UNIVERSITY OF KANSAS
WORKING PAPERS SERIES IN
THEORETICAL AND APPLIED ECONOMICS

Strategy-Proofness versus Efficiency in Exchange Economies: General Domain Properties and Applications

Biung-Ghi Ju

Paper no. 2001-02
December 20, 2001

Department of Economics
University of Kansas
Summerfield Hall
Lawrence, KS 66045-2113

Strategy-proofness versus efficiency in exchange economies: general domain properties and applications

Biung-Ghi Ju*

June 2000 (Revised December 20, 2001)

Abstract

We identify general domain properties that induce the non-existence of *efficient, strategy-proof,* and *non-dictatorial* rules in the 2-agent exchange economy. Applying these properties, we establish the impossibility result in several restricted domains; the "intertemporal exchange problem" (without saving technology) with preferences represented by the discounted sum of a temporal utility function, the "risk sharing problem" with risk averse expected utility preferences, the CES-preference domain, etc. None of the earlier studies applies to these domains.

Keywords: *strategy-proofness, efficiency,* restricted domain*,* dictatorship.

*Department of Economics, University of Kansas, Lawrence, KS66045, USA. E-mail: `bgju@ku.edu`. I am grateful to Professor William Thomson for helpful comments and suggestions. I also thank seminar participants in University of Rochester, John Duggan, and François Maniquet. I thank two anonymous referees for their various comments. All remaining errors are mine.

1 Introduction

In the "exchange economy", an allocation rule, or simply, a *rule*, associates with each profile of agents' preferences a *single* desirable allocation, a list of individual consumption bundles. We refer to the set of admissible preference profiles as the *domain*. We are interested in the following two basic requirements of rules. The first is *efficiency*, the requirement that no one can be made better off without anyone else being made worse off. The second is *strategy-proofness* (Gibbard, 1973, Satterthwaite, 1975), the requirement that truthful representation of one's preference always weakly dominates any admissible misrepresentation.

A number of earlier studies have shown impossibilities of satisfying the two requirements together with other standard equity criteria. In particular, in the 2-agent case, Dasgupta, Hammond, and Maskin (1979), Zhou (1991a), and Schummer (1997) show that there is no *efficient* and *strategy-proof* rules satisfying the minimal equity criterion, "non-dictatorship"; a rule is *dictatorial* if there is an agent, the dictator, who always receives his best bundle.[1] However, their results are not fully satisfactory because they provide no implication for various interesting allocation problems in which agents' preferences are restricted for some intuitive or technical reasons.

For example, in the "intertemporal exchange problem" (without saving technology), we often consider preferences that are represented in the additively separable form by temporal utility functions and discount factors. In the "risk sharing problem", we often consider preferences that are represented in the expected utility form by strictly convex ("risk aversion") utility indices and subjective probability distributions over states. Also, in many applications, we focus on preferences that satisfy technical conditions such as "smoothness", "continuous differentiability of utility functions", "quasilinearity", etc.

Our main objective is to strengthen the impossibility result for the 2-agent exchange economy by identifying general domain properties sufficient for the impossibility. They are satisfied by the domains considered in the earlier studies; our result simplifies the proofs by Dasgupta, Hammond, and Maskin (1979), Zhou (1991a), and Schummer (1997). More importantly, our domain properties are applicable to several restricted domains such as the intertemporal exchange problem, the risk sharing problem, the domain of "CES preferences", and the domain of quasilinear, strictly convex, and smooth preferences, etc., while none of the earlier studies applies to them.

The seminal study by Hurwicz (1972) shows that in the 2-agent and 2-good exchange economy, there exists no *efficient* and *strategy-proof* rule satisfying

[1]See also Hurwicz (1972), Satterthwaite and Sonnenschein (1981), Hurwicz and Walker (1990), and Barberà and Jackson (1995).

"individual rationality", the requirement that everyone should be at least as well off as in his endowment. Dasgupta, Hammond, and Maskin (1979) strengthen his result by replacing *individual rationality* with *non-dictatorship*. However, their conclusion crucially relies on the admissibility of "discontinuous" preferences, while Hurwicz's result pertains to preferences satisfying the classical assumptions, "continuity", "monotonicity", and "convexity".

Zhou (1991) reinforces the impossibility result by Dasgupta, Hammond, and Maskin (1979), considering the classical domain consisting of continuous, strictly monotonic, and strictly convex preferences. When preferences are strictly monotonic, this conclusion extends to any larger domain, as he remarks. A natural question addressed by Schummer (1997) is whether the impossibility applies to smaller, yet interesting, domains. He shows that the impossibility continues to hold both in the domain of "homothetic" preferences and in the domain of "linear" preferences (preferences with linear utility functions).

The arguments used by Zhou (1991) and Schummer (1997) crucially rely on the admissibility of "kinked" preferences.[2] So their results do not apply, for example, to domains consisting of only *smooth* and strictly convex preferences. On the other hand, Schummer (1997) crucially relies on the homotheticity restriction. So, his result does not apply to other restricted domains, for example, the domain consisting of only *quasilinear* and strictly convex preferences. Our domain properties do not necessarily require that kinked or homothetic preferences be admissible. They are applicable not only to all the above domains but various other restricted domains as we show in the application of our main result.

Several recent authors bring out some important domain properties in different perspectives of their studies on *strategy-proofness*. In a voting model, Barberà, Sonnenschein, and Zhou (1991) identify the unique maximal domain in which a class of rules, called "voting by committees", are *strategy-proof*. The maximal domain issue is studied also by Berga and Serizawa (1998) in the 1-dimensional public choice model. In a linear production model, Maniquet and Sprumont (1999) identify domain properties under which their characterization results apply.

Most of the earlier studies focus on "product domains", Cartesian products of families of individual preferences.[3] Product domains do not capture the interdependency, or correlation, of preferences across agents, which is common in reality. Such an interdependency arises especially when agents share identical cultural or historic background relevant to their preferences. Thus, it is standard in implementation theory to capture such an interdependency by considering non-

[2] The only exception is the linear preference domain in Schummer (1997).

[3] Or "independent domains" (Moore, 1993, p 214).

product domains: see Moore (1993) for a broad survey of literature. Therefore, we do not restrict our attention only to product domains; our domain properties are stated for possibly, non-product domains. *Strategy-proofness* is a necessary condition for the implementability in dominant strategy equilibrium both in the product domain case and in the non-product domain case. It is also sufficient in the product domain case, while it is not sufficient in the non-product domain case.

This paper is composed of five sections. In Section 2, we introduce the model and basic concepts. In Section 3, we define general domain properties and establish our main result. In Section 4, we provide several applications. We conclude in Section 5.

2 The model and basic concepts

We consider l-good exchange economies, $l \geq 2$, with social endowment $\Omega \in \mathbb{R}^l_{++}$ and two agents. Let $N \equiv \{1, 2\}$ be the set of agents. Let $Z \equiv \{z \in \mathbb{R}^{l \cdot 2}_{+} : \sum_N z_i = \Omega\}$ be the set of feasible allocations. Let $Z_0 \equiv \{z_i \in \mathbb{R}^l_+ : \mathbf{0} \leqq z_i \leqq \Omega\}$ be the set of possible consumption bundles for each agent.[4] We use z, z', z'', etc. to denote allocations: z_i denotes i's bundle at z. Notation $-i$ refers to the agent other than i; that is, $-1 \equiv 2$ and $-2 \equiv 1$.

Each agent has a **preference**, a complete and transitive binary relation over $\mathbb{R}^l_+$. Preferences are *continuous, strictly monotonic over* $\mathbb{R}^l_{++}$, and *convex*.[5] Let $\mathcal{R}$ be the class of all such preferences. A preference $R_i \in \mathcal{R}$ is **strictly monotonic** if for all $z_i, z'_i \in \mathbb{R}^l_+$. $z_i \geq z'_i$ implies $z_i \; P_i \; z'_i$. Let $I_i(z_i)$ be the set of all bundles indifferent to z_i under R_i.

A **domain** $\mathcal{D}$ is a subset of $\mathcal{R}^N$. Let $\mathcal{D}(R_{-i}) \equiv \{R' \in \mathcal{D} : R'_{-i} = R_{-i}\}$, $\mathcal{D}_i(R_{-i}) \equiv \{R_i : (R_i, R_{-i}) \in \mathcal{D}\}$, and $\mathcal{D}_i \equiv \{R_i : \text{for some } R_{-i}, \; (R_i, R_{-i}) \in \mathcal{D}\}$. Since we keep the social endowment fixed, an **economy** can be characterized by a preference profile in $\mathcal{D}$. A social choice rule, or simply a **rule**, over $\mathcal{D}$ is a function $\varphi \colon \mathcal{D} \to Z$ associating with each economy a feasible allocation.

A domain $\mathcal{D}$ is a **product domain** if for each $i \in N$, there exists $\mathcal{D}_i \subseteq \mathcal{R}$ such that $\mathcal{D} = \mathcal{D}_1 \times \mathcal{D}_2$. Product domains do not capture the interdependency

[4]We denote elements of Z_0 by z_i, z_0, x, y etc. Vector inequalities, $\leqq$, $\leq$, $<$, are defined as follows. Let $x, y \in \mathbb{R}^l$. Then $x \leqq y$ if for all $k \in \{1, \cdots, l\}$, $x_k \leq y_k$. We write $x \leq y$ if $x \leqq y$ and $x \neq y$. We write $x < y$ if for all $k \in \{1, \cdots, l\}$, $x_k < y_k$.

[5]For $R_i \in \mathcal{R}$, we use P_i and I_i to denote its strict and indifference relations respectively. A preference R_i is *strictly monotonic over* $\mathbb{R}^l_{++}$ if for all $z_i, z'_i \in \mathbb{R}^l_{++}$, $z_i \geq z'_i$ implies $z_i \; P_i \; z'_i$, where the vector inequality $z_i \geq z'_i$ means that each component of z_i is weakly larger than each component of z'_i and $z_i \neq z'_i$. It is *convex* if for all $z_i, z'_i \in \mathbb{R}^l_{++}$ with $z_i \; R_i \; z'_i$ and all $\lambda \in [0, 1]$, $\lambda z_i + (1 - \lambda) z'_i \; R_i \; z'_i$.

of preferences across agents, which is commonly observed in reality. Thus we do not restrict ourselves to product domains. However, the following general features of domains are important in our result. Let $R, R' \in \mathcal{D}$. Profile R' is a *unilateral variation of* R if $R'_1 = R_1$ or $R'_2 = R_2$. A unilateral variation of R, R', is 0-*indifference-monotonic for* i if $I'_i(0) \supseteq I_i(0)$. Profile R' is *reachable from* R *through iterative unilateral variations* if there exists a finite sequence of profiles $(R^1, \cdots, R^n)$ in $\mathcal{D}$ such that $R^1 = R$, $R^n = R'$, and for all $k \in \{2, \cdots, n\}$, R^k is a unilateral variation of R^{k-1}. A domain $\mathcal{D}$ is **everywhere reachable** if for all $R, R' \in \mathcal{D}$, R' is reachable from R through iterative unilateral variations. It is **everywhere reachable*** if for all $i \in N$ and all $R, R' \in \mathcal{D}$ with $I_i(0) \subseteq I'_i(0)$, R' is reachable from R through iterative unilateral variations 0-indifference-monotonic for i.

Note that *everywhere reachability** implies *everywhere reachability* and that every product domain is *everywhere reachable**. When R_1 and R_2 are strictly monotonic, $I_1(0) = I_2(0) = \{0\}$. So if $\mathcal{D}$ is *everywhere reachable**, then for all $R' \in \mathcal{D}$, R' is reachable from R both through iterative unilateral variations 0-indifference-monotonic for 1 and through iterative unilateral variations 0-indifference-monotonic for 2. When all preferences in $\mathcal{D}$ are strictly monotonic, then *everywhere reachability* is equivalent to *everywhere reachability**.

We next define our two main requirements of rules. Given $R \in \mathcal{D}$, an allocation $z \in Z$ is **efficient for R** if there exists no $z' \in Z$ such that for all $i \in N$, $z'_i \; R_i \; z_i$ and for some $j \in N$, $z'_j \; P_j \; z_j$. Let $P(R)$ be the set of all efficient allocations for R. For all $i \in N$, let $P_i(R) \equiv \{z_i \in \mathbb{R}^l_+ : (z_i, z_{-i}) \in Z \text{ for some } z_{-i} \in \mathbb{R}^l_+\}$. A rule $\varphi \colon \mathcal{D} \to Z$ satisfies **efficiency** if for all $R \in \mathcal{D}$, $\varphi(R) \in P(R)$.

