppose for every $e \in E$, $e \prec \overline{e}$. Then $\Phi(e) \sqsubseteq^{uw} \Phi^+(e) \sqsubseteq^{u} \Phi^+(\overline{e})$ implies $\Phi(\overline{e}) \cap \hat{X} \neq \emptyset$. Combined with $\Phi(x) \sqsubseteq^{uw} \Phi^+(x)$, it follows that $\hat{\Phi}$ is nonempty valued. As Φ is lower-isotone on upper intervals, let $\hat{e}$ be the smallest fixed point of $\hat{\Phi}$. Then $\hat{e} \in \mathcal{E}(\Phi)$. Moreover, for every $e \in \mathcal{E}(\Phi)$ that is an upper bound for E, $\overline{e} \preceq e$, whence $e \in \mathcal{E}(\hat{\Phi})$, and consequently, $\hat{e} \preceq e$. This shows that $\sup_{\mathcal{E}(\Phi)} E = \hat{e}$. Statement (2)(b) is proved similarly. ∎

Corollary 24. *For every poset X and every $\Phi : X \rightrightarrows X$, if X is CSC and $\inf_X X$ (respectively, CIC and $\sup_X X$) exists, Φ is upper-isotone (respectively, lower-isotone),* *and Φ is lower-isotone on upper intervals (respectively, upper-isotone on lower intervals), then* $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ *exists and* $\mathcal{E}(\Phi) \sqsubseteq^{*sc} \mathcal{E}(\Phi)$ *(respectively,* $\sup_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ *exists and* $\mathcal{E}(\Phi) \sqsubseteq^{*ic} \mathcal{E}(\Phi)$*).*

Proof. Similar to proof of Corollary 6. ∎

Corollary 25. *For every poset X, the following are equivalent.*

(1) X *is CSC and* $\inf_X X$ *exists.*

(2) For every $\Phi : X \rightrightarrows X$ *that is upper-isotone and is lower-isotone on upper intervals,* $\mathcal{E}(\Phi) \sqsubseteq^{*sc} \mathcal{E}(\Phi)$ *and* $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ *exists.*

(3) For every $\Phi : X \rightrightarrows X$ *that is upper-isotone and is lower-isotone on upper intervals,* $\mathcal{E}(\Phi)$ *is CSC and* $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ *exists.*

(4) For every $\Phi : X \rightrightarrows X$ *that is upper-isotone and is lower-isotone on upper intervals,* $\inf_{\mathcal{E}(\Phi)} \mathcal{E}(\Phi)$ *exists.*

Proof. Similar to proof of Corollary 7. ∎

A dual characterization holds for X is CIC and $\sup_X X$ exists.

Proposition 26. *For every poset X and T such that X is CSC and* $\inf_X X$ *(respectively, X is CIC and* $\sup_X X$*) exists, for every* $\Psi, \Phi : X \times T \rightrightarrows X$*,*

(1) If Φ is upper-isotone (respectively, lower-isotone) and for every t, Φ_t is lower-isotone on upper intervals (respectively, upper-isotone on lower intervals), then

(a) The full equilibrium correspondence $t \mapsto \mathcal{E}(\Phi_t)$ *is nonempty valued and isotone in $*sc$ (respectively, $*ic$) relation,*

(b) For every t, $\Psi_t \sqsubseteq^{uw} \Phi_t$ *(respectively,* $\Phi_t \sqsubseteq^{lw} \Psi_t$*)* $\Rightarrow \mathcal{E}(\Psi_t) \sqsubseteq^{*sc} \mathcal{E}(\Phi_t)$ *(respectively,* $\mathcal{E}(\Phi_t) \sqsubseteq^{*ic} \mathcal{E}(\Psi_t)$*), whenever* $\mathcal{E}(\Psi_t)$ *is nonempty.*

Proof. Similar to proof of Theorem 8. ∎