Pricing rules with market frictions: an axiomatic approach

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Abstract

This paper studies markets with frictions, taking the market price as the primitive concept, assumed to be exogenously given and defined axiomatically. To account for general types of frictions, we consider market pricing rules that go beyond the subadditive framework.

However, extending the model beyond subadditivity significantly broadens the set of possible arbitrage opportunities, for example, allowing for multiple buy & sell arbitrage opportunities, which would previously have been automatically excluded by subadditivity. As in the standard theory of asset pricing, the extended framework is developed under the assumption that markets have no standard arbitrage opportunities, but also no generalized ones.

Applications are provided to pricing rules represented by discounted Choquet expectations with respect to risk-neutral, nonadditive probabilities, without assuming submodularity (i.e., concavity).

Key words: Pricing rule, subadditivity, arbitrage opportunities, market frictions, bid-ask spread, Choquet pricing rule.

JEL: G12, D81, C71.

1. Introduction

The classical theory of asset pricing, as developed in frictionless markets, is built upon the assumption that market participants can trade freely without incurring any costs or constraints. However, in real financial markets, various forms of frictions, such as transaction costs, bid-ask spreads, taxes, and market illiquidity, are pervasive and can significantly impact pricing and trading strategies. A growing body of literature has thus sought to extend the theory of asset pricing to account for these frictions, as in Garman and Ohlson (1981), Prisman (1986), Ross (1987), Bensaid et al. (1992). In these papers, financial markets are defined through a family of securities that enables investors to generate payoffs by appropriately selecting the quantities to buy or sell. The market price is then derived from the prices of these underlying securities.

In contrast, another strand of literature takes the market price as the primitive concept, assumed to be exogenously given and axiomatically defined. This paper adopts such an axiomatic approach to study markets with frictions that generate pricing rules as in Araujo, Chateauneuf, and Faro (2012, 2018), Jouini and Kallal (1995, 1999, 2001), Jouini (2000), Jouini and Napp (2006), Campi, Jouini, and Porte (2013), Castagnoli, Maccheroni, and Marinacci (2002), Chateauneuf and Cornet (2018, 2022a, 2025). However, these papers mainly consider subadditive market pricing rules or Choquet pricing rules associated with a risk-neutral nonadditive probability that is submodular, as in Chateauneuf, Kast and Lapied (1996) and Chateauneuf and Cornet (2018).

Extensions beyond the subadditive framework have been considered by Cerreia-Vioglio et al. (2015a,b), Chateauneuf and Cornet (2022b), and Bastianello, Chateauneuf and Cornet (2024). The first two papers generalize the fundamental theorem of asset pricing by showing that, in the presence of markets frictions, the combination of Put-Call Parity and monotonicity is equivalent to the market pricing rule being representable as a discounted Choquet expectation with respect to a risk-neutral nonadditive probability, one that is no longer assumed to be submodular (i.e., concave). The notion of Put-Call Parity generalizes the earlier notion of Call-Put Parity introduced by Chateauneuf, Kast and Lapied (1996), which implicitly assumed that the pricing rule has no bid-ask spreads. The latter two papers demonstrate the nonvacuity of the core of the risk-neutral nonadditive probability associated with the Choquet pricing rule, in the absence of multiple buy & sell arbitrage opportunities.

Extending the model beyond subadditivity significantly enlarges the set of arbitrage opportunities, including buy & sell arbitrage opportunities that were previously automatically ruled out by subadditivity. Indeed, subadditivity of the pricing rule $f : X \to \mathbb{R}$, which requires that the cost of jointly purchasing x and x' is not greater than the cost of purchasing them separately (that is, $f(0) = 0$ and $f(x + x') \leq f(x) + f(x')$ for all payoffs x, x' in the payoff space X), implies the absence of buy & sell arbitrage. That is, it prevents the possibility of buying a payoff x and simultaneously selling the same payoff x (i.e., buying $-x$) that would yield a negative aggregate cost $f(x) + f(x') < 0$ (hence a gain), while resulting in zero aggregate payoff $x+(-x) = 0$. As in the standard theory of asset pricing, the extended theory beyond subadditivity is developed under the assumption that markets admit no arbitrage opportunities, neither standard arbitrage, nor the buy & sell arbitrage opportunities of order 2 discussed previously, nor more generally, those of higher order K as studied in this paper.

The paper is organized as follows. The model and notations are presented in Section 2.1 and pricing rules are defined in Section 2.2. Arbitrage-free and frictionless pricing rules are introduced in Sections 2.3 and 2.4 in a framework that does not require subadditivity. Section 2.6 considers a finite dimensional setting with a finite state space Ω, and our first result (Theorem 1) shows that arbitrage opportunities and market frictions of orders greater than $\#\Omega+1$ are irrelevant and can be eliminated at no cost. Section 2.7 characterizes additive pricing rules by the absence of order-3 arbitrage opportunities and of bid-ask spreads, both in the monotone case (Theorem 2) and in the nonmonotone case (Theorem 3). Section 2.8 introduces Choquet pricing rules and Section 2.11 provides another characterization of additive pricing rules (Theorem 7) in term of its associated non-additive probability (or the game). Section 2.12 presents an explicit example (Example 2) of a Choquet pricing rule that is not additive, does not have bid-ask spread, and exhibits both an arbitrage opportunity and a market friction of order 3. Finally, the conclusion in

Section 2.13 discuss how the results relate to the notion of Put-Call and Call-put parities. All proofs are colected in Section 3.

2. Arbitrage-free and frictionless pricing rules

2.1. The model and notations

This paper considers the simple framework of a stochastic two-date model: time $t = 0$ (today) is known, while time $t = 1$ (tomorrow) is uncertain. A single commodity (money) is available at both dates and in every state. Uncertainty is represented by a nonempty set Ω (finite or infinite) of *states of nature* and one, and only one, state of nature will be realized and known tomorrow. Throughout the paper, $(\Omega, \mathcal{A})$ will be a measurable space, where Ω is an arbitrary nonempty set and $\mathcal{A} \subseteq 2^{\Omega}$ is a σ-algebra of subsets of Ω, that is, $\mathcal{A}$ contains Ω and is closed under countable unions and complementation. Note that we do not assume the existence of a reference probability on $(\Omega, \mathcal{A})$ as in Riedel (2015) and Burzoni Riedel and Soner (2021).

A payoff is a random variable $x : \Omega \to \mathbb{R}$ that is measurable, i.e., $x^{-1}(I) \in \mathcal{A}$ for every interval $I \subseteq \mathbb{R}$ and bounded, i.e., $\|x\|_\infty := \sup\{|x(\omega)| : \omega \in \Omega\} < +\infty$. We denote by $X := B(\mathcal{A})$ the set of all bounded measurable functions $x : \Omega \to \mathbb{R}$ and by $B_0(\mathcal{A})$ the subspace of $X := B(\mathcal{A})$ of the simple measurable functions, i.e., measurable functions taking only finitely many values. The space $X := B(\mathcal{A})$ is a Banach lattice for the coordinate-wise order $x \geq x'$, defined by $x(\omega) \geq x'(\omega)$ for all $\omega \in \Omega$, we also define the strict order $x > x'$ by $x \geq x'$ and $x \neq x'$; we define $x \vee y$ by $(x \vee y)(\omega) := \max\{x(\omega), y(\omega)\}$ for all $\omega \in \Omega$ and $x \wedge y$ by $(x \wedge y)(\omega) := \min\{x(\omega), y(\omega)\}$ for all $\omega \in \Omega$. We denote by $X_+ := \{x \in B(\mathcal{A}) : x \geq 0\}$ the set of non-negative bounded measurable functions and by $B_0^+(\mathcal{A}) := \{x \in B_0(\mathcal{A}) : x \geq 0\}$ the corresponding set of non-negative simple measurable functions. We adopt the standard convention that, if $x(\omega) < 0$, then $|x(\omega)|$ represents the amount paid by the agent for buying the single commodity (money) if state ω occurs, and if $x(\omega) > 0$, then $|x(\omega)|$ is the gain received from selling the single commodity (money) if state ω occurs. Thus, if $x \geq 0$, the payoff x corresponds to buying $x(\omega) \geq 0$ at every state ω, while $-x$ corresponds to selling the quantity $|x(\omega)| \geq 0$ at every state; more generally, the same terminology will be used for arbitrary payoffs $x \in X$. For every measurable set $A \in \mathcal{A}$, we denote by

A^c its complement in Ω, and define $\mathbf{1}_A$ by $\mathbf{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbf{1}_A(\omega) = 0$ otherwise. By convention, we set $\mathbf{1}_\emptyset = 0$.

2.2. Pricing rules

To every payoff $x \in X$, the market assigns a price $f(x)$ to be *paid* at time $t = 0$, for the delivery of the random payoff x at time $t = 1$. We adopt the convention that $|f(x)|$ is the amount *paid* if $f(x) > 0$ and a *gain* (i.e, received amount) if $f(x) < 0$. In other words, $f(x)$ is the *buying price (or ask price)*, that is, the price *paid* today to *buy* x to be delivered tomorrow, and $-f(-x)$ is the *selling price (or bid price)*, that is, the amount *received* today for *selling* x (or equivalently buying $-x$)) to be delivered tomorrow. For a given payoff $x \in X$, the pair $(-f(x), x) \in \mathbb{R} \times X$, called total payoff, specifies the total stream of payments at time $t = 0$ and at $t = 1$ the random payoff delivered across all future states.

In this paper, will take as a primitive concept the real-valued function $f : X \to \mathbb{R}$, referred to as a pricing rule under the unique assumption that $f(0) = 0$, that is, receiving 0 tomorrow incurs no payment and yields no gain today. Standard definitions are recalled below.

Definition 1. *Let $(\Omega, \mathcal{A})$ be a measurable space, let $X := B(\mathcal{A})$ be the payoff space of real-valued bounded measurable functions defined on Ω, then a pricing rule is a real-valued function $f : X \to \mathbb{R}$ satisfying $f(0) = 0$. We say that*

f is monotone if $x_1 \leq x_2$ implies $f(x_1) \leq f(x_2)$ for all x_1, x_2 in X;
f is strictly monotone if $x_1 < x_2$ implies $f(x_1) < f(x_2)$ for x_1, x_2 in X;
f is positively homogeneous if $f(tx) = tf(x)$ for all $x \in X$ and all $t \in \mathbb{R}_+$;
f is subadditive if $f(x_1 + x_2) \leq f(x_1) + f(x_2)$ for all x_1, x_2 in X;
f is subadditive at $x \in X$ if for all positive integer K, all $(x_1, \ldots, x_K) \in X^K$
$$x_1 + \cdots + x_K = x \Rightarrow f(x_1) + \cdots + f(x_K) \geq f(x);$$
f is superadditive if $-f$ is subadditive;
f is superadditive at $x \in X$ if $-f$ is subadditive at $x \in X$;
f is sublinear if f is subadditive and positively homogeneous;
f is sublinear at $x \in X$ if, for all positive integer K,
for all $(x_1, \ldots, x_K) \in X^K$, and all $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K$,
$$\theta_1 x_1 + \cdots + \theta_K x_K = x \Rightarrow \theta_1 f(x_1) + \cdots + \theta_K f(x_K) \geq f(x);$$
f is linear if, for all x_1, x_2 in X, all α_1, α_2 in $\mathbb{R}$

$$f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2);$$

f is submodular if, for all x_1, x_2 in X

$$f(x_1 \vee x_2) + f(x_1 \wedge x_2) \leq f(x_1) + f(x_2).$$

Note that the definitions of subadditivity, superadditivity, and sublinearity at a given payoff $x \in X$ require to consider arbitrary finite collection (or K-list) of payoffs, and not just pairs as in the definition of subadditivity, superadditivity, and sublinearity. Clearly, a function $f : X \to \mathbb{R}$ is subadditive (respectively, superadditive, sublinear) if it satisfies the corresponding property at every $x \in X$.