In order to define the next requirement, let $i \in N$ have preference R_i. Let (R_i, R_{-i}) and $(R'_i, R_{-i}) \in \mathcal{D}$. Consider a rule $\varphi \colon \mathcal{D} \to Z$. Let $z \equiv \varphi(R)$ and $z' \equiv \varphi(R'_i, R_{-i})$. Agent i will have an incentive to represent his true preference as opposed to the misrepresentation R'_i if $z_i \; R_i \; z'_i$. We refer to this condition as **i's incentive compatibility condition associated with** $(\mathbf{R_i}, \mathbf{R'_i}, \mathbf{z_i})$, where R_i is i's true preference, R'_i is a misrepresentation, and z_i is the truthful outcome. Our next requirement is that the incentive compatibility condition should never be violated. Formally, a rule $\varphi \colon \mathcal{D} \to Z$ satisfies **strategy-proofness** if for all $i \in N$ and all $R, R' \in \mathcal{D}$ with $R_{-i} = R'_{-i}$, $\varphi_i(R) \; R_i \; \varphi_i(R')$.

We show that every *efficient* and *strategy-proof* rule has the following displeasing features. A rule $\varphi \colon \mathcal{D} \to Z$ is **dictatorial over** $\mathcal{D}^* \subseteq \mathcal{D}$ if there exists $i \in N$ such that for all $R \in \mathcal{D}^*$ and all $z_i \in Z_0$, $\varphi_i(R) \; R_i \; z_i$. The rule is **dictatorial** if it is dictatorial over the entire domain. Since preferences are strictly monotonic over $\mathbb{R}^l_{++}$, a rule φ is dictatorial over $\mathcal{D}^*$ if and only if there exists $i \in N$ such that for all $R \in \mathcal{D}^*$, $\varphi_i(R) = \Omega$.

We use the following notation. For all $R_i \in \mathcal{R}$ and all $z_i \in \mathbb{R}^l_+$, let $UC(R_i, z_i) \equiv \{x \in Z_0 : x \ R_i \ z_i\}$ and $UC^0(R_i, z_i) \equiv \{x \in Z_0 : x \ P_i \ z_i\}$ be the upper contour set of R_i at z_i and the strict upper contour set, respectively. Let $LC(R_i, z_i) \equiv \{x \in Z_0 : z_i \ R_i \ x\}$ and $LC^0(R_i, z_i) \equiv \{x \in Z_0 : z_i \ P_i \ x\}$ be the lower contour set of R_i at z_i and the strictly lower contour set.

3 The main result

In this section, we define general domain properties and derive our main conclusion based on these properties.

We consider domain $\mathcal{D} \subseteq \mathcal{R}^N$ that has a subdomain $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and a *reference set* $M \subseteq Z$ satisfying the following three properties. For all $i \in N$, let $M_i \equiv \{z_i \in \mathbb{R}^l_+ : (z_i, z_{-i}) \in Z \text{ for some } z_{-i} \in \mathbb{R}^l_+\}$.

The first property is that the reference set M is the Pareto set for at least one economy in $\bar{\mathcal{D}}$ with strictly monotonic preferences.

Potential efficiency: There exists $R \in \bar{\mathcal{D}}$ such that $P(R) = M$ and both R_1 and R_2 are strictly monotonic.

The second property is that each agent can always make M be the Pareto set by announcing a preference admissible in $\bar{\mathcal{D}}$.

Attainability: For all $i \in N$ and all $R_{-i} \in \bar{\mathcal{D}}_{-i}$, there exists $R_i \in \bar{\mathcal{D}}_i(R_{-i})$ such that $P(R_i, R_{-i}) = M$.

The third property is stated in terms of the following notions. Both incentive compatibility conditions associated with (R_i, R'_i, z_i) and (R'_i, R_i, z'_i) imply that $z'_i \in LC(R_i, z_i) \cap UC(R'_i, z_i)$ (also $z_i \in LC(R'_i, z'_i) \cap UC(R_i, z'_i)$). Therefore, given R_{-i} and the truthful outcome z_i for R_i, the set of incentive compatible outcomes for R'_i coincides with $LC(R_i, z_i) \cap UC(R'_i, z_i)$. We call $LC(R_i, z_i) \cap UC(R'_i, z_i)$ the **incentive compatibility set associated with $(\mathbf{R_i}, \mathbf{R'_i}, \mathbf{z_i})$**. For all $R \in \mathcal{D}$, all $i \in N$, and all $z \in P(R)$, $R'_i \in \mathcal{D}(R_{-i})$ is a **local transformation of $\mathbf{R_i}$ at $\mathbf{z_i}$ relative to $\mathbf{R_{-i}}$** if z_i is the unique *efficient* bundle for (R'_i, R_{-i}), in i's incentive compatibility set associated with (R_i, R'_i, z_i); that is, $P_i(R'_i, R_{-i}) \cap LC(R_i, z_i) \cap UC(R'_i, z_i) = \{z_i\}$. A preference R_i of agent i exhibits **crossly local dominance of $\mathbf{z'_i}$ relative to $(\mathbf{R_{-i}}, \mathbf{R'_{-i}}, \mathbf{z_i})$** if agent i with R_i prefers z'_i to every allocation that is *efficient* for (R_i, R'_{-i}) and is in $-i$'s incentive compatibility set associated with $(R_{-i}, R'_{-i}, z_{-i})$; that is $P_i(R_i, R'_{-i}) \cap \{x \in Z_0 : \Omega - x \in LC(R_{-i}, z_{-i}) \cap UC(R'_{-i}, z_{-i})\} \subset LC^0(R_i, z'_i)$.

Our next property states that for any two profiles R and R' with the Pareto set M and for any two *efficient* allocations z and z', there exist an agent $i \in N$ and his preference $\bar{R}_i$ that is a local transformation of R_i at z_i relative to R_{-i} and at the same time exhibits crossly local dominance of z'_i relative to R_{-i}, R'_{-i}, and z_i.

Transformability with crossly local dominance: For all $R, R' \in \bar{\mathcal{D}}$ and all $z, z' \in M$, if $P(R) = P(R') = M$ and $z \neq z'$, then there exist $i \in N$ and $\bar{R}_i \in \bar{\mathcal{D}}_i(R_{-i}) \cap \bar{\mathcal{D}}_i(R'_{-i})$ such that
(i) $P_i(\bar{R}_i, R_{-i}) \cap LC(R_i, z_i) \cap UC(\bar{R}_i, z_i) = \{z_i\}$;
(ii) $P_i(\bar{R}_i, R'_{-i}) \cap \{x \in Z_0 : \Omega - x \in LC(R_{-i}, z_{-i}) \cap UC(R'_{-i}, z_{-i})\} \subset LC^0(\bar{R}_i, z'_i)$.

There are domains that satisfy the above three properties and yet, over which we do have *efficient, strategy-proof*, and *non-dictatorial* rules.

Example 1 *Risk sharing with an objective probability distribution and aggregate certainty*: Let l be the number of states. Each state $k = 1, \cdots, l$ is realized with probability π_k. Each bundle $x \in \mathbb{R}^l_+$ is a state-contingent commodity. Let $\mathcal{R}_*$ be the family of all preferences $R_0 \in \mathcal{R}$ that has the following "expected utility" representation: there exists a *concave* function $u_0 \colon \mathbb{R}_+ \to \mathbb{R}$ such that for all $x, x' \in \mathbb{R}^l_+$, $x \; R_0 \; x' \iff \sum_{k=1}^{l} \pi_k u_0(x_k) \geq \sum_{k=1}^{l} \pi_k u_0(x'_k)$. Suppose *aggregate certainty,* that is, $\Omega_1 = \cdots = \Omega_l$. Let $\bar{m}$ be the constant aggregate wealth across states.

The equal division $((\frac{\bar{m}}{2}, \cdots, \frac{\bar{m}}{2}), (\frac{\bar{m}}{2}, \cdots, \frac{\bar{m}}{2}))$ is *efficient* for all profiles in $\mathcal{R}^N_*$. Let $\varphi^{\text{ed}} \colon \mathcal{R}^N_* \to Z$ be the *equal division rule*, that is, for all $R \in \mathcal{R}^N_*$, $\varphi^{\text{ed}}(R) \equiv ((\frac{\bar{m}}{2}, \cdots, \frac{\bar{m}}{2}), (\frac{\bar{m}}{2}, \cdots, \frac{\bar{m}}{2}))$. Then φ^{ed} is *efficient* and *strategy-proof* over $\mathcal{R}^N_*$.

Let $M \equiv \{z \in Z : \text{for all } i \in N,\ z_{i1} = \cdots = z_{il}\}$. We now show that $\mathcal{R}^N_*$ and M satisfy the above three properties. *Potential efficiency* and *attainability* are trivial. Let $R, R' \in \mathcal{R}^N_*$ be such that $P(R) = P(R') = M$. Let $z, z' \in M$ and $z \neq z'$. Then since M_i is a monotonic path for each $i \in N$, without loss of generality we may assume $z_1 < z'_1$. Since z_1 is on the 45°-line, there exists $\bar{R}_1 \in \mathcal{R}_*$ such that $\bar{R}_1$ is *strictly convex* and $LC(R_1, z_1) \cap UC(\bar{R}_1, z_1) = \{z_1\}$. Then (i) of *transformability with crossly local dominance* holds. Since $\bar{R}_1$ is strictly convex, $P_1(\bar{R}_1, R'_2) = M_1$. Therefore, $P_1(\bar{R}_1, R'_2) \cap \{x \in Z_0 : \Omega - x \in LC(R_2, z_2) \cap UC(R'_2, z_2)\} = \{z_1\}$. Hence (ii) also holds. □

We show that under the following additional richness properties, no rule satisfies *efficiency, strategy-proofness*, and *non-dictatorship.* We use the following notation.

For each differentiable numerical representation u_i of R_i and all $z_i \in \mathbb{R}^l_{++}$, let $\nabla u_i(z_i) \equiv (\partial u_i(z_i)/\partial x_1, \cdots, \partial u_i(z_i)/\partial x_l)$. The hyperplane through z_i normal to $\nabla u_i(z_i)$ supports $UC(R_i, z_i)$. Formally, for all $p \in \mathbb{R}^l_{++}$ and all $z_0 \in Z_0$, let $H(p,z_0) \equiv \{x \in Z_0 : p \cdot x \geq p \cdot z_0\}$ be the set of bundles lying above the hyperplane through z_0 normal to p. We say that $H(p,z_0)$ *supports* R_i *at* z_0 if $UC(R_i,z_0) \subseteq H(p,z_0)$. Similarly, for all $R_0 \in \mathcal{R}$ and all $z_0 \in Z_0$, let $\nabla R_0(z_0) \equiv \{p \in \mathbb{R}^l_{++} : UC(R_0,z_0) \subseteq H(p,z_0)\}$ be the set of all vectors normal to a hyperplane supporting R_0 at z_0. For all $R \in \mathcal{D}$ and all $z \in Z$, let $\nabla R(z) \equiv \{p \in \mathbb{R}^l_{++} : UC(R_1,z_1) \subseteq H(p,z_1) \text{ and } UC(R_2,z_2) \subseteq H(p,z_2)\}$ be the set of all vectors normal to a hyperplane supporting both R_1 at z_1 and R_2 at z_2. Note that when z is *efficient for* R, $\nabla R(z) \neq \varnothing$.

A domain is **flexible** if there exist a subdomain and a reference set satisfying *potential efficiency, attainability, transformability with crossly local dominance*, and the following two properties, F1 and F2.

Condition F1 states that for any preference and any bundle, it is admissible to flatten the preference sufficiently without changing the local structure of the preference at the bundle.

F1: For all $R \in \bar{\mathcal{D}}$, all $i \in N$, all $z \in P(R)$, and all $x \in \mathbb{R}^l_+$,
if for some $p \in \nabla R(z)$, $p \cdot z_i < p \cdot x$, then there exists $R'_i \in \bar{\mathcal{D}}_i(R_{-i})$ such that

$$P_i(R'_i, R_{-i}) \cap LC(R_i,z_i) \cap UC(R'_i,z_i) = \{z_i\} \text{ and } x \; P'_i \; z_i.$$

Note that F1 implies the admissibility of local transformation.

Condition F2 states that for each agent i and allocation $d \in M$, there exists a profile $R \in \bar{\mathcal{D}}$ whose Pareto set intersects with M_i only at $(0,\Omega)$ and $(\Omega,0)$, and such that whenever an efficient allocation $z \neq d$ for R happens to have d in its supporting hyperplane, changing i's preference is admissible so that for the new profile (R'_i, R_{-i}), such a coincidence never happens at any efficient allocation in i's incentive compatibility set associated with (R_i, R'_i, z_i), $LC(R_i,z_i) \cap UC(R'_i,z_i)$.

F2: For all $i \in N$ and all $d \in M$, there exists $R \in \bar{\mathcal{D}}$ such that
(i) $P_i(R) \cap M_i = \{0,\Omega\}$, and
(ii) if $z \in P(R) \backslash \{d\}$ and $p \cdot z_i = p \cdot d_i$ for all $p \in \nabla R(z)$, then there exists $R'_i \in \bar{\mathcal{D}}_i(R_{-i})$ such that for all $z'_i \in P_i(R'_i, R_{-i}) \cap LC(R_i,z_i) \cap UC(R'_i,z_i)$,

$$p' \cdot z'_i \neq p' \cdot d_i, \text{ for some } p' \in \nabla(R'_i, R_{-i})(z').$$

We next provide an example of *flexible* domain.

Example 2 *Homothetic preferences*: Let $\mathcal{R}_H$ be the family of all *homothetic* preferences that are *smooth*,[6] *strictly convex*, and *strictly monotonic over* $\mathbb{R}^l_{++}$.

[6]A preference $R_0 \in \mathcal{R}$ is **smooth** if for all $x \in \mathbb{R}^l_+$ with $0 < x < \Omega$, $\nabla R_0(x)$ is unique.

Let $\bar{\mathcal{D}} \equiv \mathcal{R}_H^N$ and $M \equiv \{z \in Z$: for some $\lambda \in [0,1]$, $z_1 = \lambda\Omega + (1-\lambda)\,0$ and $z_2 = \Omega - z_1\}$. Then we show that $\bar{\mathcal{D}}$ and M satisfy *potential efficiency, attainability, transformability with crossly local dominance,* F1, and F2; so $\mathcal{R}_H^N$ is *flexible*.

For all $R_i \in \mathcal{R}_H$, if $R_{-i} = R_i$, $P(R_i, R_{-i}) = M$. Hence *potential efficiency* and *attainability* hold.

In order to show *transformability with crossly local dominance*, let $R, R' \in \bar{\mathcal{D}}$ and $z, z' \in M$ be such that $P(R) = P(R') = M$ and $z \neq z'$. Without loss of generality, let $z_1 < z_1'$. When $z_1 = 0$, if we let $\bar{R}_1 = R_1$, then (i) and (ii) of *transformability with crossly local dominance* hold. Now suppose $z_1 \neq 0$. Then let R_1^{Leon} be the Leontieff-type preference with the locus of kinks equal to M_1. Then (i) and (ii) holds with $\bar{R}_1 = R_1^{\text{Leon}}$. Note that $R_1^{\text{Leon}} \notin \mathcal{R}_H$ but that $\mathcal{R}_H$ contains a sequence of preferences, which is composed of local transformations of R_1 at z_1 relative to R_{-1} and, at the same time, converges to R_1^{Leon}. Therefore, there exist a local transformation of R_1, $\bar{R}_1$, which is sufficiently close to R_1^{Leon} so that (i) and (ii) can be satisfied.

For all $x \in \mathbb{R}_{++}^l$ and all $p \in \mathbb{R}_{++}^l$, there exists a sequence of preferences in $\mathcal{R}_H$, which are supported by p at x and converge to the linear preference associated with normal vector p. Therefore F1 holds. Now we only have to verify F2. We show F2 for the 2-good case. However, our argument can be easily extended to the l-good case.

Let $i \equiv 1$ and $d \in M$. Let R be the preference such that $P_1(R) = \{z_1 : z_{11} = 0$ or $z_{12} = \Omega_2\}$ and the slope of indifference curves of R_1 over $P_1(R)$ is bounded *above* by $-\delta < 0$. Clearly R satisfies (i) of F2. Let $z \in P(R) \setminus \{d\}$ be such that for all $p \in \nabla R(z)$, $p \cdot z_1 = p \cdot d_1$. Then there exists $R_1' \in \mathcal{R}_H$ such that the slopes of indifference curves of R_1' is bounded *below* by $-\delta$. Then clearly, $P(R_1', R_2) = P(R)$ and since $P_1(R)$ is a boundary and monotonic path of the Edgeworth box, $P_1(R_1', R_2) \cap LC(R_1, z_1) \cap UC(R_1', z_1) = \{z_1\}$. Since for all $p \in \nabla R(z)$, $p \cdot z_1 = p \cdot d_1$, and the indifference curve of R_1' through z_1 is flatter at z_1 than the indifference curve of R_1, there exists $p' \in \nabla(R_i', R_{-i})(z')$ such that $p' \cdot z_1 \neq p' \cdot d_1$. Therefore, (ii) of F2 also holds. □

We next provide an example in which F1 and F2 do not hold.

Example 3 *Linear domain*: Let $\mathcal{R}_L$ be the family of preferences that are represented by linear utility functions. Let $\bar{\mathcal{D}} \equiv \mathcal{R}_L^N$. Then we show that for any reference set M, $\bar{\mathcal{D}}$ and M do not satisfy F1. Let $R_1 = R_2 \in \mathcal{R}_L$. Then $P(R) = Z$. Let $z \in P(R)$ be such that $z_1 \in \mathbb{R}_{++}^L$. Then for all $R_1' \in \mathcal{R}_L$, either $z_1 \notin P_1(R_1', R_2)$ or $R_1' = R_1$; hence there is no local transformation of R_1 at z_1 relative to $R_2(= R_1)$. □

We show in Lemma 1 that F1 and F2 imply the following more general property.

Double transformability: For all $i \in N$ and all $d \in M$, there exists $R \in \bar{\mathcal{D}}$ with $P_i(R) \cap M_i = \{0, \Omega\}$ such that for all $z \in P(R)$ with $z_i \neq d_i$, there exists $R'_i \in \bar{\mathcal{D}}_i(R_{-i})$ such that

if $z'_i \in P_i(R'_i, R_{-i}) \cap UC(R'_i, z_i) \cap LC(R_i, z_i)$, then either

(ii-1) for some $R''_i \in \bar{\mathcal{D}}_i(R_{-i})$,

$$P_i(R''_i, R_{-i}) \cap LC(R'_i, z'_i) \cap UC(R''_i, z'_i) = \{z'_i\} \text{ and } d_i \; P''_i \; z'_i,$$

or

(ii-2) for some $R'_{-i} \in \bar{\mathcal{D}}_{-i}(R'_i)$,

$$P_{-i}(R'_i, R'_{-i}) \cap LC(R_{-i}, z'_{-i}) \cap UC(R'_{-i}, z'_{-i}) = \{z'_{-i}\} \text{ and } d_{-i} \; P'_{-i} \; z'_{-i}.$$

Double transformability has wider applicability. For example, as we saw in Example 3, there is no reference path M such that $\mathcal{R}_L^N$ and M satisfy *flexibility*. However, our conclusion in Section 4.1 shows that there exists a reference path M such that $\mathcal{R}_L^N$ and M satisfy *double transformability*.

A domain $\mathcal{D}$ satisfies **rich transformability** if there exist a subdomain $\bar{\mathcal{D}}$ and a reference set M satisfying *potential efficiency, attainability, trasformability with crossly local dominance*, and *double transformability*. In Lemma 1, we show that every *flexible* domain satisfies *rich transformability*.

Lemma 1: *Every flexible domain satisfies rich transformability.*

Proof: Let $\mathcal{D}$ be *flexible* with respect to a subdomain $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and a reference set $M \subseteq Z$. Let $i \in N$ and $d \in M$. By F2, there exists $R \in \bar{\mathcal{D}}$ such that $P_i(R) \cap M_i = \{0, \Omega\}$. Let $z \in P(R)$ with $z_i \neq d_i$. We divide into three cases.

Case 1: *There exists $p \in \nabla R(z)$ such that $p \cdot z_i < p \cdot d_i$.*

Then by F1, there exists $R'_i \in \bar{\mathcal{D}}_i(R_{-i})$ such that

$$P_i(R'_i, R_{-i}) \cap LC(R_i, z_i) \cap UC(R'_i, z_i) = \{z_i\} \text{ and } d_i \; P'_i \; z_i.$$

Therefore, if we let $R''_i \equiv R'_i$, then (ii)-1 holds. $\square$

Case 2: *There exists $p \in \nabla R(z)$ such that $p \cdot z_i > p \cdot d_i$.*

By F1, there exists $R'_i \in \bar{\mathcal{D}}_i(R_{-i})$ such that $P_i(R'_i, R_{-i}) \cap LC(R_i, z_i) \cap UC(R'_i, z_i) = \{z_i\}$.

Since $p \cdot z_{-i} < p \cdot d_{-i}$, then applying F1 for (R_i', R_{-i}) and agent $-i$, there exists $R_{-i}' \in \bar{\mathcal{D}}_{-i}(R_i')$ such that

$$P_i(R_i', R_{-i}') \cap LC(R_{-i}, z_{-i}) \cap UC(R_{-i}', z_i) = \{z_{-i}\} \text{ and } d_{-i}\ P_{-i}'\ z_{-i}.$$

Therefore, if we let $R_i' \equiv R_i$, then (ii)-2 holds. □

Case 3: *For all* $p \in \nabla R(z)$, $p \cdot z_i = p \cdot d_i$.

By F2, there exists $R_i' \in \bar{\mathcal{D}}_i(R_{-i})$ such that for all $z_i' \in P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i)$,

$$p' \cdot z_i' \neq p' \cdot d_i, \text{ for some } p' \in \nabla(R_i', R_{-i})(z').$$

If $z_i' \in P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i)$ and $p' \cdot z_i' < p' \cdot d_i$, for some $p' \in \nabla(R_i', R_{-i})(z')$, then by F1, there exists $R_i'' \in \bar{\mathcal{D}}_i(R_{-i})$ such that

$$P_i(R_i'', R_{-i}) \cap LC(R_i', z_i) \cap UC(R_i'', z_i) = \{z_i'\} \text{ and } d_i\ P_i''\ z_i'.$$

Hence (ii-1) holds.

On the other hand, if $z_i' \in P_i(R_i', R_{-i}) \cap LC(R_i, z_i) \cap UC(R_i', z_i)$ and $p' \cdot z_i' > p' \cdot d_i$, for some $p' \in \nabla(R_i', R_{-i})(z')$, then $p' \cdot z_{-i}' < p' \cdot d_{-i}$. Now applying F1 for (R_i', R_{-i}) and agent $-i$, there exists $R_{-i}' \in \bar{\mathcal{D}}_{-i}(R_i')$ such that

$$P_i(R_i', R_{-i}') \cap LC(R_{-i}, z_{-i}') \cap UC(R_{-i}', z_i') = \{z_{-i}'\} \text{ and } d_{-i}\ P_{-i}'\ z_{-i}'.$$

Hence (ii-2) holds. □ *Q.E.D.*

Zhou (1991) and Schummer (1997) establish an invariance property of *efficient* and *strategy-proof* rule with respect to "Maskin monotonic" transformations of preferences.[7] Lemma 1 states an even stronger invariance property related with local transformation. Formally, a rule φ is **invariant with respect to local transformation** if for all $R \in \mathcal{D}$ and all $i \in N$, if R_i' is a local transformation of R_i at z_i relative to R_{-i}, then $\varphi(R_i', R_{-i}) = \varphi(R)$.

Lemma 2: *Every efficient and strategy-proof rule is invariant with respect to local transformation.*

Proof: Let φ be an *efficient* and *strategy-proof* rule. Let $z \equiv \varphi(R)$, $i \in N$, and R_i' be a local transformation of R_i at z_i relative to R_{-i}. Let $z' \equiv \varphi(R_i', R_{-i})$.

[7] Let $R \in \mathcal{D}$. Let $z \in P(R)$. A preference R_i' is a (strong) *Maskin monotonic transformation of* R_i *at* z if $LC(R_i', z_i) \supseteq LC(R_i, z_i)$ and $LC(R_i, z_i) \cap UC(R_i', z_i) = \{z_i\}$. Clearly, such R_i' is a local transformation of R_i at z_i relative to R_{-i}. However, there are various local transformations that are not Maskin monotonic transformations.

By the two incentive compatibility conditions associated with (R_i, R'_i, z_i) and (R'_i, R_i, z'_i), $z'_i \in LC(R_i, z_i) \cap UC(R'_i, z_i)$. Therefore by *efficiency,* $z'_i \in P_i(R'_i, R_{-i}) \cap LC(R_i, z_i) \cap UC(R'_i, z_i) = \{z_i\}$. Hence $z'_i = z_i$, that is, $\varphi(R'_i, R_{-i}) = \varphi(R)$.

Q.E.D.

This strong invariance property plays an important role in our result. As we will see in the following argument, it simplifies the proof of Zhou (1991) and Schummer (1997) and moreover leads us to the impossibility result for a number of other restricted domains.

Next, we show that if our domain has *rich transformability* with respect to a subdomain $\bar{\mathcal{D}}$ and a set M, then for a rule to be *efficient* and *strategy-proof,* it should always pick a fixed allocation for each economy in $\bar{\mathcal{D}}$ for which M is the Pareto set.

Lemma 3: *Let $\mathcal{D}$ have rich transformability with respect to $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and $M \subset Z$. Let $\varphi\colon \mathcal{D} \to Z$ be efficient and strategy-proof. Then for all $R, R' \in \bar{\mathcal{D}}$, if $P(R) = P(R') = M$, then $\varphi(R) = \varphi(R')$.*[8]

Proof: Let $\mathcal{D}$ have *rich transformability* with respect to $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and $M \subset Z$. Let φ be *efficient* and *strategy-proof.* Let $R, R' \in \bar{\mathcal{D}}$ be such that $P(R) = P(R') = M$. Let $z \equiv \varphi(R)$ and $z' \equiv \varphi(R')$.