2.3. Arbitrage-free pricing rules

Arbitrage-free pricing rules are standardly defined by the absence of arbitrage opportunities. However, when the pricing rule $f : X \to \mathbb{R}$ is neither assumed to be subadditive nor monotone, arbitrage opportunities of "order" $K \geq 2$, which would otherwise be automatically ruled out by subadditivity and monotonicity, must be explicitly excluded in the definition of arbitrage-free pricing rules as explained below.

We illustrate this point with several examples of arbitrage opportunities for a pricing rule $f : X \to \mathbb{R}$. First, consider the most basic form of arbitrage opportunity: a payoff $x \in X$ such that the total stream of payoffs across time (today and tomorrow) and across all future states involves no payment in any state, while yielding a gain in at least one state, that is, its total payoff is positive:

$$\big(-f(x), x\big) > 0, \text{ or equivalently}$$

$$[(\star)\ x \geq 0 \text{ and } f(x) < 0] \text{ or } [(\star\star)\ x > 0 \text{ and } f(x) \leq 0].$$

Our second example concerns a *"buy and sell" arbitrage opportunity,* that is, a payoff $x \in X$, for which there is a gain in simultaneously buying x and selling it **in full**, that is, $f(x) + f(-x) < 0$, its *aggregate cost* (also called *bid-ask spread*) is negative, thus yielding a gain. Equivalently, the pair $(x, -x) \in X^2$ satisfies $f(x) < -f(-x)$, that is, the *cost* $f(x)$ of *buying* x is smaller than the *gain* $-f(-x)$ of *selling* in full the same payoff x (i.e., buying $-x$). A similar situation arises (i) for payoffs $x \in X$, which yield a gain at $t = 0$ when simultaneously buying x and selling **only part of it**, that is,

$f(x) + f(-x') < 0$ for some $x' \leq x$, or (ii) for pairs $(x, -x') \in X^2$, with a nonegative aggregate payoff, i.e., $x + (-x') \geq 0$, and an aggregate gain at $t = 0$, i.e., $f(x) + f(-x') < 0$.

These situations can be generalized as *"multiple buy and sell" arbitrage opportunities,* involving arbitrary K-lists of payoffs $(x_1, \ldots, x_K) \in X^K$ that yield a gain at $t = 0$, that is, the aggregate cost is negative, i.e., $\sum_{k=1}^{K} f(x_k) < 0$, while the aggregate payoff $\sum_{k=1}^{K} x_k$ is either zero or nonnegative.

When the pricing rule f is both *subadditive and monotone,* the absence of arbitrage opportunities of order 1 as in $(\star)$ (that is, $f(x) \geq 0$ whenever $x \geq 0$) automatically rules out all such *"multiple buy and sell" arbitrage opportunities* $(x_1, \ldots, x_K) \in X^K$ of arbitrary positive order K. Indeed is such an arbitrage opportunity existed we would have:

$$0 = f(0) \leq f(\sum_{k=1}^{K} x_k) \leq \sum_{k=1}^{K} f(x_k) < 0, \text{ a contradiction.}$$

Therefore, in models assuming pricing rules f to be both *monotone and subadditive,* arbitrage-free pricing rules are often defined by requiring only the order 1 Condition $(\star)$ (or possibly the two Conditions $(\star)$ and $(\star\star)$). However, since the goal of this paper is to study pricing rules f that are neither *subadditive nor monotone*, the definitions of arbitrage-free pricing rules will explicitly require the absence of arbitrage opportunities of any positive order K as in Chateauneuf and Cornet (2022b) and Bastianello, Chateauneuf and Cornet (2024).

Definition 2. *Let $f : X \to \mathbb{R}$ be a pricing rule and let $K \in \mathbb{N}, K \geq 1$.*

• *The list $(x_1, \ldots, x_K) \in X^K$ is an arbitrage opportunity of f of order K if its aggregate total payoff is positive, that is, if*

$$(-f(x_1 - \cdots - f(x_K)), x_1 + \cdots + x_K) > 0.$$

Arbitrage opportunities will be said to have

- *zero aggregate payoff if* $\sum_{k=1}^{K} x_k = 0$ *and* $\sum_{k=1}^{K} f(x_k) < 0$,
- *nonnegative aggregate payoff if* $\sum_{k=1}^{K} x_k \geq 0$ *and* $\sum_{k=1}^{K} f(x_k) < 0$,
- *positive aggregate payoff if* $\sum_{k=1}^{K} x_k > 0$ *and* $\sum_{k=1}^{K} f(x_k) \leq 0$.

• *f is said to be arbitrage-free of order K with zero (resp. nonnegative, resp. positive) aggregate payoff if it has no arbitrage opportunity of order K with zero (resp. nonnegative, resp. positive) aggregate payoff, that is,*

$$\forall (x_1,\dots,x_K)\in X^K,\ \sum_{k=1}^K x_k = 0 \ \Rightarrow\ \sum_{k=1}^K\ f(x_k)\geq 0, \qquad (AFK)$$

$$\forall (x_1,\dots,x_K)\in X^K,\ \sum_{k=1}^K x_k \geq 0 \ \Rightarrow\ \sum_{k=1}^K\ f(x_k)\geq 0, \qquad (AFK_+)$$

$$\forall (x_1,\dots,x_K)\in X^K,\ \sum_{k=1}^K x_k > 0 \ \Rightarrow\ \sum_{k=1}^K\ f(x_k)> 0. \qquad (AFK_{++})$$

• *f satisfies the Arbitrage-free Condition AF (resp. AF_+, AF_{++}) if it has no arbitrage opportunity of every order K with zero (resp. nonnegative, positive) aggregate payoff, that is,*

$$\forall K\in\mathbb{N}, K\geq 1, \forall (x_1,\dots,x_K)\in X^K, \sum_{k=1}^K x_k = 0 \ \Rightarrow\ \sum_{k=1}^K f(x_k)\geq 0. (AF)$$

$$\forall K\in\mathbb{N}, K\geq 1, \forall (x_1,\dots,x_K)\in X^K, \sum_{k=1}^K x_k \geq 0 \ \Rightarrow\ \sum_{k=1}^K f(x_k)\geq 0. (AF_+)$$

$$\forall K\in\mathbb{N}, K\geq 1, \forall (x_1,\dots,x_K)\in X^K, \sum_{k=1}^K x_k > 0 \ \Rightarrow\ \sum_{k=1}^K f(x_k)> 0. (AF_{++})$$

The following remark considers arbitrage-free pricing rules of low orders.

Remark 1 ($K = 1, 2, 3$)**.** Let f be a pricing rule.
1. $\mathrm{AF1}_{++} \Leftrightarrow f(x) > 0$ for all $x > 0$,
$\mathrm{AF1}_{+} \Leftrightarrow f(x) \geq 0$ for all $x \geq 0$,
$\mathrm{AF1} \quad \Leftrightarrow f(0) \geq 0$ is always satisfied for a pricing rule f since $f(0) = 0$,
2. $\mathrm{AF2}_{++} \Leftrightarrow -f(-x') < f(x)$ for all x, x' such that $x' < x$,
$\mathrm{AF2}_{+} \Leftrightarrow -f(-x') \leq f(x)$ for all x, x' such that $x' \leq x$,
$\mathrm{AF2} \quad \Leftrightarrow -f(-x) \leq f(x)$ for all $x \in X$.
3. The case $K = 3$ plays an important role in the following and is worth further discussion. Consider a buy & sell arbitrage opportunity of order 3 $(x_1, x_2, x_3) \in X^3$ with a zero aggregate payoff at $t = 1$, that is, $x_1+x_2+x_3 = 0$, and a gain at $t = 0$, or equivalently, a negative aggregate cost, that is, $f(x_1) + f(x_2) + f(x_3) < 0$.

Then $f(x_1) + f(x_2) < -f(-x_1 - x_2)$, the aggregate cost, $f(x_1) + f(x_2)$, of buying x_1 and x_2 *separately*, is smaller than the gain, $-f(-x_1 - x_2)$, in selling x_1 and x_2 *jointly*. Similarly, $f(x_1 + x_2) < -f(-x_1) - f(-x_2)$, the cost, $f(x_1 + x_2)$, of buying x_1 and x_2 *jointly*, is smaller than the aggregate gain, $-f(-x_1) - f(-x_2)$, in selling x_1 and x_2 *separately*. In other words, in the first case the payoff $x_1 + x_2$ has been split in two parts, x_1 and x_2, that are bought separately, and is then sold as a whole. In the second case, the payoff $x_1 + x_2$ is bought as a whole, and it has then been split in two parts, x_1 and x_2, that are sold separately. □

The following proposition summarizes basic properties of arbitrage-free pricing rules. First, Condition AF$_{++}$ is stronger than AF$_{+}$ which is stronger than AF, which is equivalent to the subadditivity of f at 0. Second, each of the three Arbitrage-free Conditions of order K implies the respective Arbitrage-free Condition at every smaller order $k < K$. Third, Assertion (4) provides a subadditive-type characterization of Condition AF (hence also of AF$_{+}$ when f is monotone according to (2)). Finally, when f is subadditive, then f always satisfies the Arbitrage-free Condition AF and it satisfies the Arbitrage-free Condition AF$_{+}$ if f is additionally monotone.

Proposition 1. *Let $f : X \to \mathbb{R}$ be a pricing rule.*

1. *AF$_{++}$ $\Rightarrow$ AF$_{+}$ $\Rightarrow$ AF $\Leftrightarrow$ f is subadditive at* 0*,*
   AFK$_{+}$ $\Rightarrow$ AFK for all positive integer K.
2. *If f is monotone, then AF$_{+}$ $\Leftrightarrow$ AF.*
   If f is strictly monotone, then AF$_{++}$ $\Leftrightarrow$ AF$_{+}$ $\Leftrightarrow$ AF.
3. *For all integers $K \geq k \geq 1$,*
   AFK$_{++}$ $\Rightarrow$ AFk$_{++}$, AFK$_{+}$ $\Rightarrow$ AFk$_{+}$, and AFK$\Rightarrow$ AFk.
4. *Condition AF(K+1) is equivalent to* $(*)$
   $(*)\ -f(-\sum_{k=1}^{K} x_k) \leq \sum_{k=1}^{K} f(x_k)$ *for all* $(x_1, \ldots, x_K) \in X^K$.
5. *If f is subadditive, then f satisfies the Arbitrage-free Condition AF.*
6. *If f is subadditive, then for all $K \in \mathbb{N}, K \geq 2$,*
   AF$_{+}$ $\Leftrightarrow$ AFK$_{+}$ $\Leftrightarrow$ AF1$_{+}$ $\Leftrightarrow$ [$f(x) \geq 0$ for all $x \geq 0$].
7. *If f is subadditive and monotone, then f satisfies AF$_{+}$.*

The proof of Proposition 1 is given in Section 3.1.