Suppose to the contrary $z_1 \neq z'_1$. By *transformability with crossly local dominance,* there exists $\bar{R}_1 \in \bar{\mathcal{D}}_1(R_2) \cap \bar{\mathcal{D}}_1(R'_2)$ such that

(i) $P_1(\bar{R}_1, R_2) \cap LC(R_1, z_1) \cap UC(\bar{R}_1, z_1) = \{z_1\}$;

(ii) $P_1(\bar{R}_1, R'_2) \cap \{x \in \mathbb{R}^l_+ : \Omega - x \in LC(R_2, z_2) \cap UC(R'_2, z_2)\} \subset LC^0(\bar{R}_1, z'_1)$.

By (i) and Lemma 2, $\varphi(\bar{R}_1, R_2) = z$. By the incentive compatibility associated with (R_2, R'_2, z), $\varphi_2(\bar{R}_1, R'_2) \in LC(R_2, z_2) \cap UC(R'_2, z_2)$. Hence by *efficiency,*

$$\varphi_1(\bar{R}_1, R'_2) \in P_1(\bar{R}_1, R'_2) \cap \{x \in \mathbb{R}^l_+ : \Omega - x \in LC(R_2, z_2) \cap UC(R'_2, z_2)\}.$$

Therefore by (ii), $\varphi_1(\bar{R}_1, R'_2) \in LC^0(\bar{R}_1, z'_1)$: that is, $\varphi_1(R'_1, R'_2)\ \bar{P}_1\ \varphi_1(\bar{R}_1, R'_2)$. This contradicts *strategy-proofness.* *Q.E.D.*

[8] Lemma 3 corresponds to Step 4 of Proof of Theorem 1 by Zhou (1991) and Lemmas 2, 3, and Corollary 1 by Schummer (1997). Zhou and Schummer make use of Maskin monotonic transformation in the proofs. Particularly in Schummer (1997), M_i is the line segment between 0 and Ω. He uses a preference which is a Maskin monotonic transformation of both R_i at z_i and R_i at z'_i, where $z_i, z'_i \in M_i$. This preference should be kinked as far as it is homothetic and R_i has a different supporting hyperplane at z_i from the supporting hyperplane of R'_i at z'_i. In restricted domains without kinked preferences, the proof in Schummer (1997) does not work. Our proof does not necessarily require such Maskin monotonic transformation. We only use a preference that satisfies (i) and (ii) in the above proof. Our argument is based on strong invariance property established in Lemma 2. Consequently, as we show in Example 2 and in Section 4 later, Lemma 3 applies in a number of domains without kinked preferences.

We will show that the allocation commonly chosen for profiles with Pareto set M, should be either $(\Omega, 0)$ or $(0, \Omega)$. We first show that when a rule gives one agent the whole endowment at a profile, for it to be *efficient* and *strategy-proof,* it should be dictatorial over a certain neighborhood of the initial profile.[9] In this sense, dictatorship at a profile contaminates the choices made for some other profiles. For the formal description, we need the following notation.

Let $R \in \mathcal{D}$ and $i \in N$. Let $\mathcal{S}^i(R) \equiv \{R' \in \mathcal{D} : I'_{-i}(0) \supseteq I_{-i}(0)$ and there exists $R''_i \in \mathcal{D}_i$ such that $R'_{-i} \in \mathcal{D}_{-i}(R''_i)$ and $R''_i \in \mathcal{D}_i(R_{-i})\}$. Note that if $R' \in \mathcal{S}^i(R)$, then R' is reachable from R through the following three unilateral variations: $(R_i, R_{-i}) \to (R''_i, R_{-i}) \to \left(R''_i, R'_{-i}\right) \to \left(R'_i, R'_{-i}\right)$, where $R''_i \in \mathcal{D}_i$ is such that $R'_{-i} \in \mathcal{D}_{-i}(R''_i)$ and $R''_i \in \mathcal{D}_i(R_{-i})$. Since $I'_{-i}(0) \supseteq I_{-i}(0)$, all these unilateral variations are 0-indifference-monotonic for $-i$. Note also that if R' is a unilateral variation of R, which is 0-indifference-monotonic for $-i$, then $R' \in \mathcal{S}^i(R)$.

Let $\bar{\mathcal{S}}^i(R)$ be defined as follows: for all $R' \in \mathcal{D}$, $R' \in \bar{\mathcal{S}}^i(R)$ if and only if there exists a finite sequence $(R^1, \cdots, R^n)$ of profiles in $\mathcal{D}$, $n \geq 2$, such that $R^1 \equiv R$, $R^n \equiv R'$, and $R^2 \in \mathcal{S}^i(R^1), \cdots, R^n \in \mathcal{S}^i(R^{n-1})$. We call $\bar{\mathcal{S}}^i(R)$ the **contamination set relative to R and i**. Then every $R' \in \bar{\mathcal{S}}^i(R)$ is reachable from R through iterative unilateral variations 0-indifference-monotonic for $-i$ and conversely.

When $\mathcal{D}$ is *everywhere reachable** and R_{-i} is strictly monotonic, every $R' \in \mathcal{D}$ is reachable from R through iterative unilateral variations 0-indifference-monotonic for $-i$. Hence $\bar{\mathcal{S}}^i(R) = \mathcal{D}$.

Lemma 4: *Let $\varphi \colon \mathcal{D} \to Z$ be efficient and strategy-proof. If there exist $i \in N$ and $R \in \mathcal{D}$ such that $\varphi_i(R) = \Omega$, then φ is dictatorial over $\bar{\mathcal{S}}^i(R)$.*

Proof: Let $\varphi \colon \mathcal{D} \to Z$ be *efficient* and *strategy-proof.* We only have to show that for all $R \in \mathcal{D}$, all $i \in N$, and all $R' \in \mathcal{S}^i(R)$, $\varphi_i(R) = \Omega \Rightarrow \varphi_i(R') = \Omega$.

Let R, i, and R' be given as above. By definition of $\mathcal{S}^i(R)$, $I'_{-i}(0) \supseteq I_{-i}(0)$ and there exists $R''_i \in \mathcal{D}_i$ such that $R'_i \in \mathcal{D}_i(R'_{-i})$, $R'_{-i} \in \mathcal{D}_{-i}(R''_i)$, and $R''_i \in \mathcal{D}_i(R_{-i})$. Since R_i is strictly monotonic over $\mathbb{R}^l_{++}$, then by i's incentive compatibility condition relative to (R_i, R''_i, Ω), $\varphi_i(R''_i, R_{-i}) = \Omega$ and so $\varphi_{-i}(R''_i, R_{-i}) = 0$. By $-i$'s incentive compatibility condition relative to $(R_{-i}, R'_{-i}, 0)$, $\varphi_{-i}\left(R''_i, R'_{-i}\right) \; I_{-i} \; 0$. Since $I'_{-i}(0) \supseteq I_{-i}(0)$, $\varphi_{-i}\left(R''_i, R'_{-i}\right) \; I'_{-i} \; 0$. Therefore, by *efficiency,* $\varphi_{-i}\left(R''_i, R'_{-i}\right) = 0$ and so $\varphi_i\left(R''_i, R'_{-i}\right) = \Omega$. Finally, by i's incentive compatibility condition relative to (R''_i, R'_i, Ω), $\varphi_i\left(R'_i, R'_{-i}\right) = \Omega$. *Q.E.D.*

Let $\mathcal{D}$ have *rich transformability* with respect to $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and $M \subset Z$. Given

[9] By using the term "neighborhood" of a profile R, we do not mean an "open" set containing R. It simply means a set containing R.

$(\bar{\mathcal{D}}, M)$, we call $\bigcup_{R \in \{R' \in \bar{\mathcal{D}}\colon P(R')=M\}} \bar{\mathcal{S}}^i(R)$ **i's minimal contamination set relative to** $\boldsymbol{(\bar{\mathcal{D}}, M)}$.

Proposition 1: *Assume that domain $\mathcal{D}$ satisfies rich transformability with respect to $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and $M \subset \mathbb{R}^l_+$. Then if a rule over $\mathcal{D}$ is efficient and strategy-proof, then for some $i \in N$, the rule is dictatorial over i's minimal contamination set relative to $(\bar{\mathcal{D}}, M)$.*[10]

Proof: Let $\varphi\colon \mathcal{D} \to Z$ be *efficient* and *strategy-proof.* Let d be an allocation such that for all $R' \in \bar{\mathcal{D}}$, if $P(R') = M$, then $\varphi(R') = d$. By *potential efficiency* and Lemma 3, d is well-defined. Let $R \in \bar{\mathcal{D}}$ be such that $P(R) = M$. By Lemmas 3 and 4, we only have to show that $d_1 \in \{0, \Omega\}$.

Suppose to the contrary that $d_1 \notin \{0, \Omega\}$. By *double transformability*, there exists $R \in \bar{\mathcal{D}}$ such that
(i) $P_1(R) \cap M_1 = \{0, \Omega\}$ and
(ii) if $z \in P(R)$ with $z_1 \neq d_1$, then there exists $R'_1 \in \bar{\mathcal{D}}_1(R_2)$ such that for all $z'_1 \in P_1(R'_1, R_2) \cap LC(R_1, z_1) \cap UC(R'_1, z_1)$, either
(ii-1) for some $R''_1 \in \bar{\mathcal{D}}_1(R_2)$,

$$P_1(R''_1, R_2) \cap LC(R'_1, z'_1) \cap UC(R''_1, z'_1) = \{z'_1\} \text{ and } d_1 \ P''_1 \ z'_1$$

or
(ii-2) for some $R'_2 \in \bar{\mathcal{D}}_2(R'_1)$,

$$P_2(R'_1, R'_2) \cap LC(R_2, z'_2) \cap UC(R'_2, z'_2) = \{z'_2\} \text{ and } d_2 \ P'_2 \ z'_2.$$

Let $z \equiv \varphi(R)$. Then clearly $z \in P(R)$ and by (i), $z_1 \neq d_1$. Therefore by *efficiency* and *strategy-proofness* and (ii), there exists $R'_1 \in \bar{\mathcal{D}}_1(R_2)$ such that either (ii-1) or (ii-2) holds at $z'_1 \equiv \varphi_1(R'_1, R_2)$.

When (ii-1) holds, there exists $R''_1 \in \mathcal{D}_1(R_2)$ such that $d_1 \ P''_1 \ z'_1$ and $P_1(R''_1, R_2) \cap LC(R'_1, z'_1) \cap UC(R''_1, z'_1) = \{z'_1\}$. By Lemma 2, $\varphi(R''_1, R_2) = z'$. By *attainability,* there exists $\bar{R}_1 \in \mathcal{D}_1(R_2)$ such that $P(\bar{R}_1, R_2) = M$. Since $\varphi(\bar{R}_1, R_2) = d$, $\varphi_1(\bar{R}_1, R_2) \ P''_1 \ \varphi_1(R''_1, R_2)$. This contradicts *strategy-proofness.*

When (ii-2) holds, similarly we derive a contradiction. *Q.E.D.*

When the domain satisfies *everywhere reachability** in addition, the minimal contamination set in Proposition 1 coincides with the entire domain. Therefore, dictatorial rules are the only *efficient* and *strategy-proof* rules.

[10] Proposition 1 corresponds to Steps 2, 5, and 6 in Proof of Theorem 1 by Zhou (1991) and Proof of Theorem 1 by Schummer (1997). Zhou and Schummer construct kinked preferences and make use of the invariance of *strategy-proof* and *efficient* rules with respect to Maskin monotonic transformations. Our proof is simpler and works well without kinked preferences. This is because our proof makes use of the stronger invariance property in Lemma 2.

Theorem 1: *Assume that domain $\mathcal{D}$ satisfies rich transformability and everywhere reachability*. Then a rule over $\mathcal{D}$ is efficient and strategy-proof if and only if it is dictatorial.*

Proof: Let $\mathcal{D}$ satisfy *rich transformability* and *everywhere reachability**. Then there exist $\bar{\mathcal{D}} \subseteq \mathcal{D}$ and $M \subset \mathbb{R}^l_+$ satisfying *potential efficiency, attainability, transformability with crossly local dominance,* and *double transformability.* Let $\varphi\colon \mathcal{D} \to Z$ be *efficient* and *strategy-proof.* Then by Proposition 1, there exists $i \in N$ such that φ is dictatorial over i's minimal contamination set relative to $(\bar{\mathcal{D}}, M)$. By *potential efficiency,* there exists $R \in \bar{\mathcal{D}}$ such that $P(R) = M$ and both R_1 and R_2 are strictly monotonic. Then since R_{-i} is strictly monotonic and $\mathcal{D}$ satisfies *everywhere reachability**, $\bar{\mathcal{S}}^i(R) = \mathcal{D}$. Therefore φ is dictatorial.

Q.E.D.

Remark 1: Theorem 1 applies to product domains with *rich transformability.* Also it applies to *everywhere reachable* domains with *rich transformability*, when preferences are strictly monotonic.

Remark 2: Both *rich transformability* and *everywhere reachability** are essential in Theorem 1. Example 1 shows that without *rich transformability*, the impossibility does not hold. Without *everywhere reachability**, the impossibility does not hold either. The following example shows this. Let $\mathcal{D}_a$ and $\mathcal{D}_b$ be such that $\mathcal{D} \equiv \mathcal{D}_a \cup \mathcal{D}_b$. Suppose that for all $R \in \mathcal{D}_a$ and all $R' \in \mathcal{D}_b$, R' cannot be reached from R through iterative unilateral variations. Now let φ be dictatorial over $\mathcal{D}_a$ and agent 1 be the dictator over $\mathcal{D}_a$. Let φ be dictatorial over $\mathcal{D}_b$ and agent 2 be the dictator over $\mathcal{D}_b$. Then φ is *efficient, strategy-proof,* and *non-dictatorial.*

4 Applications

In this section, we apply our result in Section 3 to "intertemporal exchange problem", "risk sharing problem", and two restricted domains, the "CES domain" and the "quasilinear domain".

4.1 Intertemporal exchange

In this section, we apply our main result to the following *intertemporal exchange problem.*

Let T be the number of periods, $T \geq 2$. For each $t = 1, \cdots, T$, let $\Omega_t > 0$ be the endowment of a single consumption good at period t. Suppose that there exists no saving technology. Then an allocation $(z_i)_N \in \mathbb{R}^{T \times N}_+$ is *feasible* if for all $t = 1, \cdots, T$, $\sum_i z_{it} \leq \Omega_t$.

Each agent $i \in N$ has a preference R_i represented by a *temporal utility function* $u_i \colon \mathbb{R}_+ \to \mathbb{R}$ and a *discount factor* $\delta_i \in (0,1)$ as follows: for all $x, y \in \mathbb{R}^T_+$,

$$x \; R_i \; y \iff \sum_{t=1}^{T} \delta_i^{t-1} u_i\left(x_t\right) \geq \sum_{t=1}^{T} \delta_i^{t-1} u_i\left(y_t\right).$$

When the temporal utility function u_i is *concave* (respectively, *strictly concave*), R_i is *convex* (respectively, *strictly convex*). Let $\mathcal{R}_{\text{IE-convex}}$ be the class of all such preferences represented by *concave*, *strictly monotonic*, and *continuous* temporal utility functions. Let $\mathcal{R}_{\text{IE-lin}}$ be the class of preferences in $\mathcal{R}_{\text{IE-convex}}$ with the *linear* temporal utility function, $u^{\text{lin}}(m) = m$, for all $m \in \mathbb{R}_+$. Let $\mathcal{R}_{\text{IE-s.convex}}$ be the set of preferences in $\mathcal{R}_{\text{IE-convex}}$ with *strictly concave* temporal utility functions.

We first consider $\mathcal{R}^N_{\text{IE-lin}}$, in which the temporal utility function is fixed by the linear function u^{lin} and the only variable parameter of each agent's preference is his discount factor.

We show that $\mathcal{R}^N_{\text{IE-lin}}$ satisfies *rich transformability* and so does $\mathcal{R}^N_{\text{IE-convex}}$. Therefore, the impossibility result in Theorem 1 applies to both domains.

Let $P^{\lrcorner} \equiv \{z \in Z \colon$ for some $t \in \{1, \cdots, T\}$, $z_1 = (\Omega_1, \cdots, \Omega_{t-1}, z_{1t}, 0, \cdots, 0)$ and $z_2 = (0, \cdots, 0, z_{2t}, \Omega_{t+1}, \cdots, \Omega_T)\}$. Let $P^{\ulcorner} \equiv \{z \in Z \colon$ for some t, $z_2 = (\Omega_1, \cdots, \Omega_{t-1}, z_{2t}, 0, \cdots, 0)$ and $z_1 = (0, \cdots, 0, z_{1t}, \Omega_{t+1}, \cdots, \Omega_T)\}$. Note that for each $i \in N$, both $P^{\lrcorner}_i$ and $P^{\ulcorner}_i$ are monotonic path from 0 to $(\Omega_1, \cdots, \Omega_T)$.

Proposition 2: *Both domains* $\mathcal{R}^N_{\text{IE-lin}}$ *and* $\mathcal{R}^N_{\text{IE-convex}}$ *in the intertemporal exchange problem satisfy rich transformability.*[11]

Proof: For each $\delta \in (0,1)$, the linear preference with discount factor δ is denoted by R^{lin}_δ. Then R^{lin}_δ is represented by the following utility function U^{lin}_δ: for all $x \in \mathbb{R}^T_+$, $U^{\text{lin}}_\delta(x) \equiv \sum_t \delta^{t-1} x_t$.

We make use of the following claim, which states that when both agents have linear preferences, the Pareto set is equal to $P^{\lrcorner}$ (respectively, $P^{\ulcorner}$) if and only if agent 2 is more (respectively, less) patient than agent 1. We omit the proof.

Claim 1: *For all* $\delta_1, \delta_2 \in (0,1)$, *(i)* $P\left(R^{\text{lin}}_{\delta_1}, R^{\text{lin}}_{\delta_2}\right) = P^{\lrcorner} \iff \delta_1 < \delta_2$; *(ii)* $P\left(R^{\text{lin}}_{\delta_1}, R^{\text{lin}}_{\delta_2}\right) = P^{\ulcorner} \iff \delta_1 > \delta_2$.

[11] In the two period case, $T = 2$, for each agent, there are infinitely many admissible linear preferences in $\mathcal{R}_{\text{IE-lin}}$. Schummer (1997) shows that in the 2-good exchange economy case, given any domain with at least four admissible linear preferences for each agent, dictatorial rules are the only *efficient* and *strategy-proof* rules. Therefore his result applies. Schummer (1997) extends this result for the 2-good case to the l-good case using specific preferences in which commodities are partitioned into two groups with identical marginal utilities. Such preferences are not admissible in $\mathcal{R}^N_{\text{IE-lin}}$, since marginal utility decreases in the rate of discount factor over periods. Therefore, when $T \geq 3$, Schummer's result does not apply.

Let $M \equiv P^{\lrcorner}$ and $\bar{\mathcal{D}} \equiv \mathcal{R}^N_{\text{lin}}$. Then both *potential efficiency* and *attainability* follow from Claim 1.

Transformability with crossly local dominance: Let $(\delta_1, \delta_2), (\delta'_1, \delta'_2) \in (0,1) \times (0,1)$ and $z, z' \in M$ be such that $z \neq z'$ and $P\left(R^{\text{lin}}_{\delta_1}, R^{\text{lin}}_{\delta_2}\right) = P(R^{\text{lin}}_{\delta'_1}, R^{\text{lin}}_{\delta'_2}) = M$. Without loss of generality, we assume $z_1 \leq z'_1$.

By Claim 1, $\delta_1 < \delta_2$ and $\delta'_1 < \delta'_2$. There exists $\delta^*_1 \in (0,1)$ such that $\delta^*_1 \leq \min\{\delta_1, \delta'_1\}$. Then by Claim 1, $P(R^{\text{lin}}_{\delta^*_1}, R^{\text{lin}}_{\delta_2}) = P^{\lrcorner}$. Since $P^{\lrcorner}$ is a monotonic path, $P_1(R^{\text{lin}}_{\delta^*_1}, R^{\text{lin}}_{\delta_2}) \cap LC(R^{\text{lin}}_{\delta_1}, z_1) \cap UC(R^{\text{lin}}_{\delta^*_1}, z_1) = \{z_1\}$. Also by Claim 1, $P_1(R^{\text{lin}}_{\delta^*_1}, R^{\text{lin}}_{\delta'_2}) = P^{\lrcorner}$. Hence $P_1(R^{\text{lin}}_{\delta^*_1}, R^{\text{lin}}_{\delta'_2}) \cap \{x \in Z_0 : \Omega - x \in LC(R^{\text{lin}}_{\delta_2}, z_2) \cap UC(R^{\text{lin}}_{\delta'_2}, z_2)\} = \{z\}$. Since $z_1 \leq z'_1$, $z_1 \in LC^0(R^{\text{lin}}_{\delta^*_1}, z'_1)$. □

Double transformability: Let $d \in M$. Let $\delta_1 > \delta_2$. Then $P\left(R^{\text{lin}}_{\delta_1}, R^{\text{lin}}_{\delta_2}\right) = P^{\ulcorner}$ and so $P_i\left(R^{\text{lin}}_{\delta_1}, R^{\text{lin}}_{\delta_2}\right) \cap M_i = \{0, \Omega\}$ for each $i \in N$. Let $z \in P\left(R^{\text{lin}}_{\delta_1}, R^{\text{lin}}_{\delta_2}\right)$ be such that $z_1 \neq d_1$. When $U^{\text{lin}}_{\delta_1}(z_1) < U^{\text{lin}}_{\delta_1}(d_1)$, if we let $R'_1 = R''_1 \equiv R^{\text{lin}}_{\delta_1}$, then (ii-1) of *double transformability* holds. When $U^{\text{lin}}_{\delta_2}(z_2) < U^{\text{lin}}_{\delta_2}(d_2)$, if we let $R'_1 \equiv R^{\text{lin}}_{\delta_1}$ and $R'_2 \equiv R^{\text{lin}}_{\delta_2}$, then (ii-2) holds.

Now assume $U^{\text{lin}}_{\delta_1}(z_1) \geq U^{\text{lin}}_{\delta_1}(d_1)$ and $U^{\text{lin}}_{\delta_2}(z_2) \geq U^{\text{lin}}_{\delta_2}(d_2)$. Then $d_1 \in P^{\lrcorner} \backslash \{0, \Omega\}$. So d is not *efficient* for $\left(R^{\text{lin}}_{\delta_1}, R^{\text{lin}}_{\delta_2}\right)$. Hence either $U^{\text{lin}}_{\delta_1}(z_1) > U^{\text{lin}}_{\delta_2}(d_1)$ or $U^{\text{lin}}_{\delta_2}(z_2) > U^{\text{lin}}_{\delta_2}(d_2)$.

We consider the case $U^{\text{lin}}_{\delta_1}(z_1) > U^{\text{lin}}_{\delta_2}(d_1)$ and $U^{\text{lin}}_{\delta_2}(z_2) \geq U^{\text{lin}}_{\delta_2}(d_2)$ (the same argument applies in the other case). Since $U^{\text{lin}}_{\delta_2}(z_2) = \sum_t \delta_2^{t-1}(\Omega_t - z_{1t}) \geq \sum_t \delta_2^{t-1}(\Omega_t - d_{1t}) = U^{\text{lin}}_{\delta_2}(d_2)$, then $U^{\text{lin}}_{\delta_2}(z_1) \leq U^{\text{lin}}_{\delta_2}(d_1)$. Since $U^{\text{lin}}_{\delta_1}(z_1) > U^{\text{lin}}_{\delta_1}(d_1)$, $U^{\text{lin}}_{\delta_2}(z_1) \leq U^{\text{lin}}_{\delta_2}(d_1)$, and $\delta_1 > \delta_2$, then there exists $\delta'_2 \in (\delta_2, \delta_1)$ such that $U^{\text{lin}}_{\delta'_2}(z_1) > U^{\text{lin}}_{\delta'_2}(d_1)$. Then, $U^{\text{lin}}_{\delta'_2}(z_2) < U^{\text{lin}}_{\delta'_2}(d_2)$. Now let $R'_1 \equiv R^{\text{lin}}_{\delta_1}$ and $R'_2 \equiv R^{\text{lin}}_{\delta'_2}$. Then $P(R'_1, R'_2) = P^{\ulcorner}$ and (ii-2) holds. □ *Q.E.D.*

We now consider the domain of preferences associated with *strictly concave* temporal utility functions. We show that this domain also satisfies *rich transformability.*

Proposition 3: *The domain $\mathcal{R}^N_{\text{IE-s.convex}}$ in the intertemporal exchange problem satisfies rich transformability.*

Proof: The following proof is similar to the proof of Proposition 2. In what follows, we assume $T = 2$. However, our proof can be extended to the general case.

Let $\rho > 0$ be given. Let $u\colon \mathbb{R}_+ \to \mathbb{R}$ be the following temporal utility function: for all $m \in \mathbb{R}_+$, $u(m) = -e^{-\rho m}$. Throughout the proof we use the following notation. For each discount factor $\delta \in (0,1)$, let $U_\delta\colon \mathbb{R}^2_+ \to \mathbb{R}$ be the utility function associated with u and δ and let R_δ be the corresponding preference.