We end this section with the definition of the lower monotone envelope of a pricing rule $f : X \to \mathbb{R}$, a concept that will be used throughout the paper in the absence of monotonicity assumption on pricing rules. Let $f : X \to \mathbb{R}$ be a pricing rule, then the function $f_+ : X \to \mathbb{R} \cup \{-\infty\}$ is defined by:

$$f_+(x) := \inf\{f(x') : x' \geq x, x' \in X\} \text{ for } x \in X.$$

The function f_+ is called the lower monotone envelope of f and the following proposition summarizes basic properties of f_+. Note that, at this stage

f_+, cannot be considered yet as a pricing rule since it does not satisfy the two following conditions required previously on pricing rules, namely that (i) $f_+(0) = 0$ and (ii) f_+ is finite-valued. Indeed, the function f_+ may take the value $-\infty$, that is, there may exist payoffs $x \in X$ such that $f(x) = -\infty$, thus yielding an infinite gain. This situation is typically eliminated whenever f is assumed to be arbitrage-free as proved by the following proposition.

Proposition 2 (Monotonicity and f_+)**.** *Let $f : X \to \mathbb{R}$ be a pricing rule.*

1. *f_+ is the greatest monotone[1] function below f,*
   that is, f_+ is monotone, $f_+ \leq f$, and $f_+ \geq g$ for all monotone function $g : X \to \mathbb{R} \cup \{-\infty, +\infty\}$, $g \leq f$.
2. *$f_+(0) = 0$ if and only if $f(x) \geq 0$ for all $x \geq 0$.*
3. *f satisfies AF+ $\Leftrightarrow$ f_+ is a (finite-valued) pricing rule that satisfies AF.*
4. *Assume f is monotone. Then $f = f_+$.*
   f satisfies AF+ $\Leftrightarrow$ f satisfies AF.
   f is bounded on a neighborhood of 0, more precisely
   $$f(-\mathbf{1}_\Omega) \leq f(x) \leq f(\mathbf{1}_\Omega) \text{ for all } x \in B_\infty(0, 1).$$

The proof of Proposition 2 is given in Section 3.1.

2.4. Frictionless pricing rules

A pricing rule is said to be frictionless if it exhibits no market frictions of any positive order K. Various forms of market frictions (of order K) are introduced analogously to the treatment of arbitrage opportunities. In fact, we will establish a one-to-one correspondence between arbitrage opportunities and market frictions, through the notion of the dual *pricing rule* f^* associated with the original pricing rule f.

For $K = 2$ a buy & sell market friction of f is a 2-list $(x, -x)$, or simply a payoff $x \in X$, for which simultaneously buying x and selling it *in full* (thus, $x + (-x) = 0$) generates a positive aggregate cost, that is, $f(x) + f(-x) > 0$. Equivalently, this condition can be written as $-f(-x) < f(x)$, that is, the *cost*, $f(x)$, of *buying* x is greater than the *gain*, $-f(-x)$, of *selling* it. A

[1] The monotonicity of a function $g : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ (hence in particular for f_+) is defined in the same way as for finite-valued functions, that is, $x_1 \leq x_2 \Rightarrow g(x_1) \leq g(x_2)$.

similar situation arises when, simultaneously more of the payoff is sold than bought. Formally, we consider a pair $(x, -x') \in X^2$, where x is bought and $x' \geq x$ is sold simultaneously, while generating a nonpositive aggregate payoff, $x + (-x') \leq 0$, and a positive aggregate cost at time $t = 0$, i.e., $f(x) + f(-x') > 0$.

These notions can be generalized to define *"multiple buy and sell" market frictions,* as K-lists of payoffs $(x_1, \ldots, x_K) \in X^K$ (with K being an arbitrary positive integer) generating a positive aggregate cost at $t = 0$, i.e., $\sum_{k=1}^{K} f(x_k) > 0$, and an aggregate payoff $\sum_{k=1}^{K} x_k$ that is either zero or nonpositive.

The following definition formally introduces the notions of market frictions and frictionless pricing rules that we have heuristically presented before.

Definition 3. *Let $f : X \to \mathbb{R}$ be a pricing rule and let $K \in \mathbb{N}, K \geq 1$.*

- *The list $(x_1, \ldots, x_K) \in X^K$ is a market friction of f of order K*

- with zero aggregate payoff if $\sum_{k=1}^{K} x_k = 0$ *and* $\sum_{k=1}^{K} f(x_k) > 0$,

- with nonnegative aggregate payoff if $\sum_{k=1}^{K} x_k \leq 0$ *and* $\sum_{k=1}^{K} f(x_k) > 0$.

- *f is said to be frictionless of order K with zero (resp. nonnegative, resp. positive) aggregate payoff if it has no market friction of order K with zero (resp. nonnegative, resp. positive) aggregate payoff, that is,*

$$\forall (x_1, \ldots, x_K) \in X^K,\ \sum_{k=1}^{K} x_k = 0 \ \Rightarrow\ \sum_{k=1}^{K} f(x_k) \leq 0, \qquad (FLK)$$

$$\forall (x_1, \ldots, x_K) \in X^K,\ \sum_{k=1}^{K} x_k \leq 0 \ \Rightarrow\ \sum_{k=1}^{K} f(x_k) \leq 0, \qquad (FLK_+)$$

- *f satisfies the Frictionless Condition FL (resp. FL_+) if it has no market friction of every order K with zero (resp. nonegative) aggregate payoff, that is*

$$\forall K \in \mathbb{N}, K \geq 1, \forall (x_1, \ldots, x_K) \in X^K, \sum_{k=1}^{K} x_k = 0 \ \Rightarrow\ \sum_{k=1}^{K} f(x_k) \leq 0. (FL)$$

$$\forall K \in \mathbb{N}, K \geq 1, \forall (x_1, \ldots, x_K) \in X^K, \sum_{k=1}^{K} x_k \leq 0 \ \Rightarrow\ \sum_{k=1}^{K} f(x_k) \leq 0. (FL_+)$$

- *The pricing rule $f : X \to \mathbb{R}$ is said to have at $x \in X$*

- a positive bid-ask spread if $f(x) + f(-x) > 0$, or equivalently if $(x, -x)$ is

a market friction[2] *of order 2 (or if x is a buy & sell market friction),*

- a negative bid-ask spread if $f(x)+f(-x) < 0$, or equivalently if $(x, -x)$ is an arbitrage opportunity of order 2 (or x is a buy & sell arbitrage opportunity),

The pricing rule f is said to have a zero (resp. nonnegative, positif, nonpositive, negative) bid-ask spread if $f(x) + f(-x) = 0$ (resp. $\geq 0, > 0, \leq 0, < 0$) for all $x \in X$.

The following remark considers frictionless pricing rules of low orders $K = 1, 2, 3$ and is the analogue of Remark 1, which addressed arbitrage-free pricing rules.

Remark 2 ($K = 1, 2$)**.** Let f be a pricing rule.
1. FL1$_+$ $\Leftrightarrow$ $f(x) \leq 0$ for all $x \leq 0$,
   FL1 $\Leftrightarrow$ $f(0) \leq 0$ is always satisfied for a pricing rule f (since $f(0) = 0$),
2. FL2$_+$ $\Leftrightarrow$ $-f(-x') \geq f(x)$ for all x, x' such that $x' \geq x$,
   FL2 $\Leftrightarrow$ $-f(-x) \geq f(x)$ for all $x \in X$.
2'. [AF2$_+$ and FL2$_+$] $\Rightarrow$ [AF2 and FL2] $\Leftrightarrow$ f has no Bid-ask Spread,
   and the three conditions are equivalent if f is monotone. [3]

Note that in the case $K = 2$, the no bid-ask spread property of f intuitively means that there is neither gain nor loss in simultaneously buying and selling a payoff x, since the (ask) buying price $f(x)$ of x is equal to its (bid) selling price $-f(-x)$.

3. The case $K = 3$ can be interpreted similarly. Consider a 3-list of payoffs $(x_1, x_2, x_3) \in X^3$ such that $x_1 + x_2 + x_3 = 0$, that is neither an arbitrage opportunity, nor a market friction, then the aggregate cost is zero, and assume it has a zero aggregate cost, $f(x_1) + f(x_2) + f(x_3) = 0$. Writing the above equality equivalently as $f(x_1) + f(x_2) = -f(-x_1 - x_2)$ can also be interpreted as follows: the cost $f(x_1) + f(x_2)$ of buying x_1 and x_2 *separately*, is equal to the gain, $-f(-x_1 - x_2)$, from jointly selling x_1 and x_2. $\square$

[2] No need here and below for a couple $(x, -x)$ to precise "with zero aggregate payoff" since $x + (-x) = 0$.

[3] This follows from the two equivalence properties [AF2$_+$ $\Leftrightarrow$ AF2] and [FL2$_+$ $\Leftrightarrow$ FL2] that hold when f is monotone by Proposition 1 and Proposition 3.

Proposition 3. *Let $f : X \to \mathbb{R}$ be a pricing rule.*

1. *$FL_+ \Rightarrow FL \Leftrightarrow f$ is superadditive at* 0*,*
   $FLK_+ \Rightarrow FLK$ for all positive integer K;
2. *If f is monotone, then $FL_+ \Leftrightarrow FL$;*
3. *For all integers $K \geq k \geq 1$*
   FLk_{++}, $FLK_{\ +} \Rightarrow FLk_+$, and $FLK \Rightarrow FLk$.
4. *FL(K+1) is equivalent to* $(*)$
   $$(*)\ -f(-\sum_{k=1}^{K} x_k) \geq \sum_{k=1}^{K} f(x_k) \text{ for all } (x_1, \ldots, x_K) \in X^K.$$
5. *If f is superadditive, then f satisfies the Market Friction Condition FL;*
6. *If f is superadditive, then for all $K \in \mathbb{N}, K \geq 2$*
   $FL_+ \Leftrightarrow FLK_+ \Leftrightarrow FL1_+ \Leftrightarrow [f(x) \leq 0$ for all $x \leq 0]$
7. *If f is superadditive and monotone, then f satisfies FL_+.*

The proof of Proposition 3 follows directly from Proposition 1, together with the following result, which establishes a duality between the notions of Arbitrage-free and Market Frictionless pricing rules. This duality is formulated via the dual pricing rule $f^* : X \to \mathbb{R}$ associated with a given pricing rule $f : X \to \mathbb{R}$, and it is defined by

$$f^*(x) = -f(-x) \text{ for all } x \in X.$$

The following result summarizes the key properties of f^*, and formally shows the relationship between the Arbitrage-free Conditions for f and the Market Frictions Conditions for f^*, and vice versa.

Proposition 4. *Let $f : X \to \mathbb{R}$ be a pricing rule.*

1. *f^* is a pricing rule, that is, $f^*(0) = 0$, and satisfies $(f^*)^* = f$.*
2. *f is monotone if and only if f^* is monotone.*
3. *f is subadditive if and only if f^* is superadditive.*
4. *Let K be a positive integer.*
   f satisfies AFK (resp. FLK) $\Leftrightarrow f^*$ satisfies FLK (resp. AFK),
   f satisfies AFK_+ (resp. FLK_+) $\Leftrightarrow f^*$ satisfies FLK_+ (resp. AFK_+).

The proof of Proposition 4 is given in Section 3.1.

2.5. Various

Proposition 5. *Let $f : X \to \mathbb{R}$ be a pricing rule and let $x \in X$.*

1. f is linear at $x \iff f$ is sublinear at x and at $-x \iff f$ and $-f$ are sublinear at x

Proof. (1) If f is sublinear at $x \in X$ and at $-x$, for all positive integer K, for all $(x_1, \ldots, x_K) \in X^K$, and all $(\theta_1, \ldots, \theta_K) \in \mathbb{R}_+^K$,

$$\theta_1 x_1 + \cdots + \theta_K x_K = x \Rightarrow \theta_1 f(x_1) + \cdots + \theta_K f(x_K) \geq f(x);$$
$$\theta_1(-x_1) + \ldots \theta_K(-x_K) = -x \Rightarrow \theta_1 f(x_1) + \cdots + \theta_K f(x_K) \geq f(x);$$

□

2.6. Bounding the order of arbitrage-free and frictionless pricing rules

When the pricing rule f is subadditive and monotone, it is arbitrage-free if and only if f is arbitrage-free of order 1. More precisely, the Arbitrage-free Conditions AFK and and AFK+ hold for every positive integer K (see Proposition 1). In other words, the assumption of subadditivity on f ensures that all arbitrage opportunities of any positive order K, with zero or nonnegative aggregate payoffs, are automatically excluded.