For each $x \in \mathbb{R}^2$, let $MRS\,(x; U_\delta)$ be the ratio of the marginal utility of period 1 consumption and the marginal utility of period 2 consumption at x, that is, $MRS\,(x; U_\delta) = \frac{1}{\delta}\frac{e^{\rho x_2}}{e^{\rho x_1}}$. Then clearly, MRS is minimized at $(\Omega_1, 0)$ and maximized at $(0, \Omega_2)$ and it is easy to show the following claim.

Claim 1: *For all* $\delta_1, \delta_2 \in (0,1)$, *(i)* $P\,(R_{\delta_1}, R_{\delta_2}) = P^{\lrcorner} \iff \frac{\delta_2}{\delta_1} \geq e^{\rho(\Omega_1+\Omega_2)}$;
(ii) $P\,(R_{\delta_1}, R_{\delta_2}) = P^{\ulcorner} \iff \frac{\delta_1}{\delta_2} \geq e^{\rho(\Omega_1+\Omega_2)}$.

Let $\bar{\mathcal{D}}$ be the set of all preference profiles $(R_{\delta_1}, R_{\delta_2})$, where $\delta_1 \in \left(0, e^{-\rho(\Omega_1+\Omega_2)}\right)$ and $\delta_2 \in (0,1)$. So in $\bar{\mathcal{D}}$, the temporal utility function is known and the only unknown parameter of each preference is the discount factor. Let $M \equiv P^{\lrcorner}$. We show that $\bar{\mathcal{D}}$ and M satisfy *rich transformability.*

Potential efficiency holds by (i) of Claim 1. Since agent 1's discount factor is smaller than $e^{-\rho(\Omega_1+\Omega_2)}$, *attainability* also holds. *Transformability with crossly local dominance* can be shown by using Claim 1 and the same argument as in the proof of Proposition 2. We now show *double transformability.*

Double transformability: Let $d \in M$. Let $\delta_1 \in \left(0, e^{-\rho(\Omega_1+\Omega_2)}\right)$ and $\delta_2 \in (0,1)$ be such that $\frac{\delta_1}{\delta_2} \geq e^{-\rho(\Omega_1+\Omega_2)}$. Then By Claim 1, $P\,(R_{\delta_1}, R_{\delta_2}) = P^{\ulcorner}$ and so for each $i \in N$, $P_i\,(R_{\delta_1}, R_{\delta_2}) \cap M_i = \{0, \Omega\}$. Let $z \in P\,(R_{\delta_1}, R_{\delta_2})$ be such that $z_1 \neq d_1$. When $U_{\delta_1}(z_1) < U_{\delta_1}(d_1)$, if we let $R_1' = R_1'' \equiv R_{\delta_1}$, then (ii-1) of *double transformability* holds. When $U_{\delta_2}(z_2) < U_{\delta_2}(d_2)$, if we let $R_1' \equiv R_{\delta_1}$ and $R_2' \equiv R_{\delta_2}$, then (ii-2) holds.

Now assume $U_{\delta_1}(z_1) \geq U_{\delta_1}(d_1)$ and $U_{\delta_2}(z_2) \geq U_{\delta_2}(d_2)$. Since $d_1 \in P^{\lrcorner}\backslash\{0, \Omega\}$, d is not *efficient* for $(R_{\delta_1}, R_{\delta_2})$. Hence either $U_{\delta_1}(z_1) > U_{\delta_1}(d_1)$ or $U_{\delta_2}(z_2) > U_{\delta_2}(d_2)$. In what follows, we consider the case $U_{\delta_1}(z_1) > U_{\delta_1}(d_1)$ and $U_{\delta_2}(z_2) \geq U_{\delta_2}(d_2)$ (the same argument applies in the other case).

Since $U_{\delta_1}(z_1) > U_{\delta_1}(d_1)$, $z \in P^{\ulcorner}$, and $d \in P^{\lrcorner}$, then $\delta_1 > \frac{-e^{-\rho d_{11}}+e^{-\rho z_{11}}}{-e^{-\rho z_{12}}+e^{-\rho d_{12}}}$, $d_{11} > z_{11}$, and $d_{12} < z_{12}$ (so $d_{21} < z_{21}$ and $d_{22} > z_{22}$). Let $\delta_1' \in (0,1)$ be such that $\delta_1' < \frac{-e^{-\rho d_{11}}+e^{-\rho z_{11}}}{-e^{-\rho z_{12}}+e^{-\rho d_{12}}}$ (since $d_{11} > z_{11}$ and $d_{12} < z_{12}$, then there exists sufficiently small δ_1' satisfying the inequality). Then $U_{\delta_1'}(d_1) > U_{\delta_1'}(z_1)$ and $\delta_1' < \delta_1$.

Let $\delta_2' \equiv \min\{\delta_2, \delta_1'/e^{\rho(\Omega_1+\Omega_2)}\}$. Then $\frac{\delta_1'}{\delta_2'} \geq e^{\rho(\Omega_1+\Omega_2)}$. Therefore by Claim 1, $P(R_{\delta_1'}, R_{\delta_2'}) = P^{\ulcorner}$. Let $R_1' \equiv R_{\delta_1'}$ and $R_2' \equiv R_{\delta_2'}$. Then (ii-2) of *double transformability* holds. $\square$ *Q.E.D.*

Remark 3: In the proof of Proposition 3, we use only the restricted family of *smooth* preferences, $\bar{\mathcal{D}}$, in which each preference has a temporal utility function u of the following form: there exists $\rho \in (0,1)$ such that $u\,(m) = -e^{-\rho m}$ for all $m \in \mathbb{R}_+$. Therefore any domain including $\bar{\mathcal{D}}$ satisfies *rich transformability*; for example, the domain consisting of all profiles of *smooth* preferences in $\mathcal{R}^N_{\text{IE-s.convex}}$.

It may be the case that both agents share a common cultural background relevant to impatience. Then it is appealing to assume that their impatience levels are not too different; that is, the difference of their discount factors is bounded by a fixed positive number. For each $\mu > 0$, let $\mathcal{D}_{\text{IE-lin}\&|\delta_1-\delta_2|<\mu}$ be the family of linear preference profiles $(R_1, R_2) \in \mathcal{R}^N_{\text{lin}}$ such that the difference between the two discount factors δ_1 and δ_2 for R_1 and R_2 respectively is less than μ; that is, $|\delta_1 - \delta_2| < \mu$. Similarly, let $\mathcal{D}_{\text{IE-convex}\&|\delta_1-\delta_2|<\mu}$ be the family of preference profiles $(R_1, R_2) \in \mathcal{R}^N_{\text{IE-convex}}$ such that the difference between the two discount factors δ_1 and δ_2 for R_1 and R_2 respectively is less than μ. Similarly, we define $\mathcal{D}_{\text{IE-s.convex}\&|\delta_1-\delta_2|<\mu}$. The proof of Propositions 2 and 3 can be modified to derive the same conclusion for these domains. It is easy to show that the three domains satisfy *everywhere reachability**.

Corollary 1: *All three non-product domains,* $\mathcal{D}_{\text{IE-lin}\&|\delta_1-\delta_2|<\mu}$, $\mathcal{D}_{\text{IE-convex}\&|\delta_1-\delta_2|<\mu}$, *and* $\mathcal{D}_{\text{IE-s.convex}\&|\delta_1-\delta_2|<\mu}$, *satisfy rich transformability and everywhere reachability**.

Corollary 2: *Let* $\mathcal{D}$ *be one of the following domains in the intertemporal exchange problem,* $\mathcal{R}^N_{\text{IE-lin}}, \mathcal{R}^N_{\text{IE-convex}}, \mathcal{R}^N_{\text{IE-s.convex}}, \mathcal{D}_{\text{IE-lin}\&|\delta_1-\delta_2|<\mu}, \mathcal{D}_{\text{IE-convex}\&|\delta_1-\delta_2|<\mu}$, *and* $\mathcal{D}_{\text{IE-s.convex}\&|\delta_1-\delta_2|<\mu}$, *where* $\mu > 0$*. Then a rule over* $\mathcal{D}$ *is efficient and strategy-proof if and only if it is dictatorial.*

4.2 Risk sharing

In this section, we consider the following *risk sharing problem.*

Let S be the number of states, $S \geq 2$. For each $s = 1, \cdots, S$, let $\Omega_s > 0$ be the endowment at state s. We consider the problem of allocating these endowments prior to the realization of state. An allocation is a list of state contingent consumption bundles indexed by agents, $z \equiv (z_i)_{i \in N} \in \mathbb{R}^{S \times N}_+$.

Each agent $i \in N$ has a preference R_i that is represented by a subjective probability distribution, or belief, $\pi_i \equiv (\pi_{is})_s \in \Delta^{S-1}$ and a utility index $u_i \colon \mathbb{R}_+ \to \mathbb{R}$ in the expected utility form as follows: for all $x, y \in \mathbb{R}^S_+$

$$x \; R_i \; y \iff \sum_{s=1}^{S} \pi_{is} u_i (x_s) \geq \sum_{s=1}^{S} \pi_{is} u_i (y_s) .$$

We assume that $\pi_i > 0$ and that u_i is *strictly increasing* and *continuous.* We further assume that u_i is *concave.* Let $\mathcal{R}_{\text{Risk}}$ be the family of all such expected utility preferences. Preference R_i is **risk averse** if u_i is *strictly concave.* It is **risk neutral** if u_i is the linear function u^{lin}, that is, for all $m \in \mathbb{R}_+$, $u^{\text{lin}}(m) = m$. Let

$\mathcal{R}_{\text{Risk-aver}}$ be the family of all *risk averse* preferences. Let $\mathcal{R}_{\text{Risk-neut}}$ be the family of all *risk neutral* preferences.

Let $R_0 \in \mathcal{R}_{\text{IE-convex}}$ be the preference in Section 4.1, which is represented by a temporal convex utility function u_0 and discount factor $\delta \in (0,1)$. Then R_0 is represented by the following utility function U: for all $x \in \mathbb{R}^T_+$, $U(x) \equiv \sum_t \delta^{t-1} u_0(x_t)$. Therefore, when $T = S$, R_0 coincides with the preference in $\mathcal{R}_{\text{Risk}}$ with utility index u_0 and the following belief,

$$\left(\frac{1}{\sum_t \delta^{t-1}}, \frac{\delta}{\sum_t \delta^{t-1}}, \cdots, \frac{\delta^T}{\sum_t \delta^{t-1}}\right).$$

Therefore, $\mathcal{R}^N_{\text{IE-lin}} \subseteq \mathcal{R}^N_{\text{Risk-neut}} \subseteq \mathcal{R}^N_{\text{Risk}}$ and $\mathcal{R}^N_{\text{IE-s.convex}} \subseteq \mathcal{R}^N_{\text{Risk-aver}}$. Therefore, it follows directly from Propositions 2 and 3 that:

Proposition 4: *All three domains, $\mathcal{R}^N_{\text{Risk-neut}}$, $\mathcal{R}^N_{\text{Risk}}$, and $\mathcal{R}^N_{\text{Risk-aver}}$, in the risk sharing problem satisfy rich transformability.*

Remark 4: It follows from the proof of Proposition 3 that the following restricted domains in the risk sharing problem also satisfy *rich transformability*.

For each $\rho > 0$, let $u^\rho_{CARA} \colon \mathbb{R}_+ \to \mathbb{R}$ be such that for all $m \in \mathbb{R}_+$, $u^\rho_{CARA}(m) \equiv -e^{-\rho m}$. The utility index u^ρ_{CARA} exhibits constant "Arrow-Pratt coefficient of absolute risk aversion" equal to ρ; that is, for all $m \in \mathbb{R}_+$, $-\frac{d^2 u^\rho_{CARA}(m)/dm^2}{du^\rho_{CARA}(m)/dm} = \rho$.[12] Let $\mathcal{R}_{\text{Risk-CARA},\rho}$ be the family of all expected utility preferences represented by $\pi \in \Delta^{S-1}$ and u^ρ_{CARA}. Then in the domain $\mathcal{R}^N_{\text{Risk-CARA},\rho}$, agents' utility indices are fixed and the only unknown factors of preferences are their beliefs. It follows directly from the proof of Proposition 3, that for each $\rho > 0$, $\mathcal{R}^N_{\text{Risk-CARA},\rho}$ satisfies *rich transformability.*

Every domain including $\mathcal{R}^N_{\text{Risk-CARA},\rho}$ for some $\rho > 0$ satisfies *rich transformability.* For example, the domain consisting of profiles of expected utility preferences that are *smooth* and *risk averse* satisfies *rich transformability.*

When both agents share an information on the state space, their beliefs will be affected commonly by this information. Then, agents' beliefs may not be too far from each other. For each $\mu > 0$, let $\mathcal{D}_{\text{Risk-neut}\&|\pi_1-\pi_2|<\mu}$ be the family of risk neutral preference profiles $(R_1, R_2) \in \mathcal{R}^N_{\text{Risk-neut}}$ such that the difference between the two beliefs π_1 and π_2 for R_1 and R_2 respectively is less than μ; that is, $|\pi_1 - \pi_2| < \mu$. Similarly, let $\mathcal{D}_{\text{Risk}\&|\pi_1-\pi_2|<\mu}$ be the family of preference profiles $(R_1, R_2) \in \mathcal{R}^N_{\text{Risk}}$ such that the difference between the two beliefs π_1 and π_2 for R_1 and R_2 respectively is less than μ. Similarly, we define $\mathcal{D}_{\text{Risk-aver}\&|\pi_1-\pi_2|<\mu}$. As in

[12]CARA stands for the constant absolute risk aversion.