Our next result identifies an important situation in which arbitrage opportunities of high orders become irrelevant, even without assuming subadditivity of the pricing rule f. Specifically, when the payoff space is finite dimensional, that is, when the state space Ω is finite, all arbitrage opportunities and market frictions of order greater than $\#\Omega + 1$ are irrelevant. This is established in the result below under the additional assumption that f is positively homogeneous, a property satisfied, for example, in Section 2.8 when f is a Choquet pricing rule.

Theorem 1. *Let Ω be finite and let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be positively homogeneous.*
1. Then f satisfies the Arbitrage-free Condition AF++ (resp. AF+, AF) if and only if it satisfies Condition AFK++ (resp. AFK+, AFK) for all positive order $K \leq \#\Omega + 1$.
2. Then f satisfies the Frictionless Condition FL++ (resp. FL+, FL) if and only if it satisfies Condition FLK+ (resp. FLK+, FLK) for all positive order $K \leq \#\Omega + 1$.

The proof of Theorem 1 is given in Section 3.2.

2.7. Characterizing additive and linear pricing rules

When the pricing rule f is both arbitrage-free and frictionless, it follows that f is additive and linear, a classical characterization of complete markets. This is established in Theorem 2 when f satisfies Conditions AF+ and FL+ and in Theorem 3 when f satisfies the weaker Conditions AF and FL but under an additional but mild continuity-type assumption.

Importantly, both Theorem 2 and Theorem 3 demonstrate that it is not necessary to impose arbitrage-free and frictionless conditions for all orders $K \geq 1$. It suffices to verify these conditions for $K = 3$ and even one of these two Conditions (arbitrage-free or frictionless) needs only to be satisfied at order 2. However, the restriction to order 2 for both conditions is generally insufficient to guarantee additivity, as shown by Example 2, even when f is a Choquet pricing rule.

Theorem 2. *Let f be a pricing rule, the following assertions are equivalent:*

1. f satisfies the Conditions AF+ and FL+;

2. f satisfies the Conditions AF3+ and FL2+;

2'. f satisfies the Conditions AF2+ and FL3+;

3. f is additive and monotone;

4. f is linear and monotone.

The proof of Theorem 2 is given in Section 3.3 as a consequence of the following result that considers the case of nonmonotone pricing rules.

Theorem 3. *Let $f : X \to \mathbb{R}$ be a pricing rule.*

(a) The following assertions are equivalent:

(i) f satisfies the Conditions AF and FL;

(ii) f satisfies the Conditions AF3 and FL2;

(ii') f satisfies the Conditions AF2 and FL3;

(iii) f is additive.

(b) Moreover, if f is bounded on some neighborhood of 0 (which holds if f is either continuous at 0 or monotone), then (1)-(3) are also equivalent to

(iv) f is linear.

The proof of Theorem 3 is given in Section 3.3. We point out that the additional assumption in Part (b) is very mild and is satisfied either under monotonicity or under continuity of the pricing rule f, respectively as in Theorem 2 and Theorem 7 hereafter, which will be both proved as a direct consequence of Theorem 3.

One can immediately deduce the following results from Theorem 3.

Corollary 1. *Let $f : X \to \mathbb{R}$ be a pricing rule that satisfies AF3.*

Then f is additive if and only if $f(x) + f(-x) \leq 0$ for all $x \in X$.

Thus, if f is not additive, then f has some positive bid-ask spreads, that is, there exists $x \in X$ such that $f(x) + f(-x) > 0$.

Corollary 2. *Let $f : X \to \mathbb{R}$ be a pricing rule such that $f(x) + f(-x) \leq 0$ for all x (hence in particular if f has no bid-ask spread).*

Then f is additive if and only if f satisfies AF3.

Thus, if f is not additive it has an arbitrage opportunity of order 3 with zero aggregate payoff.

2.8. Choquet pricing rules

The set Ω is now equipped with an algebra $\mathcal{A} \subseteq 2^{\Omega}$ (not assumed to be σ-algebra), of subsets of Ω, that is, $\mathcal{A}$ contains the whole set Ω and is stable by union and complementation. A **game** is a set-function $v : \mathcal{A} \to \mathbb{R}$ such that $v(\emptyset) = 0$ (and v is not assumed to be a capacity). Following Aumann and Shapley (1974), the variation norm of a game v is defined as follows:

$$\|v\| := \sup \left\{ \textstyle\sum_{k=1}^{K} |v(A_k) - v(A_{k-1})| \,:\, (A_k)_{k=0}^{K} \text{ finite chain in } \mathcal{A} \right\},$$

where, by finite chain $(A_k)_{k=0}^{K}$ in $\mathcal{A}$, we mean that $A_k \in \mathcal{A}$ for all $k = 1, \dots, K$ and $\emptyset = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_K = \Omega$. We denote by:

$$bv(\mathcal{A}) := \left\{ v : \mathcal{A} \to \mathbb{R} \,:\, v(\emptyset) = 0 \text{ and } \|v\| < +\infty \right\},$$

the set of games having a finite variation norm. The variation norm is indeed a norm on the space $bv(\mathcal{A})$, which is a Banach space for this norm. Moreover, $bv(\mathcal{A})$ contains the class of finite games, the class of capacities, i.e., monotone games, and also the set $ba(\mathcal{A})$ of bounded charges, i.e., additive games of bounded variation. [4] For all these properties we refer to Aumann and Shapley (1974), Denneberg (1994), and Marinacci and Montrucchio (2004).

[4] Indeed $bv(\mathcal{A})$ contains the class of all finite games (i.e., Ω is finite) since there are finitely many finite chains. Moreover, $bv(\mathcal{A})$ contains all capacities $v : \mathcal{A} \mapsto \mathbb{R}$ since, using

A *capacity* v on the measurable space $(\Omega, \mathcal{A})$ is a game $v : \mathcal{A} \mapsto \mathbb{R}$, which is monotone, that is, for all $A, B \in \mathcal{A}$, $A \subseteq B$ implies $v(A) \leq v(B)$. The capacity is said to be normalized, also called *non-additive probability*, if $v(\Omega) = 1$. A *probability* $\mu : \mathcal{A} \mapsto \mathbb{R}$ is a normalized capacity which is (finitely) additive, that is, $A \cap B = \emptyset$ implies $\mu(A \cup B) = \mu(A) + \mu(B)$.

Consider a game $v \in bv(\mathcal{A})$, the function f is said to be a *Choquet pricing rule* with respect to v if, for all $x \in X$,

$$f(x) = \int_{\Omega}^{C} x\, dv := \int_{-\infty}^{0} (v(\{x \geq t\}) - v(\Omega))\, dt + \int_{0}^{+\infty} v(\{x \geq t\})\, dt,$$

where $\{x \geq t\} = \{\omega \in \Omega | x(\omega) \geq t\}$. The notation $\int^{C}$ indicates that we consider a Choquet integral. We will use the notation $\int$ for the standard integral with respect to a bounded charge $\mu \in ba(\mathcal{A})$, and in this case, the pricing rule f is *linear*. We recall some properties of the Choquet pricing rule $f : X \to \mathbb{R}$ with respect to the game $v \in bv(\mathcal{A})$. For general references on Choquet integral, we refer to Denneberg (1994), Grabisch (2016), and Marinacci and Montrucchio (2004).

- The game $v \in bv(\mathcal{A})$, called the *risk-neutral game* is uniquely defined by f and satisfies $v(A) = f(\mathbf{1}_A)$ for all $A \subseteq \Omega$;
- f is positively homogeneous;
- f is constant additive, i.e., $f(x + t\mathbf{1}_\Omega) = f(x) + tf(t\mathbf{1}_\Omega)$ for all x and all $t \in \mathbb{R}$, hence in particular $f(-\mathbf{1}_\Omega) = f(\mathbf{1}_\Omega)$;
- f is continuous.

2.9. Additive pricing rules

This result is taken from Aliprantis and Border Theorem 5.54 page 197.

Theorem 4. *Let X be a Hausdorff locally convex topological vector space and let $f : X \to \mathbb{R}$ be sublinear. Then f is linear if and only if it dominates exactly*

---

the monotonicity of v, for all finite chains (A_k), one gets
$\sum_{k=1}^{K} |v(A_k) - v(A_{k-1})| = \sum_{k=1}^{K} v(A_k) - v(A_{k-1}) = v(\Omega) - v(\emptyset) = v(\Omega)$.

Consequently, $\|v\| = v(\Omega) < \infty$. An additive game μ is called a charge (or a signed charge) and we point out that the total variation norm of a charge $\|\mu\|$ is exactly the variation norm of the game μ defined above.

one linear functional on X, that is there exists a unique linear functional $p : X \to \mathbb{R}$ such that $p(x) \le f(x)$ for all x.

We will use the following theorem in Aliprantis and Border Theorem 5.53 page 195.

Theorem 5 (Hahn-Banach Theorem)**.** *Let X be a vector space and let $f : X \to \mathbb{R}$ be a convex function and let M be a vector subspace of X.*

If $\varphi : M \to \mathbb{R}$ is a linear function dominated by f on M, that is, $\varphi(x) \le f(x)$ for all $x \in M$, then there exists a linear extension (not generally unique) $\hat{\varphi} : X \to \mathbb{R}$ of φ on X which is also dominated by f on X.

Proof. First let $f : X \to \mathbb{R}$ be a sublinear functional. First, assume that f is linear and there exists $p : X \to \mathbb{R}$ linear such that $p(x) \le f(x)$ for all $x \in X$. Then $-p(x) = p(-x) \le f(-x) = -f(x)$, thus $f(x) \le p(x)$ for all $x \in X$. Hence, $p = f$.

Now assume that f dominates exactly one linear functional on X. Note that the sublinear function f is linear if and only if $f(-x) \ge -f(x)$ for each $x \in X$. So if we assume by way of contradiction that f is not linear, then there exists some $x_0 \neq 0$ such that$-f(-x_0) < f(x_0)$. Let $M = \{\lambda x_0 : \lambda \in \mathbb{R}\}$, the vector subspace generated by x_0, and define the linear functionals $p, q : M \to \mathbb{R}$ by $p(\lambda x_0) = \lambda f(x_0)$ and $q(\lambda x_0) = -\lambda f(-x_0)$. Since $q(x_0) = -f(-x_0) < f(x_0) = p(x_0)$, we see that $p \neq q$.

Next, notice that $p = f$ and $q = f$ on the subspace M; indeed, $z \in M$, then $z := \lambda x_0$ for some $\lambda \in \mathbb{R}$, thus $p(z) := \lambda f(x_0) = f(z)$ and $q(z) := -\lambda f(-x_0) = f(z)$. This proves in particular that f dominates both p and q on the subspace M. Now by the Hahn-Banach Theorem, the two distinct linear functionals p and q have linear extensions to all of X that are dominated by f, a contradiction. □

2.10. Arbitrage-free and frictionless Choquet pricing rules

We have shown previously that a pricing rule f is additive if and only if it is arbitrage-free and frictionless. More precisely, additivity holds under the low order Arbitrage-free and Frictionless Conditions AF3+ and FL2+ (Theorem 2) when f is monotone, and under Conditions AF3 and FL2 (Theorem 3) when f is not assumed to be monotone.

In this section, we consider the case where f is a Choquet pricing rule with respect to a game v. The main result (Theorem 2) characterizes the additivity of f in terms of properties of its associated game v.

We begin with two preliminary results that address the Arbitrage-free Conditions AF2 and AF3, respectively. Let $v : \mathcal{A} \to \mathbb{R}$ be a game, then the dual game $v^* : \mathcal{A} \to \mathbb{R}$ is defined by:

$$v^*(A) := v(\Omega) - v(A^c) \text{ for } A \in \mathcal{A},$$

and we recall the two following standard properties (i) $v^*(\emptyset) = 0$ and $v^*(\Omega) = v(\Omega)$, and (ii) $f^*(\mathbf{1}_A) = v^*(A)$ for all $A \in \mathcal{A}$, where f^* denotes the dual pricing rule of f, as defined earlier.

The first characterization of order 2 is taken from Chateauneuf and Cornet (2022b) and Bastianello, Chateauneuf and Cornet (2024).

Proposition 6. *Let f be a Choquet pricing rule with respect to $v \in bv(\mathcal{A})$.*

1. *f satisfies AF2* $\Leftrightarrow [v(A) + v(A^c) \geq v(\Omega)\ \forall A \in \mathcal{A}] \Leftrightarrow v \geq v^*$;
2. *f satisfies FL2* $\Leftrightarrow [v(A) + v(A^c) \leq v(\Omega)\ \forall A \in \mathcal{A}] \Leftrightarrow v \leq v^*$;
3. *No bid-ask spread* $\Leftrightarrow [v(A) + v(A^c) = v(\Omega)\ \forall A \in \mathcal{A}] \Leftrightarrow v = v^*$.

We introduce the following submodular-type assumption on v

(**) $v^*(A \cup B) + v^*(A \cap B) \leq v(A) + v(B)$ for all A, B in $\mathcal{A}$

that will be used in the next result in order to characterize the additivity of the pricing rule f. This assumption is related to the characterization of the arbitrage-free Condition AF3 (Condition (*) of Proposition 1)

(*) $f^*(x_1 + x_2) \leq f(x_1) + f(x_2)\ \ \forall (x_1, x_2) \in X^2$.

Note that the Condition (*) is satisfied when f is subadditive, whereas Condition (**) is satisfied when v is a submodular game.

Proposition 7. *Let be Choquet pricing rule with respect to $v \in bv(\mathcal{A})$.*

1. *AF3 $\Rightarrow$ (**) $\Rightarrow v^* \leq v$.*
2. *AF3 and FL2 $\Leftrightarrow$ (**) and $v \leq v^* \Leftrightarrow v$ additive $\Leftrightarrow f$ additive.*

The proof of Proposition 9 is provided in Section 3.4.

Theorem 6. *Let $f : X \to \mathbb{R}$ Choquet pricing rule with respect to a game $v \in bv(\mathcal{A})$, then the following conditions are equivalent:*

1. f satisfies the Conditions AF and FL;

2.1. f satisfies the Conditions AF3 and FL2;

2.2. f satisfies the Conditions AF2 and FL3;

2.3. $v \le v^*$ and v satisfies

(**) $v^*(A \cup B) + v^*(A \cap B) \le v(A) + v(B)$ for all A, B in $\mathcal{A}$;

3.1. v is additive;

3.2. f is additive;

4. f is linear.

The proof of Theorem 7 is provided in Section 3.4. We emphasize that Assumption (2.1) in Theorem 7 cannot be weakened by only requiring that f satisfies the Arbitrage-free and Frictionless Conditions of order 2 (i.e., AF2 and FL2). This point is established in Example 2, which exhibits a monotone Choquet pricing rule f that is not linear, and has no bid-ask spreads (thus satisfies the arbitrage-free and frictionless Conditions of orders 1 and 2, namely, AF1, AF2, FL1 and FL2). The example also exhibits explicit arbitrage opportunities and market frictions of order 3, thus implying that both AF3 and FL3 are not satisfied.

We now deduce from Theorem 7 the following results.

Corollary 3. *Let $f : X \to \mathbb{R}$ be Choquet pricing rule that satisfies AF3. Then f is linear if and only if $f(x) + f(-x) \le 0$ for all $x \in X$.*

Thus, if f is not linear, then f has some positive bid-ask spreads, that is, there exists $x \in X$ such that $f(x) + f(-x) > 0$.

Corollary 4. *Let $f : X \to \mathbb{R}$ be a Choquet pricing rule such that $f(x) + f(-x) \le 0$ for all x (hence in particular if f has no bid-ask spread).*

Then f is linear if and only if f satisfies AF3.

Thus, if f is not linear it has an arbitrage opportunity of order 3.

2.11. Arbitrage-free and frictionless Choquet pricing rules

We have shown previously that a pricing rule f is additive if and only if it is arbitrage-free and frictionless. More precisely, additivity holds under the low order Arbitrage-free and Frictionless Conditions AF3+ and FL2+ (Theorem 2) when f is monotone, and under Conditions AF3 and FL2 (Theorem 3) when f is not assumed to be monotone.

In this section, we consider the case where f is a Choquet pricing rule with respect to a game v. The main result (Theorem 2) characterizes the additivity of f in terms of properties of its associated game v.

We begin with two preliminary results that address the Arbitrage-free Conditions AF2 and AF3, respectively. Let $v : \mathcal{A} \to \mathbb{R}$ be a game, then the dual game $v^* : \mathcal{A} \to \mathbb{R}$ is defined by:

$$v^*(A) := v(\Omega) - v(A^c) \text{ for } A \in \mathcal{A},$$

and we recall the two following standard properties (i) $v^*(\emptyset) = 0$ and $v^*(\Omega) = v(\Omega)$, and (ii) $f^*(\mathbf{1}_A) = v^*(A)$ for all $A \in \mathcal{A}$, where f^* denotes the dual pricing rule of f, as defined earlier.

The first characterization of order 2 is taken from Chateauneuf and Cornet (2022b) and Bastianello, Chateauneuf and Cornet (2024).

Proposition 8. *Let f be a Choquet pricing rule with respect to $v \in bv(\mathcal{A})$.*

1. *f satisfies AF2* $\Leftrightarrow [v(A) + v(A^c) \geq v(\Omega)\ \forall A \in \mathcal{A}] \Leftrightarrow v \geq v^*$;
2. *f satisfies FL2* $\Leftrightarrow [v(A) + v(A^c) \leq v(\Omega)\ \forall A \in \mathcal{A}] \Leftrightarrow v \leq v^*$;
3. *No bid-ask spread* $\Leftrightarrow [v(A) + v(A^c) = v(\Omega)\ \forall A \in \mathcal{A}] \Leftrightarrow v = v^*$.

We introduce the following weak submodularity assumption on v

(**) $v^*(A \cup B) + v^*(A \cap B) \leq v(A) + v(B)$ for all A, B in $\mathcal{A}$

that will be used in the next result in order to characterize the additivity of the pricing rule f. This assumption is related to the characterization of the arbitrage-free Condition AF3 (Condition (*) of Proposition 1)

(*) $f^*(x_1 + x_2) \leq f(x_1) + f(x_2)\ \ \forall (x_1, x_2) \in X^2$.

Note that the Condition (*) is satisfied when f is subadditive, whereas Condition (**) is satisfied when v is a submodular game.

v submodular $\Rightarrow (*) \Rightarrow (**) \iff (***) \Rightarrow v^* \leq v$ where

$(***)\ \ f^*(x_1 \vee x_2) + f^*(x_1 \wedge x_2) \leq f(x_1) + f(x_2)\ \ \forall (x_1, x_2) \in X^2$.

Proof. • [[(**) $\Rightarrow$ $(***)$] Let $(x_1, x_2) \in X^2$. Since f is constant additive, it is sufficient to prove the assertion for all $x \geq 0$ and $y \geq 0$. First one has, for all $t \in \mathbb{R}$

$$\{x \vee y \geq t\} = \{x \geq t\} \cup \{y \geq t\} \text{ and } \{x \wedge y \geq t\} = \{x \geq t\} \cap \{y \geq t\}.$$

Hence, from (**), v satisfies

$$v^*(x \vee y \geq t) + v^*(x \wedge y \geq t) \leq v(x \geq t) + v(y \geq t).$$

Consequently

$$\begin{aligned} f^*(x \vee y) + f^*(x \wedge y) &= \int_0^\infty v^*(x \vee y \geq t)\, dt + \int_0^\infty v^*(x \wedge y \geq t)\, dt \\ &= \int_0^\infty [v^*(x \vee y \geq t) + v^*(x \wedge y \geq t)]\, dt \\ &\leq \int_0^\infty v(x \geq t)\, dt + \int_0^\infty v(y \geq t)\, dt = f(x) + f(y). \end{aligned}$$

• [$(***)$ $\Rightarrow$ (**)] In $(***)$, take $x = \mathbf{1}_A$ and $y = \mathbf{1}_B$, since $\mathbf{1}_A \vee \mathbf{1}_B = \mathbf{1}_{A \cup B}$ and $\mathbf{1}_A \wedge \mathbf{1}_B = \mathbf{1}_{A \cap B}$, we get

$$\begin{aligned} v^*(A \cup B) + v^*(A \cap B) &= f^*(\mathbf{1}_{A \cup B}) + f^*(\mathbf{1}_{A \cap B}) \\ &= f^*(\mathbf{1}_A \vee \mathbf{1}_B) + f^*(\mathbf{1}_A \wedge \mathbf{1}_B) \\ &\leq f(\mathbf{1}_A) + f(\mathbf{1}_B) = v(A) + v(B). \end{aligned}$$

□

Proposition 9. *v submodular $\Rightarrow$ v balanced $\Rightarrow$ (**) v weakly submodular $\Rightarrow v^* \leq v$*

We have $\frac{1}{2}\mathbf{1}_{A \cup B} + \frac{1}{2}\mathbf{1}_{A \cap B} + \mathbf{1}_A + \mathbf{1}_B = \mathbf{1}_\Omega$

[v submodular and $v^* \geq v$] $\iff$ [(**) and $v^* \geq v$] $\iff$ v additive.

Let f be a pricing rule

f subadditive $\Rightarrow$ (**) v weakly subadditive $\Rightarrow f^* \leq f$

[f subadditive and $f^* \geq f$] $\iff$ [(*) and $f^* \geq f$] $\iff$ f additive.

Let be Choquet pricing rule with respect to $v \in bv(\mathcal{A})$.

1. *v submodular $\Rightarrow$ [(*) and $v^* \leq v$].*

1. *[(*) and $v \leq v^*$] $\Leftrightarrow$ v additive.*
1. *AF3 $\Rightarrow$ (**) $\Rightarrow v^* \leq v$.*
2. *AF3 and FL2 $\Leftrightarrow$ (**) and $v \leq v^*$ $\Leftrightarrow$ v additive $\Leftrightarrow$ f additive.*

:KEEP Example of $v = v^*$ but v is not additive

Example 1. [$v = v^*$ but v is not additive]Note first that for $\Omega := \{1, 2\}$, then $v = v^*$ implies v additive.Indeed, $v(\{1\}) = v^*(\{1\}) = v(\Omega) - v(\{2\})$. Thus, $v(\{1\}) + v(\{2\}) = v(\Omega)$, that is v is additive.