Corollary 1, the same conclusion as Proposition 4 can be derived in these domains. It is easy to show that the three domains satisfy *everywhere reachability**.

Corollary 3: *All three non-product domains,* $\mathcal{D}_{\text{Risk-neut}\&|\pi_1-\pi_2|<\mu}$, $\mathcal{D}_{\text{Risk}\&|\pi_1-\pi_2|<\mu}$, *and* $\mathcal{D}_{\text{Risk-aver}\&|\pi_1-\pi_2|<\mu}$, *satisfy rich transformability and everywhere reachability**.

Corollary 4: *Let* $\mathcal{D}$ *be one of the following domains in the risk sharing problem,* $\mathcal{R}^N_{\text{Risk-neut}}$, $\mathcal{R}^N_{\text{Risk}}$, $\mathcal{R}^N_{\text{Risk-aver}}$, $\mathcal{D}_{\text{Risk-neut}\&|\pi_1-\pi_2|<\mu}$, $\mathcal{D}_{\text{Risk}\&|\pi_1-\pi_2|<\mu}$, *and* $\mathcal{D}_{\text{Risk-aver}\&|\pi_1-\pi_2|<\mu}$, *where* $\mu > 0$. *Then a rule over* $\mathcal{D}$ *is efficient and strategy-proof if and only if it is dictatorial.*

Remark 5: An interesting case not considered in this paper is when all agents have a common prior. Domains with the common prior restriction do not satisfy *rich transformability*, as our discussion in Example 1 shows.[13] However, in such a common prior case, Ju (2001) shows that the same result as Theorem 1 applies when aggregate uncertainty holds.

4.3 Other restricted domains

In this section, we show that the domain of "CES preferences" and the domain of "quasilinear", strictly convex, and smooth preferences satisfy *flexibility* and *rich transformability* respectively.

Let $(a_1, \cdots, a_l) \in \mathbb{R}^l_{++}$. For each $\rho \in (-\infty, 0)$, the *CES function* $u\colon \mathbb{R}^l_+ \to \mathbb{R}$ *parametrized by* $((a_1, \cdots, a_l), \rho)$ is defined as follows: for all $x \in \mathbb{R}^l_{++}$, $u(x) \equiv (\sum_k a_k x_k^\rho)^{1/\rho}$. For each $\rho \in (0, 1)$, the *CES function* $u\colon \mathbb{R}^l_+ \to \mathbb{R}$ *parametrized by* $((a_1, \cdots, a_l), \rho)$ is defined as follows: for all $x \in \mathbb{R}^l_+$, $u(x) \equiv (\sum_k a_k x_k^\rho)^{1/\rho}$. Finally, for $\rho \equiv 0$, the *CES function* $u\colon \mathbb{R}^l_+ \to \mathbb{R}$ *parametrized by* $((a_1, \cdots, a_l), \rho)$ is defined as follows: for all $x \in \mathbb{R}^l_+$, $u^\rho(x) \equiv x_1^{a_1} \times \cdots \times x_k^{a_k}$. A preference R_i is a *CES preference* if R_i is represented by a *CES* function. Let $\mathcal{R}_{CES}$ be the class of all *CES* preferences. We refer to $\mathcal{R}^N_{CES}$ as the **CES-domain**.

Proposition 5: *The CES-domain is flexible.*

Proof: Let $M \equiv \{z \in Z : z_1 \in \overline{0, \Omega}\}$. We show that $\mathcal{R}^N_{CES}$ and M satisfy *potential efficiency, attainability, transformability with crossly local dominance,*

[13]In Example 1, we showed that when aggregate certainty holds, there exists a *efficient, strategy-proof*, and *non-dictatorial* rules over the domain of expected utility preferences with a common prior. Therefore, by Theorem 1, the domain violates *rich transformability**; in fact, the domain violate *double transformability*. We can also show that even if aggregate uncertainty holds, the domain violate *rich transformability**.

F1, and F2. The first four properties can be shown similarly to Example 2. We are left with F2. In what follows, we only consider the 2-good case and show F2; our argument can be extended to the l-good case.[14] We use the following property of Pareto set for homothetic preferences.

Fact 1 (Thomson, 1995): *Let $l = 2$. When R_1 and R_2 are homothetic preferences in $\mathcal{R}$, $P(R)$ is "doubly visible", that is, for all $z_1, z_1' \in \mathbb{R}^l_{++}$, if $z_1, z_1' \in P_1(R)$ and $z_{11} < z_{11}'$, then*
$[z_{12}/z_{11} \geq z_{12}'/z_{11}'$ *and* $z_{22}/z_{21} \geq z_{22}'/z_{21}']$ *or* $[z_{12}/z_{11} \leq z_{12}'/z_{11}'$ *and* $z_{22}/z_{21} \leq z_{22}'/z_{21}']$.

Let $i \in N$ and $d_i \in M_i$. Without loss of generality we set $i \equiv 1$. We show that for some $R \in \mathcal{R}^N_{CES}$,

(i) $P_1(R) \cap M_1 = \{0, \Omega\}$ and

(ii) if $z \in P(R) \setminus \{d\}$ and $p \cdot z_1 = p \cdot d_1$ for all $p \in \nabla R(z)$, then there exists $\bar{R}_1 \in \mathcal{R}_{CES}$ such that for all $\bar{z}_1 \in P_1(\bar{R}_1, R_2) \cap LC(R_1, z_1) \cap UC(\bar{R}_1, z_1)$, $p \cdot \bar{z}_1 \neq p \cdot d_1$ for some $p \in \nabla(\bar{R}_1, R_2)(\bar{z})$.

Clearly, (i) holds. If $d_1 \in \{0, \Omega\}$, (ii) holds vacantly.

Assume that $d_1 \notin \{0, \Omega\}$. Without loss of generality, let $\Omega \equiv (1, \cdots, 1)$. Let $u_1 \colon \mathbb{R}^l_+ \to \mathbb{R}$ and $u_2 \colon \mathbb{R}^l_+ \to \mathbb{R}$ be defined as follows: for all $x \in \mathbb{R}^l_+$, $u_1(x) \equiv (ax_1^{\rho_1} + x_2^{\rho_1})^{1/\rho_1}$ and $u_2(x) \equiv (bx_1^{\rho_2} + x_2^{\rho_2})^{1/\rho_2}$, where $\rho_1, \rho_2 \in (0, 1)$ and $a, b \in \mathbb{R}_{++}$. Let R_1 and R_2 be the two preferences represented by u_1 and u_2 respectively. Let $\rho_1, \rho_2 \in (0, 1)$ and $a, b \in \mathbb{R}_{++}$ be chosen in such a way that $P_1(R) \cap \overline{0, \Omega} = \{0, \Omega\}$, $P_1(R) \setminus \{0\} \subset \mathbb{R}^l_{++}$, and for all $z_1' \in P_1(R) \setminus \{0, \Omega\}$, $z_1' \in \mathbb{R}^l_{++}$ and $z_{12}'/z_{11}' > d_{12}/d_{11}$.

Let $z \in P(R)$ be such that $z_1 \neq d_1$ and for all $p' \in \nabla R(z)$, $p' \cdot z_1 = p' \cdot d_1$. Then clearly, $z_1 \notin \{0, \Omega\}$. Then since $P_1(R) \setminus \{0\} \subset \mathbb{R}^l_{++}$, $z_1 \in \mathbb{R}^l_{++}$ and $z_2 \in \mathbb{R}^l_{++}$. Let $p \equiv \nabla u_1(z_1)$. Since $d_1 \in \overline{0, \Omega}$ and $(\Omega_1, \Omega_2) = (1, 1)$, then for all $i \in N$, $d_{i1} = d_{12}$. Since $p \in \mathbb{R}^l_{++}$ and $p \cdot z_1 = p \cdot d_1$, then $z_1 \not\leq d_1$ and $z_1 \not\geq d_1$.

---

[14] For the l-good case, we simply use the following relations between some special preference profiles in the l-good case and their counterparts in the 2-good case.

Let $a, b \in \mathbb{R}_{++}$ and $(c_2, \cdots, c_l) \in \mathbb{R}^{l-1}_{++}$. Let $U_1 \colon \mathbb{R}^l_+ \to \mathbb{R}$ and $U_2 \colon \mathbb{R}^l_+ \to \mathbb{R}$ be defined as follows: for all $x \in \mathbb{R}^l_+$, $U_1(x) \equiv \left(a(x_1/\Omega_1)^{\rho_1} + \sum_{k=2}^l c_k (x_k/\Omega_k)^{\rho_1}\right)^{1/\rho_1}$; $U_2(x) \equiv \left(b(x_1/\Omega_1)^{\rho_2} + \sum_{k=2}^l c_k (x_k/\Omega_k)^{\rho_2}\right)^{1/\rho_2}$. Let $u_1 \colon \mathbb{R}^2_+ \to \mathbb{R}$, and $u_2 \colon \mathbb{R}^2_+ \to \mathbb{R}$ be defined as follows: for all $(x_1, x_2) \in \mathbb{R}^2_+$, $u_1(x_1, x_2) \equiv \left(a(x_1/\Omega_1)^{\rho_1} + \left(\sum_{k=2}^l c_k\right)(x_2/\Omega_2)^{\rho_1}\right)^{1/\rho_1}$; $u_2(x_1, x_2) \equiv \left(b(x_1/\Omega_1)^{\rho_2} + \left(\sum_{k=2}^l c_k\right)(x_2/\Omega_2)^{\rho_2}\right)^{1/\rho_2}$. Then for all $x \in \mathbb{R}^l_{++}$, (i) if $(x, \Omega - x)$ is efficient in l-good economy (U_1, U_2, Ω), then $\frac{x_2}{\Omega_2} = \frac{x_3}{\Omega_3} = \cdots = \frac{x_l}{\Omega_l}$ and $((x_1, x_2), (\Omega_1 - x_1, \Omega_2 - x_2))$ is efficient in 2-good economy $(u_1, u_2, (\Omega_1, \Omega_2))$, and (ii) if $((x_1, x_2), (\Omega_1 - x_1, \Omega_2 - x_2))$ is efficient in 2-good economy $(u_1, u_2, (\Omega_1, \Omega_2))$, then $((x_1, x_2, \frac{\Omega_3}{\Omega_2} x_2 \cdots, \frac{\Omega_l}{\Omega_2} x_2), (\Omega_1 - x_1, \Omega_2 - x_2, \Omega_3(1 - \frac{x_2}{\Omega_2}), \cdots, \Omega_l(1 - \frac{x_2}{\Omega_2})))$ is efficient in l-good economy (U_1, U_2, Ω).

Let $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_{++}$ be such that $u_2(1-\bar{x}_1, 1-\bar{x}_2) = u_2(z_{21}, z_{22})$ and $\bar{x}_1 < z_{11}$ (since $z_1, z_2 \in \mathbb{R}^l_{++}$, there exists such $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2_{++}$). Let $(\bar{p}_1, \bar{p}_2) \in \mathbb{R}^2_{++}$ be a vector normal to $\overline{(\bar{x}_1, \bar{x}_2), (d_{11}, d_{12})}$. Then

$$(\star) \ \ (\bar{p}_1, \bar{p}_2) \cdot (\bar{x}_1, \bar{x}_2) = (\bar{p}_1, \bar{p}_2) \cdot (d_{11}, d_{12}) < (\bar{p}_1, \bar{p}_2) \cdot (z_{11}, z_{12}).$$

Then there exists a CES function $\bar{u}$ such that $\bar{u}_1(\bar{x}_1, \bar{x}_2) = \bar{u}_1(z_{11}, z_{12})$ and $\nabla \bar{u}_1(\bar{x}_1, \bar{x}_2) = (\bar{p}_1, \bar{p}_2)$. Without loss of generality, we assume that $\bar{u}_1(x_1, x_2) \equiv \left(\bar{a} x_1^{\bar{\rho}_1} + x_2^{\bar{\rho}_1}\right)^{1/\bar{\rho}_1}$, where $\bar{a} \in \mathbb{R}_{++}$ and $\bar{\rho}_1 \in (-\infty, 1)$. Then $\nabla \bar{u}_1(x) \equiv K \cdot (\bar{a} x_1^{\bar{\rho}-1}, x_2^{\bar{\rho}-1})$, where $K \equiv \left(\bar{a} x_1^{\bar{\rho}} + x_2^{\bar{\rho}}\right)^{1/\bar{\rho}-1}$, for all $x \in \mathbb{R}^l_{++}$.