Thus, our counterexample should consider at least 3 states. Let $v : 2^\Omega \to \mathbb{R}$ be defined by

$v(\{1\}) = v(\{2\}) = 1/4, v(\{1, 2\}) = 1.$

Then, $v = v^*$ but v is not additive since $3/4 = v(\{1, 2\}) \neq v(\{1\}) + v(\{2\}) = 1/4 + 1/4.$ $\square$

Take $\Omega := \{1, 2, 3\}$, let $v : 2^\Omega \to \mathbb{R}$ be defined by

$v(\{1\}) = v(\{2\}) = v(\{3\}) = 1/4, v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 3/4, v(\{1, 2, 3\}) = 1.$

Then, $v = v^*$ but v is not additive since $3/4 = v(\{1, 2\}) \neq v(\{1\}) + v(\{2\}) = 1/4 + 1/4.$ $\square$

The proof of Proposition 9 is provided in Section 3.4.

Theorem 7. *Let $f : X \to \mathbb{R}$ Choquet pricing rule with respect to a game $v \in bv(\mathcal{A})$, then the following conditions are equivalent:*

1. f satisfies the Conditions AF and FL;

2.1. f satisfies the Conditions AF3 and FL2;

2.2. f satisfies the Conditions AF2 and FL3;

2.3. $v \leq v^*$ and v satisfies

(**) $v^*(A \cup B) + v^*(A \cap B) \leq v(A) + v(B)$ for all A, B in $\mathcal{A}$;

3.1. v is additive;

3.2. f is additive;

4. f is linear.

The proof of Theorem 7 is provided in Section 3.4. We emphasize that Assumption (2.1) in Theorem 7 cannot be weakened by only requiring that f satisfies the Arbitrage-free and Frictionless Conditions of order 2 (i.e., AF2 and FL2). This point is established in Example 2, which exhibits a monotone Choquet pricing rule f that is not linear, and has no bid-ask spreads (thus satisfies the arbitrage-free and frictionless Conditions of orders 1 and 2, namely, AF1, AF2, FL1 and FL2). The example also exhibits explicit arbitrage opportunities and market frictions of order 3, thus implying that both AF3 and FL3 are not satisfied.

We now deduce from Theorem 7 the following results.

Corollary 5. *Let $f : X \to \mathbb{R}$ be Choquet pricing rule that satisfies AF3. Then f is linear if and only if $f(x) + f(-x) \leq 0$ for all $x \in X$.*

Thus, if f is not linear, then f has some positive bid-ask spreads, that is, there exists $x \in X$ such that $f(x) + f(-x) > 0$.

Corollary 6. *Let $f : X \to \mathbb{R}$ be a Choquet pricing rule such that $f(x) + f(-x) \leq 0$ for all x (hence in particular if f has no bid-ask spread).*

Then f is linear if and only if f satisfies AF3.

Thus, if f is not linear it has an arbitrage opportunity of order 3.

2.12. An example

This section presents an Example 2 which exhibits a pricing rule that is *not linear* but is monotone, has no bid-ask spread (or equivalently satisfies AF2 and FL2), and has an arbitrage opportunity and a market friction of order 3.

Example 2. Let $\Omega := \{1, 2, 3\}$, $v : 2^\Omega \to \mathbb{R}$ defined by $v(\emptyset) = 0$, $v(\Omega) = 1$,

$v(\{1\}) = v(\{2\}) = v(\{3\}) = 1/4, v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 3/4$, and let $f : \mathbb{R}^\Omega \to \mathbb{R}$ be the Choquet pricing rule associated with v.

Then, the following holds:

1. f is monotone.
2. f is 2-arbitrage-free and 2-frictionless, more precisely:
   - ⋆ $f(x) \geq 0$ for all $x \geq 0$, i.e., AF1 and AF1$_+$ hold;
   - ⋆ f has no bid-ask $[f(x) + f(-x) = 0$ for all $x]$, i.e., AF2 and FL2 hold;
   - ⋆ AF2+ and FL2+ hold.
3. f has an arbitrage opportunity $(\mathbf{1}_1, \mathbf{1}_2, -\mathbf{1}_{12})$ of order 3, (i.e., buying $\mathbf{1}_1$, $\mathbf{1}_2$ separately and selling $\mathbf{1}_{12}$ yields a zero payoff tomorrow and a gain today) and f has a market friction $(-\mathbf{1}_1, -\mathbf{1}_2, \mathbf{1}_{12})$ of order 3.
4. f satisfies PCP and CPP (which will be defined in the next section).
5. f is not subadditive, hence f is not linear.

Proof. • (1) f is monotone since v is monotone.

• (2) The first assertion that f is nonnegative is clear. The second assertion that f has no bid-ask, is a consequence of Proposition 8, since $v(A)+v(A^c) = v(\Omega) = 1$ for all $A \subseteq \Omega$. The third assertion is a consequence of AF2, FL2 and the monotonicity of f that imply AF2+ and FL2+ by Claim 3.1.

• (3) Indeed, $\mathbf{1}_1 + \mathbf{1}_2 - \mathbf{1}_{12} = 0$ and $f(-\mathbf{1}_{12}) = -f(\mathbf{1}_{12})$ from (2), thus

$$f(\mathbf{1}_1) + f(\mathbf{1}_2) + f(-\mathbf{1}_{12}) = v(1) + v(2) - v(12) = -1/4 < 0.$$

This proves that $(\mathbf{1}_1, \mathbf{1}_2, -\mathbf{1}_{12})$ is an arbitrage opportunity of order 3 of f. Consequently, using the fact that $f(x) + f(-x) = 0$ for all x, one gets

$$f(-\mathbf{1}_1) + f(-\mathbf{1}_2) + f(\mathbf{1}_{12}) = -f(\mathbf{1}_1) - f(\mathbf{1}_2) - f(-\mathbf{1}_{12}) > 0$$

which shows that $(-\mathbf{1}_1, -\mathbf{1}_2, \mathbf{1}_{12})$ is a friction of order 3 of f.

• (4) f satisfies PCP and CPP from Theorem 8 since f is a monotone Choquet pricing rule that has no bid-ask spread from (2).

• (5) f is not subadditive since, for $x_1 := \mathbf{1}_1$ and $x_2 := \mathbf{1}_2$, one has:

$$3/4 = v(12) = f(x_1 + x_2) > f(x_1) + f(x_2) = v(1) + v(2) = 2/4. \qquad \square$$

2.13. Conclusion

Choquet pricing rules have been characterized by the notion of Call-Put Parity (CPP) and Put-Call Parity (PCP), introduced respectively by Chateauneuf, Kast and Lapied (1996) and Cerreia-Vioglio et al. (2015a). See also Wang et al. (1997), Castagnoli, Maccheroni, and Marinacci (2002), Chateauneuf and Cornet (2022b), and Bastianello, Chateauneuf and Cornet (2024). We recall the definitions of PCP, CPP, together with the following characterizing result that also gives the relationship between the two conditions PCP and CPP.

Put-Call Parity (PCP). For all $x \in X$ and all $k \in \mathbb{R}_+$

$$f(x) = f((x - k\mathbf{1}_\Omega)^+) + f(-(k\mathbf{1}_\Omega - x)^+) + f(k\mathbf{1}_\Omega).$$

Call-Put Parity (CPP). For all $x \in X$ and all $k \in \mathbb{R}_+$

$$f((k\mathbf{1}_\Omega - x)^+) = f((x - k\mathbf{1}_\Omega)^+) + f(-x) + f(k\mathbf{1}_\Omega).$$

Theorem 8. *Let $f : X \to \mathbb{R}$ be a monotone pricing rule.*

(a) Then the following conditions are equivalent:

(i) f satisfies PCP;

(ii) f is a Choquet pricing rule.

(b) Moreover, f satisfies Condition CPP if and only if it satisfies PCP and it has no bid-ask spread.

Choquet pricing rules provide key examples of non-linear pricing rules that can account for market frictions without requiring the subadditivity of the pricing rule. As such, the generalized definitions of arbitrage-free and frictionless pricing rules developed in this paper apply directly to Choquet pricing rules.

Furthermore, the no bid-ask spread property arises naturally in the context of the Call-Put Parity CPP, which characterizes pricing rules that are given by a Choquet integral and satisfy no bid-ask spread. See Bastianello, Chateauneuf and Cornet (2024). According to Theorem 7, if a pricing rule satisfies the Parity CPP and is arbitrage-free, then it must be linear, implying that all forms of market frictions and arbitrage opportunities of any order (as defined in this paper) are automatically excluded.

Additionally, Corollary 6 shows that for a pricing rule satisfying the Parity CPP, linearity can be deduced solely from the absence of arbitrage opportunities of order 3, emphasizing that higher-order conditions are not necessary in this case.

3. Proofs

3.1. Proofs of the Propositions

Proof of Proposition 1. • (1) [$\mathrm{AF}_{++} \Rightarrow \mathrm{AF}_{+}$] Let $K \in \mathbb{N}, K \geq 1$, let $(x_1, \ldots, x_K) \in X^K, x_1 + \cdots + x_K \geq 0$. Then, $nx_1 + \cdots + nx_K + \mathbf{1}_\Omega > 0$ for positive $n \in \mathbb{N}$. Hence, $nf(x_1) + \cdots + nf(x_K) + f(\mathbf{1}_\Omega) > 0$ by AF_{++}. Dividing by n, and letting $n \to +\infty$, at the limit we get $f(x_1) + \cdots + f(x_K) \geq 0$.

The proofs of the other Assertions of (1) are immediate. □

• (2) ⋆ From (1) we get the implication [$\mathrm{AF}_{+} \Rightarrow \mathrm{AF}$]. Conversely, assume that f is monotone and satisfies AF, we now show it also satisfies AF_{+}. Let $(x_1, \ldots, x_K) \in X^K$ such that $x := \sum_{k=1}^{K} x_k \geq 0$, then $\sum_{k=1}^{K} x_k + (-x) = 0$.

Thus, from AF we get $\sum_{k=1}^K f(x_k) + f(-x) \geq 0$. Hence, since $-x \leq 0$ and f is monotone we get $\sum_{k=1}^K f(x_k) \geq -f(-x) \geq f(0) = 0$.

$\star$ From (1) we get the implication [$\text{AF}_{++} \Rightarrow \text{AF}_{+}$]. Conversely, assume that f is strictly monotone and satisfies AF_{+}, we now show it also satisfies AF_{++}. Let $(x_1, \ldots, x_K) \in X^K$ such that $x := \sum_{k=1}^K x_k > 0$, then $\sum_{k=1}^K x_k + (-x) = 0$. Thus, from AF_{+} we get $\sum_{k=1}^K f(x_k) \geq -f(-x) > f(0) = 0$ since $-x < 0$ and f is strictly monotone. $\square$

• (3) We prove the result for $K \geq 2$ and $k = K - 1 \geq 1$ and the result will then follow by induction. Assume f satisfies AFK_{++} (resp. AFK_{+}, resp. AFK) and let $(x_1, \ldots, x_{K-1}) \in X^{K-1}$ such that $\sum_{k=1}^{K-1} x_k > 0$ (resp. ≥ 0, resp. $= 0$). Then the list $(x_1, \ldots, x_{K-1}, 0) \in X^K$ (thus with $x_K = 0$) satisfies $\sum_{k=1}^K x_k > 0$ (resp. ≥ 0, resp. $= 0$), thus $\sum_{k=1}^{K-1} f(x_k) = \sum_{k=1}^K f(x_k) > 0$ (resp. ≥ 0, resp. $= 0$) since f satisfies AFK_{++} (resp. AFK_{+}, resp. AFK) and $f(x_K) = f(0) = 0$. Thus, f satisfies AF(K-1)_{++} (resp. AF(K-1)_{+}, resp. AF(K-1)). $\square$

• (4) The proof is immediate.

• (5) Let $K \geq 1$ and let $(x_1, \ldots, x_K) \in X^K$ such that $\sum_{k=1}^K x_k = 0$, then $0 = f(0) = f(\sum_{k=1}^K x_k) \leq \sum_{k=1}^K f(x_k)$ since f is subadditive. Thus, f satisfies the Arbitrage-free Condition AF. $\square$

• (6) From (3), for all $K \in \mathbb{N}, K \geq 2$ one has

$$\text{AF}_{+} \Rightarrow \text{AFK}_{+} \Rightarrow \text{AF1}_{+} \Leftrightarrow [f(x) \geq 0 \text{ for all } x \geq 0].$$

Assume f is subadditive and $f(x) \geq 0$ for all $x \geq 0$, the above implication will be equivalences if we show that f satisfies Condition AF_{+}. Let $K \geq 1$ and let $(x_1, \ldots, x_K) \in X^K$ such that $x := \sum_{k=1}^K x_k \geq 0$, then $0 \leq f(x) = f(\sum_{k=1}^K x_k) \leq \sum_{k=1}^K f(x_k)$. Thus f satisfies Condition AF_{+}.

• (7) Assume f is subadditive and monotone. Then, for all $x \geq 0$, one has $f(x) \geq f(0) = 0$ since f is monotone. Hence, from (6), f satisfies Condition AF_{+} since f is subadditive. $\square$

Proof of Proposition 2. • (1) Clearly the inequality $f_+ \leq f$ holds true.

Moreover, f_+ is monotone since $x_1 \leq x_2$ implies

$\{x'_1 : x'_1 \geq x_1, x'_1 \in X\} \supseteq \{x'_2 : x'_2 \geq x_2, x'_2 \in X\}$, hence

$f_+(x_1) := \inf\{f(x'_1) : x'_1 \geq x_1\} \leq \inf\{f(x'_2) : x'_2 \geq x_2\} = f_+(x_2)$.

Finally, let $g : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be monotone and satisfy $g \leq f$. Then $x \leq x'$ implies $g(x) \leq g(x') \leq f(x')$, thus

$$g(x) \leq \inf\{f(x') : x' \geq x, x' \in X\} =: f_+(x). \qquad \square$$

• (2) First $0 = f_+(0) := \inf\{f(x') : x' \geq 0, x' \in X\}$ implies $f(x') \geq 0$ for all $x' \geq 0$. Conversely, if $f(x') \geq 0$ for all $x' \geq 0$, then $f_+(0) \geq 0$. But $f_+(0) \leq f(0) = 0$. Hence $f_+(0) = 0$.

• (3) is a direct consequence of the following claim, also of interest for itself.

Claim 3.1. *Let $f : X \to \mathbb{R}$ be a pricing rule.*
(i) f satisfies AF1+ $\Leftrightarrow$ f_+ satisfies AF1. [5]
(ii) For all integer $K \geq 2$,
f satisfies AFK+ $\Leftrightarrow$ f_+ is a (finite-valued) pricing rule that satisfies AFK.

Proof of Claim 3.1. • (i) Let $K = 1$ both assertions "f satisfies AF1+" and "f_+ satisfies AF1" are equivalent to $f(x) \geq 0$ for all $x \geq 0$.

• (ii) $\star$ [$\Rightarrow$] Let $K \in \mathbb{N}, K \geq 2$ and assume f satisfies AFK+. First, we prove that $f_+(0) = 0$. From AF+, $f(x) \geq 0$ for all $x \geq 0$. Thus,

$$0 \leq f_+(0) := \inf\{f(x') : x' \geq 0\} \leq f(0) = 0.$$

Second, we show that f_+ is finite valued. This follows from the following claim that, for all x, $f_+(x) \geq -f(-x) > -\infty$. To prove the claim, notice first that f satisfies AF2+ by Assertion (3) of Proposition 1. Then, for all x, and all $x' \geq x$, Condition AF2+ and $x' - x \geq 0$ imply $f(x') + f(-x) \geq 0$. Hence, $f_+(x) := \inf\{f(x') : x' \geq x, x' \in X\} \geq -f(-x) > -\infty$.

Finally, we show that f_+ satisfies AFK. Indeed, for all $(x_1, \ldots, x_K) \in X^K$ such that $x_1 + \cdots + x_K = 0$, and all $y_k \geq x_k$ $(k = 1, \ldots, K)$, then $y_1 + \cdots + y_K \geq x_1 + \cdots + x_K = 0$. But f satisfies AFK+, hence $f(y_1) + \cdots + f(y_K) \geq 0$, and

$$f_+(x_1) + \cdots + f_+(x_K) := \textstyle\sum_{k=1}^{K} \inf\{f(y_k) : y_k \geq x_k, y_k \in X\} \geq 0.$$

This proves that f_+ satisfies AFK. $\square$

[5] Note that for $K = 1$ if f satisfies AF1+, then f_+ may take the value $-\infty$.

[$\Leftarrow$] Let $K \in \mathbb{N}, K \geq 1$ and assume f_+ is finite valued and satisfies AFK. Let $(x_1, \dots, x_K) \in X^K$ such that $u := x_1 + \cdots + x_K \geq 0$, then $x_1 - u + \cdots + x_K = 0$. Since f_+ is monotone and satisfies AFK we deduce that

$$\sum_{k=1}^K f(x_k) \geq \sum_{k=1}^K f_+(x_k) \geq f_+(x_1 - u) + \cdots + f_+(x_K) \geq 0.$$

This proves that f satisfies AFK+. $\square$

• (4) Assume f is monotone.

$\star$ [$f = f_+$] Let $x \in X$, one always has $f_+(x) \leq f(x)$ and for all $x' \geq x$ one has $f(x) \leq f(x')$, since f is monotone. Thus,

$$f(x) \leq \inf\{f(x') : x' \geq x, x' \in X\} = f_+(x).$$

$\star$ [f satisfies AF$_+$ $\Leftrightarrow$ f satisfies AF] Follows from (3) since $f = f_+$.

$\star$ [f is bounded on a neighborhood of 0] Indeed, for all $x \in B_\infty(0,1)$, that is, $-\mathbf{1}_\Omega \leq x \leq \mathbf{1}_\Omega$ one has $f(-\mathbf{1}_\Omega) \leq f(x) \leq f(\mathbf{1}_\Omega)$ since f is monotone. $\square$

Proof of Proposition 4. • (1) First, $f^*(0) = 0$ since $f^*(0) = -f(-0) = 0$ and second, $(f^*)^* = f$ clearly holds. $\square$

• (2) Assume f is monotone and $x_1 \leq x_2$. Then, $-x_1 \geq -x_2$ implies $f(-x_1) \geq f(-x_2)$ since f is monotone. Thus, $f^*(x_1) := -f(-x_1) \leq -f(-x_2) = f^*(x_2)$, that is, f^* is monotone. Conversely, if f^* is monotone, then $f = (f^*)^*$ is also monotone as proved previously. $\square$

• (3) Assume f is subadditive. For all x_1, x_2, one has $f(-x_1 - x_2) \leq f(-x_1) + f(-x_2)$, since f is subadditive. Thus $f^*(x_1 + x_2) = -f(-x_1 - x_2) \geq -f(-x_1) - f(-x_2) = f^*(x_1) + f^*(x_2)$, that is, f^* is superadditive. The converse assertion is proved in a similar way. $\square$

• (4) We prove the second equivalence. Let $(x_1, \dots, x_K) \in \mathbb{R}^K$ $(K \geq 1)$.

Assume f satisfies AFK$_+$ and $\sum_{k=1}^K x_k \leq 0$, then $\sum_{k=1}^K (-x_k) \geq 0$, hence $\sum_{k=1}^K f(-x_k) \geq 0$ and $\sum_{k=1}^K f^*(x_k) = \sum_{k=1}^K -f(-x_k) \leq 0$. That is, f^* satisfies FLK$_+$.

Conversely, assume f^* satisfies FLK$_+$, and $\sum_{k=1}^K x_k \geq 0$, then $\sum_{k=1}^K (-x_k) \leq 0$, hence $\sum_{k=1}^K f^*(-x_k) \leq 0$ and $\sum_{k=1}^K f(x_k) = \sum_{k=1}^K -f^*(-x_k) \geq 0$. That is, f satisfies AFK$_+$.

The proof of the other equivalence in (4) is similar. $\square$

3.2. Proof of Theorem 1

Proof. In the following we let $S := 1 + \#\Omega$.

• (1) The necessity part is immediate and we now prove the sufficiency part. Assume AFS++ (resp. AFS+, AFS) holds, then we have proved [(3) of Proposition 1] that, for all $K \leq S$, AFK++ (resp. AFK+, AFK) holds. We need now to show that AFK++ (resp. AFK+, AFK) also holds for $K > S$. Indeed, let $(x_1, \ldots, x_K) \in (\mathbb{R}^\Omega)^K$ such that $x_1 + \cdots + x_K > 0$ (resp. $\geq 0, = 0$) and let

$$w := \big(w_0, w_1\big) = \big(- f(x_1) - \cdots - f(x_K), x_1 + \cdots + x_K\big) \; \in \mathbb{R} \times \mathbb{R}^\Omega.$$

We first consider the case where $w = 0$, which can only happen with AFK+ and AFK; then $w = 0$ implies $f(x_1) + \cdots + f(x_K) = 0$, hence AFK+ and AFK clearly holds true.

Assume now that $w \neq 0$, we first notice that w belongs to the convex cone C generated by the vectors $(-f(x_1), x_1), \ldots, (-f(x_K), x_K)$ in $\mathbb{R} \times \mathbb{R}^\Omega$. Hence, by Caratheodory's Theorem [e.g., see Rockafellar (1970), Corollary 17.1.2, page 156], the nonzero vector w can be written as a nonegative linear combination of at most S of these vectors, say the first ones. That is, there exist $\theta_1 \geq 0, \ldots, \theta_S \geq 0$ such that

$$w = \big(- \theta_1 f(x_1) - \cdots - \theta_S f(x_S), \theta_1 x_1 + \cdots + \theta_S x_S\big).$$

Since $w_1 := x_1 + \cdots + x_K > 0$, (resp. $\geq 0, = 0$), we deduce that

$$\theta_1 x_1 + \cdots + \theta_S x_S = x_1 + \cdots + x_K > 0 \text{ (resp. } \geq 0, = 0).$$

$$\begin{aligned} -w_0 := f(x_1) + \cdots + f(x_K) &= \theta_1 f(x_1) + \cdots + \theta_S f(x_S) \\ &= f(\theta_1 x_1) + \cdots + f(\theta_S x_S) \quad \text{[since } f \text{ is positively homogeneous]} \\ &> 0 \text{ (resp. } \geq 0, \geq 0) \text{ by AFS++ (resp. AFS+, AFS).} \end{aligned}$$

Hence, $f(x_1) + \cdots + f(x_K) > 0$ (resp. $\geq 0, \geq 0$), thus AFK++ (resp. AFK+, AFK) holds. □

• (2) This is a consequence of the following sequence of equivalence properties:

f satisfies	FL++ (resp. FL+, FL)	
$\Leftrightarrow$ f^* satisfies	AF++ (resp. AF+, AF)	[by Prop. 4]
$\Leftrightarrow$ f^* satisfies	AFK++ (resp. AFK+, AFK) for $K \leq S$	[by Part (1)]
$\Leftrightarrow$ f satisfies	FLK++ (resp. FLK+, FLK) for $K \leq S$	[by Pro. 4]

□

3.3. Proof of Theorem 2 and Theorem 3

We first provide the proof of Theorem 3.