By $(\star)$, $\nabla \bar{u}_1(\bar{x}) \cdot \bar{x} = \nabla \bar{u}_1(\bar{x}) \cdot d_1$. By Fact 1, we can show that for all $\bar{z}_1 \in P_1(\bar{R}_1, R_2) \cap LC(R_1, z_1) \cap UC(\bar{R}_1, z_1)$, $\bar{x}_2/\bar{x}_1 > \bar{z}_{12}/\bar{z}_{11} > d_{12}/d_{11} = 1$. Clearly, $\nabla \bar{u}_1(\bar{x}) \cdot \bar{z}_1 > \nabla \bar{u}_1(\bar{x}) \cdot x = \nabla \bar{u}_1(\bar{x}) \cdot d_1$. Then $\bar{a}\bar{x}_1^{\bar{\rho}-1}\bar{z}_{11} + x_2^{\bar{\rho}-1}\bar{z}_{12} > \bar{a}\bar{x}_1^{\bar{\rho}-1} d_{11} + \bar{x}_2^{\bar{\rho}-1} d_{12}$, that is, $\bar{a}\bar{z}_{11} - \bar{a}d_{11} + (\frac{\bar{x}_1}{\bar{x}_2})^{1-\bar{\rho}}(\bar{z}_{12} - d_{12}) > 0$. Since $\bar{z}_{11} < d_{11}$, $\bar{z}_{12} > d_{12}$. Since $\frac{\bar{x}_1}{\bar{x}_2} < \frac{\bar{z}_{11}}{\bar{z}_{12}}$, $\bar{a}\bar{z}_{11} - \bar{a}d_{11} + (\frac{\bar{z}_{11}}{\bar{z}_{12}})^{1-\bar{\rho}}(\bar{z}_{12} - d_{12}) > 0$. Therefore, $\nabla \bar{u}_1(\bar{z}_1) \cdot \bar{z}_1 > \nabla \bar{u}_1(\bar{z}_1) \cdot d_1$. *Q.E.D.*

A preference $R_0 \in \mathcal{R}$ is **quasilinear with respect to the numeraire good k** $\in \{1, \cdots, l\}$ if for all $x, y \in \mathbb{R}^l_+$ and all $\alpha \in \mathbb{R}$, whenever $x + \alpha e_k, y + \alpha e_k \in \mathbb{R}^l_+$, where e_k is the unit vector with zero components except at the k^{th} component, $x \ I_0 \ y \Rightarrow (x + \alpha e_k) \ I_0 \ (y + \alpha e_k)$. Let $\mathcal{R}_Q$ be the family of *quasilinear*, *strictly convex*, and *smooth* preferences with respect to a common numeraire good.

Proposition 6: *Domain $\mathcal{R}^N_Q$ satisfies rich transformability.*

Proof: Let $\rho \in (0,1)$. For each $a > 0$, let $u^a \colon \mathbb{R}^l_+ \to \mathbb{R}$ be such that: for all $x \in \mathbb{R}^l_+$, $u^a(x) \equiv a \cdot \frac{x_1}{\Omega_1} + \sum_{k=2}^{l} \left(\frac{x_k}{\Omega_k} + 1\right)^{\rho}$. Let $\mathcal{R}_{Q,\rho}$ be the set of preferences represented by u^a for some $a > 0$. Let $M^1 \equiv \{z \in \bar{Z} : z_{11} = 0$ or $\frac{z_{12}}{\Omega_2} = \frac{z_{13}}{\Omega_3} = \cdots = \frac{z_{1l}}{\Omega_l} = 1\}$, and $M^2 \equiv \{z \in Z : \frac{z_{11}}{\Omega_1} = 1$ or $\frac{z_{12}}{\Omega_2} = \frac{z_{13}}{\Omega_3} = \cdots = \frac{z_{1l}}{\Omega_l} = 0\}$. Let $\bar{\mathcal{D}} \equiv \mathcal{R}^N_{Q,\rho}$ and $M \equiv M^1$. We show that $\bar{\mathcal{D}}$ and M satisfy *potential efficiency, attainability, transformability with crossly local dominance* and *double transformability.*

It is easy to show the following claim.

Claim 1: *Let R_1, R_2 be represented by u^{a_1}, u^{α_2} respectively. Then for all $i \in N$,*

$$P(R) = M^i \text{ if and only if } a_i \leq a_{-i} \cdot 2^{1-\rho}$$

Potential efficiency and *attainability* are trivial.

Transformability with crossly local dominance: Let $R, R' \in \bar{\mathcal{D}}$ and $z_i, z'_i \in M_i$ be such that $P(R) = P(R') = M$ and $z_i \neq z'_i$. Without loss of generality, assume $z_1 \leq z_1'$. For all $i \in N$, let R_i be represented by u^{a_i} and R_i' by $u^{a_i'}$, where $a_i, a_i' > 0$. Then by Claim 1, $a_1 \leq a_2 \cdot 2^{1-\rho}$ and $a_1' \leq a_2' \cdot 2^{1-\rho}$.

Let $\bar{a}_1 \equiv \min\{a_1, a_1'\}$. Let $\bar{R}_1$ be represented by $u^{\bar{a}_1}$. Then since $\bar{a}_1 \leq a_2 \cdot 2^{1-\rho}$ and $\bar{a}_1 \leq a_2' \cdot 2^{1-\rho}$, then by Claim 1, $P(\bar{R}_1, R_2) = P(\bar{R}_1, R_2') = M$. Since M_1 is a monotone path and all preferences are strictly monotonic, $P_1(\bar{R}_1, R_2) \cap LC(R_1, z_1) \cap UC(\bar{R}_1, z_1) = \{z_1\}$. Hence part (i) of *transformability with crossly local dominance* holds. Similarly, $P_1(\bar{R}_1, R_2') \cap \{x \in Z_0 : \Omega - x \in LC(R_2, z_2) \cap UC(R_2', z_2)\} = \{z_1\}$. Since $z_1 \leq z_1'$ and $\bar{R}_1$ is strictly monotonic, $z_1' \ \bar{P}_1 \ z_1$. Therefore part (ii) of *transformability with crossly local dominance* also holds. □

Double transformability: Let $d_1 \in M_1$. Let $R \in \bar{\mathcal{D}}$ be such that $P(R) = M^2$. Let $z \in P(R)$ and $z_1 \neq d_1$. Then clearly, $P_1(R) \cap M_1 = \{0, \Omega\}$: that is, part (i) of *double transformability* holds.

When $d_1 \geq z_1$ or $d_1 \leq z_1$, if we let $R_1' \equiv R_1$, then $P_1(R_1', R_2) \cap LC(R_1, z_1) \cap UC(R_1', z_1) = \{z_1\}$. Therefore when $d_1 \geq z_1$, by strict monotonicity of R_1, (ii-1) of *double transformability* is satisfied with respect to $R_1'' \equiv R_1$. When $d_1 \leq z_1$, (ii-2) of *double transformability* is satisfied with respect to $R_2' \equiv R_2$.

Now assume that $d_1 \ngeq z_1$ and $d_1 \nleq z_1$. Then $d_{11} < z_{11}$ and $d_{12} > z_{12}$. Let $a_1, a_2 > 0$ be such that R_1 is represented by u^{a_1} and R_2 is represented by u^{a_2}. Then by Claim 1, $a_2 \leq a_1 \cdot 2^{1-\rho}$.

Let $a_2' > (l-1) \cdot \frac{(z_{22}+1)^\rho - (d_{22}+1)^\rho}{d_{21} - z_{21}}$. Let $a_1' \geq \max\{a_1, \frac{a_2'}{2^{1-\rho}}\}$. Let R_1' be represented by $u^{a_1'}$ and R_2' be represented by $u^{a_2'}$. Then since $a_1' \geq a_1$, $a_1' \cdot 2^{1-\rho} \geq a_2$. Hence by Claim 1, $P_1(R_1', R_2) = M^2$. Since M_1^2 is a monotone path through z_1 and all preferences are strictly monotonic, $P_1(R_1', R_2) \cap LC(R_1, z_1) \cap UC(R_1', z_1) = \{z_1\}$.

Since $a_1' \geq a_2'/2^{1-\rho}$, by Claim 1, $P(R_1', R_2') = M^2$. Since M_2^2 is a monotone path through z_2 and all preferences are strictly monotonic, $P_2(R_1', R_2') \cap LC(R_2, z_2) \cap UC(R_2', z_2) = \{z_2\}$. Since $a_2' > (l-1) \cdot \frac{(z_{22}+1)^\rho - (d_{22}+1)^\rho}{d_{21} - z_{21}}$, $d_2 \ P_2' \ z_2$. Therefore, (ii-2) of *double transformability* is satisfied. □

Q.E.D.

Corollary 5: *Let $\mathcal{D} \in \{\mathcal{R}_{CES}^N, \mathcal{R}_Q^N\}$. Then a rule over $\mathcal{D}$ is efficient and strategy-proof if and only if it is dictatorial.*

5 Concluding remarks

1. In several other economic environments, a number of authors have reported the same impossibility results as in the 2-agent exchange economy. Among others are Walker (1980), Zhou (1991b), Schummer (1999), Serizawa (1998), and Le Breton and Weymark (1999). Identification of general domain properties that induce their impossibility results will be an interesting research agenda.

2. Our conclusion makes use of the strong invariance property of *efficient* and *strategy-proof* rules, which is established in Lemma 2. The same invariance property will hold in other economic environments, for example, exchange economies with more than two agents, classical production economies, public goods economies, etc. Application of the strong invariance property may lead to simpler proofs and extensions of the existing results, for example, by Walker (1980), Satterthwaite and Sonnenschein (1981), Serizawa (1998), and Schummer (1999).

References

[1] Barberà, S. and Jackson, M. (1995), "Strategy-proof exchange", *Econometrica* **63**:51-87.

[2] Barberà, S., H. Sonnenschein, and L. Zhou (1991), "Voting by committees", *Econometrica* **59**(3).

[3] Berga, D. and S. Serizawa, "Maximal domain for strategy-proof rules with one public good", *Journal of Economic Theory* **78**:157-166.

[4] Dasgupta, P., Hammond, P., and Maskin, E. (1979), "The implementation of social choice rules", *Review of Economic Studies* **46**:153-170.

[5] Gibbard, A. (1973), "Manipulation of voting scheme: a general result", *Econometrica* **41**: 587-602.

[6] Hurwicz, L. (1972), "On informationally decentralized systems," in McGuire and Radner (eds) *Decision and Organization*, North-Holland, Amsterdam, pp. 297-336.

[7] Hurwicz, L. and M. Walker (1990), "On the generic non-optimality of dominant strategy allocation mechanisms", *Econometrica* **58**:683-704.

[8] Ju, B.-G. (2001), "Strategy-proof risk sharing", mimeo.

[9] Le Breton, M. and A. Sen (1999), "Separable preferences, strategyproofness, and decomposability", *Econometrica* **67**(3).

[10] Le Breton, M. and J. Weymark (1999), "Strategy-proof social choice with continuous separable preferences", *Journal of Mathematical Economics* **32**: 47-85.

[11] Maniquet, F. and Y. Sprumont (1999), "Efficient strategy-proof allocation functions in linear production economies", *Economic Theory*.

[12] Mas-Colell, A., M.D. Whinston, and J.R. Green (1995), *Microeconomic Theory*, New York: Oxford University Press.

[13] Maskin (1999), "Nash equilibrium and welfare optimality", *Review of Economic Studies* 66:23-38.

[14] Moore, J. (1993), "Implementation, contracts, and renegotiation in environments with complete information", J.-J. Laffont (ed.) *Advances in Economic Theory,* Cambridge: Cambridge University Press, pp.182-282.

[15] Satterthwaite, M. (1975), "Strategy-proofness and Arrow's conditions: existence and correspondence theorems for voting procedures and social welfare functions", *Journal of Economic Theory* **10**:187-217.

[16] Satterthwaite, M. and H. Sonnenschein (1981), "Strategy-proof allocation mechanisms at differentiable points", *Review of Economic Studies.*

[17] Schummer, J. (1997), "Strategy-proofness versus efficiency on restricted domains of exchange economies", *Social Choice and Welfare* **14**:47-56.

[18] Schummer, J. (1999), "Strategy-proofness versus efficiency for small domains of preferences over public goods", *Economic Theory* **13**:709-722.

[19] Serizawa, S. (1998), "Inefficiency of strategy-proof rules for pure exchange economies", Working Paper.

[20] Thomson, W. (1995), *The Theory of Fair Allocation.*

[21] Walker, M. (1980), "On the nonexistence of a dominant strategy mechanism for making optimal public decisions", *Econometrica* **48**:1521-1540.

[22] Zhou, L. (1991a), "Inefficiency of strategy-proof allocation mechanisms in pure exchange economies", *Social Choice and Welfare* **8**:247-254.

[23] Zhou, L. (1991b), "Impossibility of strategy-proof mechanisms in economies with pure public goods", *Review of Economic Studies* **58**:107-119.