Proof of Theorem 3. • The first part of Theorem 3 is a consequence of the following implications $[(iii) \Rightarrow (i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (ii')]$, which are proved in the following.

$\star[(iii) \Rightarrow (i)]$ and $[(i) \Rightarrow (ii)]$ The proofs are immediate. $\square$

$\star[(ii) \Rightarrow (iii)$: f additive$)]$ Let x, y in X, $(x+y) - x - y = 0$ implies

$$\begin{aligned} & f(x+y) - f(x) - f(y) \\ & \qquad \geq f(x+y) + f(-x) + f(-y) \qquad \text{[since } f \text{ is 2-frictionless]} \\ & \qquad \geq 0 \qquad \text{[since } f \text{ is 3-arbitrage-free].} \end{aligned}$$

Similarly, since $-(x+y) + x + y = 0$, we have

$$\begin{aligned} & -f(x+y) + f(x) + f(y) \\ & \qquad \geq f(-(x+y)) + f(x) + f(y) \qquad \text{[since } f \text{ is 2-frictionless]} \\ & \qquad \geq 0 \qquad \text{[since } f \text{ is 3-arbitrage-free].} \end{aligned}$$

Consequently, $f(x+y) = f(x) + f(y)$, which proves that f is additive. $\square$

$\star[(iii) \Leftrightarrow (ii')]$ This is a consequence of the equivalence $[(iii) \Leftrightarrow (ii)]$ proved previously, that we apply to $-f$, noticing that $-f$ satisfies (ii) if and only if f satisfies (ii'), and $-f$ is additive if and only if f is additive. $\square$

• We now prove the second part of Theorem 3. Clearly f linear implies that f is additive and for the converse implication, we only need to prove that

$$f(tx) = tf(x) \text{ for all } x \in X \text{ and all } t \in \mathbb{R}.$$

The above statement is a consequence of the following claims. Let $x \in X$

$\star$ $f(tx) = tf(x)$ for all $t \in \mathbb{N}$. We prove it by induction. First, the equality holds for $p = 1$. Suppose now it holds up to $p \geq 1$ we prove it also holds for $p + 1$. Indeed, since f is additive

$$f((p+1)x) = f(px) + f(x) = pf(x) + f(x) = (p+1)f(x),$$

which ends the proof of the first claim. $\square$

$\star$ $f(tx) = tf(x)$ for all l $t \in \mathbb{Q}$. We first prove it for $t \geq 0$. Let $t = p/q$ with $p \in \mathbb{N}$ and $q \in \mathbb{N}$, $q \neq 0$. From the previous claim, $pf(x) = f(px) = f(q\frac{p}{q}x) = qf(\frac{p}{q}x)$. Thus $f(\frac{p}{q}x) = \frac{p}{q}f(x)$.

We now prove it for $t \leq 0$. From the additivity of f and the previous claim $0 = f(0) = f((-t)x+tx) = (-t)f(x)+f(tx)$. Thus, $f(tx) = tf(x)$. $\square$

$\star$ $f(tx) = tf(x)$ for all $t \in \mathbb{R}$. Let $x \in X$ and $t \in \mathbb{R}$. There exists a sequence $(t_n) \subseteq \mathbb{Q}$ such that $(t_n) \to t$. We let $\varepsilon^n := t^n x - tx$ and the additivity of f and the previous claims imply

$$f(tx) - t^n f(x) + f(\varepsilon^n) = f(tx) - f(t^n x) + f(\varepsilon^n) = 0.$$

Then we get $f(tx) - tf(x) = 0$, by taking the limit, when $n \to \infty$, in the above equality if we know that $\lim_{n\to\infty} f(\varepsilon^n) = 0$. The proof will thus be complete by showing this last assertion in the following claim. From the assumption that f is bounded on some neighborhood of 0, there exist a positive integer m and $\rho > 0$ such that $|f(x)| \leq \rho$ for all $x \in B_\infty(0, 1/m)$.

$\star$ $|f(\varepsilon^n)| \leq 2m\rho\|\varepsilon^n\|$ for all n. First, for all n such that $\varepsilon^n = 0$, the inequality clearly holds. Second for all n such that $\varepsilon^n \neq 0$, we can choose $\theta^n \in \mathbb{Q}$ such that $0 < m\|\varepsilon^n\|_\infty < \theta^n < 2m\|\varepsilon^n\|_\infty$. Thus, $\varepsilon^n/\theta^n \in B_\infty(0, 1/m)$ implies $|f(\varepsilon^n/\theta^n)| \leq \rho$. But, $\theta^n \in \mathbb{Q}$ and the first part of the proof imply

$$|f(\varepsilon^n)| = |f(\theta^n(\varepsilon^n/\theta^n))| = |\theta^n f(\varepsilon^n/\theta^n)| \leq \theta^n\rho \leq 2m\rho\|\varepsilon^n\|. \qquad \square$$

We can now deduce the proof of Theorem 2 from Theorem 3.

Proof of Theorem 2. We prove the following implications:

$$[(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1)] \text{ and } [(2) \Leftrightarrow (2')].$$

•[(1) $\Rightarrow$ (2)] The proof is immediate.
• [(2) $\Rightarrow$ (3)] Clearly (2) implies that f satisfies AF3 and FL2. Thus f is additive from Theorem 3. Moreover, AF3$_+$ implies AF1$_+$, that is, $f(x) \geq 0$ for all $x \geq 0$, which together with f additive, implies that f is monotone. $\square$
• [(3) $\Rightarrow$ (4)] Follows from Theorem 3 since the monotone function f is bounded on a neighborhood of 0 by Assertion (4) of Proposition 2. $\square$
• [(4) $\Rightarrow$ (1)] Let $K \in \mathbb{N}, K \geq 1$, let $(x_1, \ldots, x_K) \in X^K$, such that $\sum_{k=1}^K x_k \geq 0$ (resp. ≤ 0). Since f is linear, $\sum_{k=1}^K f(x_k) = f(\sum_{k=1}^K x_k) \geq f(0) = 0$

(resp. $\leq f(0) = 0$) since $\sum_{k=1}^{K} x_k \geq 0$ (resp. ≤ 0), f is monotone and $f(0) = 0$. $\square$

• $[(2) \Leftrightarrow (2')]$ Both Assertions are equivalent to (1) since $[(2) \Leftrightarrow (1)]$ from above and the implications $[(1) \Rightarrow (2') \Rightarrow (3) \Rightarrow (1)]$ can be proved in the same way as previously. $\square$

3.4. Proof of Theorem 7

We first give the proof of Proposition 9.

Proof of Proposition 9. • (1) [v submodular $\Rightarrow$ (*) and $v^* \leq v$] Assume v is submodular.

First, we show that $v^* \leq v$. Take $B = A^c$, then $v(\Omega)+v(\emptyset) \leq v(A)+v(A^c)$. Thus $v^*(A) := v(\Omega) - v(A^c) \leq v(A)$ that is, $v^* \leq v$.

Second we show that Condition (*) holds. Since $v^* \leq v$ and v is submodular, Condition (*) holds true since one has, for all A, B in $\mathcal{A}$

$$v^*(A \cup B) + v^*(A \cap B) \leq v(A \cup B) + v(A \cap B) \leq v(A) + v(B).$$

• [(*) and $v \leq v^* \Rightarrow v$ submodular] From (*) and $v \leq v^*$ one deduces that v is submodular since, for all A, B in $\mathcal{A}$

$$v(A \cup B) + v(A \cap B) \leq v^*(A \cup B) + v^*(A \cap B) \leq v(A) + v(B).$$

• (1) [AF3 $\Rightarrow$ (**)] From AF3, that is Condition (*), taking $x_1 := \mathbf{1}_A$, $x_2 := \mathbf{1}_B$, with A, B in $\mathcal{A}$, one gets

$$f^*(\mathbf{1}_A + \mathbf{1}_B) := -f(-\mathbf{1}_A - \mathbf{1}_B) \leq f(\mathbf{1}_A) + f(\mathbf{1}_B) = v(A) + v(B).$$

But recalling that $\mathbf{1}_A + \mathbf{1}_B = \mathbf{1}_{A\cup B} + \mathbf{1}_{A\cap B}$ and the two functions $\mathbf{1}_{A\cup B}$ and $\mathbf{1}_{A\cap B}$ are comonotonic, from the comonotonic additivity of f^* which is a Choquet pricing rule with respect to v^* (since f which is a Choquet pricing rule with respect to v), we get

$$\begin{aligned} f^*(\mathbf{1}_A + \mathbf{1}_B) &= f^*(\mathbf{1}_{A\cup B} + \mathbf{1}_{A\cap B}) \\ &= f^*(\mathbf{1}_{A\cup B}) + f^*(\mathbf{1}_{A\cap B}) = v^*(A \cup B) + v^*(A \cap B). \end{aligned}$$

Consequently, we get $v^*(A \cup B) + v^*(A \cup B) \leq v(A) + v(B)$. $\square$

$\star$ [(**) $\Rightarrow v^* \leq v$] Take $B = A^c$ in (**), then $v^*(\Omega) + v^*(\emptyset) \leq v(A) + v(A^c)$. Thus $v^*(A) := v(\Omega) - v(A^c) = v^*(\Omega) - v(A^c) \leq v(A)$, that is, $v^* \leq v$. $\square$

• (2) We prove successively that

$$\text{AF3 and FL2} \Rightarrow [(**) \text{ and } v \leq v^*] \Rightarrow v \text{ additive} \Rightarrow f \text{ additive}$$

and the last implication [f additive $\Rightarrow$ AF3 and FL2] is immediate.

$\star$ The first implication follows from (1) since FL2 is equivalent to $v \leq v^*$ by Proposition 8.

$\star$ [(**) and $v \leq v^* \Rightarrow v$ additive] From (1) we know that (**) implies $v^* \leq v$, thus $v^* = v$. From (**) and $v = v^*$ we get, for A, B in $\mathcal{A}$:

$$\begin{aligned}
&0 \geq v(A \cup B) + v(A \cap B) - v(A) - v(B) \text{ and} \\
&0 \geq v^*(A^c \cup B^c) + v^*(A^c \cap B^c) - v^*(A^c) - v^*(B^c) \\
&\quad = [v^*(\Omega) - v([A^c \cup B^c]^c)] + [v^*(\Omega) - v([A^c \cap B^c]^c)] \\
&\qquad - [v^*(\Omega) - v(A)] - [v^*(\Omega) - v(B)] \\
&\quad = -v(A \cap B) - v(A \cup B) + v(A) + v(B)
\end{aligned}$$

Thus $v(A)+v(B) = v(A \cup B)+v(A \cap B) = v(A \cup B)$ whenever $A \cap B = \emptyset$. This proves that v is additive.

$\star$ [v additive $\Rightarrow$ f additive] This is a standard property of the Choquet integral. □

Proof of Theorem 7. • The Choquet pricing rule $f : X \to \mathbb{R}$ with respect to a game $v \in \mathrm{bv}(\mathcal{A})$ is continuous, hence admits a neighborhood of 0 on which it is bounded, thus the assumptions of Theorem 3 are satisfied. From Proposition 9, Conditions (2.2), (2.3), (**), (3.1), and (3.2) are equivalent, and from Theorem 3, Condition (3.2) is equivalent to (4),(1), and (2.1). □

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